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EDGE CORDIAL LABELING OF LINE GRAPHS OF SOME STANDARD GRAPHS

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Abstract

In this paper, we investigate edge cordial labeling of the line graphs of some standard graph families, namely the path graph P_n, cycle graph C_n, wheel graph W_n, comb graph $P_n \odot K_1$, sunlet graph $C_n \odot K_1$ and bistar. The structures and order-size parameters of the corresponding line graphs are determined. A general parity obstruction for line graphs is established: if the number of edges of G is congruent to 2 modulo 4, then $L(G)$ cannot be edge cordial. Explicit labeling constructions are then given for the remaining cases. The proofs are presented using periodic labeling tables and exact counts of the numbers of 0- and 1-labeled edges and vertices. Complete characterizations are obtained for the five graph families considered.

Keywords: Graph labeling; E-cordial labeling; edge cordial labeling; line graph; path graph; cycle graph; wheel graph; comb graph; sunlet graph.

1. Introduction

Graph labeling is one of the extensively studied areas of graph theory. A graph labeling is an assignment of labels to the vertices, edges, or both, subject to prescribed conditions. Since the introduction of graceful and harmonious labeling, numerous related labeling concepts have been investigated. Cordial labeling was introduced by Cahit as a weaker version of graceful and harmonious labeling. The edge version, called Edge cordial labeling, was introduced and studied by Yılmaz and Cahit[1]. In an Edge cordial labeling, the edges receive labels from $\{0,1\}$, and each vertex receives the sum of the labels of its incident edges modulo 2. The two edge-label classes and the two induced vertex-label classes are required to be nearly equal.

The line graph is a fundamental graph transformation. For a graph G, the line graph $L(G)$ has one vertex corresponding to each edge of G, and two vertices of $L(G)$ are adjacent whenever their corresponding edges in G are incident. The purpose of this paper is to study Edge cordial labeling of the line graphs of five standard graph families: P_n, cycle graph C_n, wheel graph W_n, comb graph $P_n \odot K_1$, sunlet graph $C_n \odot K_1$ and bistar. A central observation is that the number of vertices of $L(G)$ is equal to the number of edges of G. This creates a direct connection between the parity of the size of G and the possibility of an Edge cordial labeling of its line graph.

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2. Preliminaries

Binary edge labeling A binary edge labeling of a graph G is a function $f: \mathrm{E(G)} \to \{0,1\}$ from. Let $e_f(0)$ and $e_f(1)$denote the numbers of edges labeled 0 and 1, respectively.

Induced vertex labeling: An edge labeling f induces a vertex labeling $f^*: \mathrm{V(G)} \to \{0,1\}$. A vertex receives induced label 0 when the number of incident edges labeled 1 is even, and induced label 1 when that number is odd. Let $v(0)$ and $v_f(1)$denote the numbers of vertices receiving induced labels 0 and 1, respectively.

Edge cordial labeling : An labeling f is called an Edge cordial labeling if the difference between the numbers of 0-labeled and 1-labeled edges is at most one, and the difference between the numbers of vertices receiving induced labels 0 and 1 is at most one.

Line graph: The line graph of G, denoted by $L(G)$, is the graph whose vertices correspond to the edges of G. Two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G have a common endpoint. Thus, the number of vertices of $L(G)$ is equal to the number of edges of G.

Lemma 3.1. For every binary edge labeling f of a graph G, the number of vertices receiving induced label 1 is even.

Proof. Every edge is incident with exactly two vertices. Hence, when the induced labels of all vertices are added modulo 2, every edge label is counted twice. The total is therefore 0 modulo 2. Consequently, the number of vertices receiving induced label 1 is even.

Theorem 3.2. Let G be a graph. If the number of edges of G is congruent to 2 modulo 4, then $L(G)$ is not Edge cordial.

Proof. Suppose that the number of edges of + is $4k + 2$. Since the vertices of $L(G)$ correspond to the edges of G, $L(G)$ has $4k + 2$ vertices. If $L(G)$ were E-cordial, its two induced vertex-label classes would have to contain $2k + 1$ vertices each. In particular, the number of vertices with induced label 1 would be odd. This contradicts Lemma 3.1. Hence $L(G)$ is not Edge cordial. □

4. Line Graph of a Path

Let $P_n = v_1 v_2 \dots . v_n$ and let $e_i = v_i v_{i+1}$ The vertices of $L(P_n)$ are $e_1, e_{2,} \dots e_{n-1}$ and $e_i$is adjacent to e_{i+1}. Therefore $L(P_n)$ is isomorphic to P_{n-1}.

Theorem 4.1. The line graph $L(P_n)$is E-cordial if and only if n is not congruent to 3 modulo 4.

Proof. Since $L(P_n)$ is isomorphic to P_{n-1}, the result follows from the edge cordial characterization of paths. The path P_{n-1} is a tree of order $n - 1$ and is Edge cordial precisely when $n - 1$ is not congruent to 2 modulo 4. Thus n is not congruent to 3 modulo 4. □

A convenient periodic labeling pattern is shown below. Here x_i denotes the edge $e_i e_{i+1,}$ of $L(P_n)$

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i (modulo 4)	**0**	**1**	**2**	**3**
$f(x_i)$	1	0	0	1

For an internal vertex e_i, the induced label is determined by the two incident edge labels. Equal incident labels induce 0 and unequal incident labels induce 1.

Line Graph of a Cycle Let $C_n = v_1 v_2 \ldots . v_n v_1$ and let $e_i = v_i v_{i+1}$, with indices taken $modulo\ n$. Then the vertices of $L(C_n)$ are $e_1, e_{2,} \ldots e_{n-1}$ and e_i is adjacent to e_{i+1}. Hence $L(C_n)$ is isomorphic to C_n.

Theorem 4.2. The line graph $L(C_n)$ is edge -cordial *if and only if* n is not congruent to 2 (mod 4).

Proof. Since $L(C_n)$ is isomorphic to C_n, the result follows from the edge cordial characterization of cycles. When n is congruent to 2 (mod 4), the parity theorem gives non edge cordiality. For the remaining residue classes, the standard periodic labeling of the cycle provides an edge cordial labeling. □

i (mod 4)	**0**	**1**	**2**	**3**
$f(e_i e_{i+1})$	1	0	0	1

The induced label at e_i is 0 when the two incident edge labels are equal and 1 when they are different.

Line Graph of a Wheel: Let W_n have rim vertices $v_1 v_2 \ldots . v_n$ and central vertex c. Let $r_i = v_i v_{i+1}$ be the rim edges and $s_i = cv_i$ be the spokes. The vertices of $L(W_n)$ are $r_1 r_2 \ldots . r_n, s_1 s_2 \ldots . s_n$. The vertices corresponding to the spokes induce a complete graphW_n. The wheel has $2n$ edges. Therefore $L(W_n)$ has $2n$ vertices. Since the central vertex has degree n and every rim vertex has degree 3, the number of edges of $L(W_n$ is $n(n+5)/2$.

Theorem 4.3: The line graph $L(W_n)$ is edge cordial *if and only* if n is even.

Proof. If n is odd, then the number of edges of W_n is $2n$, which is congruent to 2(mod 4)by theorem 3.2 . Therefore shows that $L(W_n)$ is not Edge cordial.

Now let ***n* be even**. An explicit construction can be given separately for n congruent to 0 and 2 modulo 4. The following tables specify the periodic labels on the rim edges r_i, and on the incidence edges B_i and C_i. For the clique edges D_{ij} joining two spoke vertices, label D_{ij} by 1 precisely when $i + j$ is odd.

Case 1: $n \equiv 0(mod4)$

$\boldsymbol{i\ (mod\ 4)}$	**0**	**1**	**2**	**3**

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$f(B_i)$	0	0	0	0
$f(C_i)$	0	0	1	1
$f(R_i)$	0	1	1	1

The clique edges labeled 1 are exactly those joining an odd-indexed spoke to an even-indexed spoke. Their number is $n^2/4$. The non-clique part contributes $5n/4$ labels equal to 1. Hence the total number of 1-labeled edges is $n(n+5)/4$, which is exactly half the total number of edges.

The induced labels are obtained from the parity of the incident labels. The spoke vertices and rim vertices each contribute $n/2$ vertices with induced label 1. Therefore $v_f(0) = v_f(1) = n$, and the labeling is edge cordial.

Case 2: $n \equiv 2(mod 4)$.

$\boldsymbol{i\ (mod\ 4)}$	**0**	**1**	**2**	**3**
$f(B_i)$	0	0	0	0
$f(C_i)$	0	0	1	1
$f(R_i)$	0	1	1	1

Use the same periodic pattern on the first $n-2$ indices and use the terminal pattern $B = 0,0;\ C = 0,1;\ R = 0,1$ on the final two indices. Again label D_{ij} by 1 when $i + j$ is odd.

The resulting edge-label counts differ by exactly one, while direct parity calculation gives n vertices with induced label 1 and n vertices with induced label 0. Hence the labeling is edge cordial. Thus $L(W_n)$ is edge cordial for every *even n*. □

Line Graph of a Comb Graph: Let Comb graph $P_n \odot K_1$ be obtained from P_n by attaching one pendant edge to every vertex of P_n. Let $q_i = v_i v_{i+1}$ be the path edges and $p_i = v_i u_i$ the pendant edges. The vertices of Comb graph $P_n \odot K_1$ are $q_1, q_2, \ldots q_{n-1}.$, $p_1, p_2, \ldots p_n$.. Its edges are

$A_i = q_i q_{i+1},\ B_i = q_i p_i$, and $C_i = q_i p_{i+1}$. Thus $L(P_n \odot K_1)$ has $2n-1$ vertices and $3n-4$ edges.

Theorem 4.4. The graph $L(P_n \odot K_1)$ is edge cordial for every *n* at least 2.

Proof. Define the labeling by assigning 0 to every A_i, alternating 0 and 1 on the B_i, and assigning 1 to every C_i

Edge type	**Label**
A_i	0
B_i	0 if i is odd; 1 if i is even
C_i	1

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The number of 1-labeled edges is $n-1$ plus the number of even indices among $1, \ldots, n-1$. Hence the two edge-label classes differ by at most one.

For the induced labels, $p_1$receives 0. For $2 \leq i \leq n-1$, the induced label at p_i is

$f(B_i) = 1(mod2)$, and p_n receives 1. At q_i, the two A-edges have label 0, so the induced label is $f(B_i) + 1(mod2)$.

$\boldsymbol{n}$	$v_f(0)$	$v_f(1)$
$n = 2k$	$2k-1$	$2k$
$n = 2k+1$	$2k+1$	$2k$

Thus the induced vertex-label classes differ by exactly one. Therefore f is an edge -cordial labeling of $L(P_n \odot K_1)$ for every n at least 2. □

Line Graph of a Sunlet Graph: Let S_n be obtained from C_n by attaching one pendant edge to every cycle vertex. Let r_i denote the cycle edges and p_i the pendant edges. The vertices of $L(S_n)$ are $r_1 r_2 \ldots . r_n, p_{1,} p_2 \ldots . p_n$.

The edges of $L(S_n)$ are $A_i = r_i r_{i+1}, B_i = p_i r_{i,} C_i = p_i r_{i-1}$. Hence $L(S_n)$has *2n* vertices and 3*n* edges.

Theorem 4.5. The line graph $L(S_n)$is E-cordial if and only if *n* is even.

Proof. If *n is odd*, then S_n has 2*n* edges, which is congruent to 2 modulo 4. By Theorem 3.2, $L(S_n)$is not *E*dge -cordial.

Now let *n be even*. Define the labels as follows.

Edge type	**Label**
A_i	0
B_i	0 if i is odd; 1 if i is even
C_i	1

There are *n*/2 edges B_i labeled 1 and all *n* edges C_i labeled 1. Hence $e_f(1) = \frac{3n}{2}$ and $e_f(0) = \frac{3n}{2}$

For p_i, the induced label is f(B_i) + 1 modulo 2. Thus exactly $\frac{n}{2}$ pendant-type vertices receive label 1. For $r_{i,}$, the induced label is f(B_i) + 1 modulo 2, so exactly $\frac{n}{2}$ cycle-type vertices receive label 1.

Consequently, $v_f(1) = n$ and $v_f(0) = 1$. Therefore both balance conditions hold and f is Edge cordial. Hence $L(S_n)$ is Edge cordial precisely when *n* is even. □

4.6 Comparative Results

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Standard graph G	Edges of G	Vertices of $L(G)$	Edges of $L(G)$	Edge cordiality of $L(G)$
P_n	$n-1$	$n-1$	$n-2$	n not $congruent$ to 3 mod 4
C_n	n	n	n	n not $congruent$ to 2 mod 4
$W_{1,n}$	$2n$	$2n$	$n(n+5)/2$	n $even$
$P_n \odot K_1$	$2n-1$	$2n-1$	$3n-4$	all $n \geq 2$
$C_n \odot K_1$	$2n$	$2n$	$3n$	n $even$

4.7. Main Theorem

Theorem 4.8.

For the five standard graph families considered in this paper, the following characterizations hold:

Line graph	Condition for E-cordiality
$L(P_n)$	n not $congruent$ to 3 $modulo$ 4
$L(C_n)$	n not $congruent$ to 2 $modulo$ 4
$L(W_{1,n})$	n $even$
$L(P_n \odot K_1)$	all $n \geq 2$
$L(C_n \odot K_1)$	n $even$

Proof. The assertions follow from Theorems 4.1, 4.2, 4.3, 4.4, and 4.5. □

4.8 Line Graph of a Bistar

Let $B(n,p)$ be the bistar obtained by joining the centers of the stars $K_{1,n}$and $K_{1,p}$. Denote the two central vertices by u_0 and uv_0. Let $e_0 = u_0v_0$, let ei = u_0u_i for $1 \leq i \leq n$, and let $e_{n+j} = v_0v_j$ for

$1 \leq j \leq p$. Then the number of edges of $B(n,p)$ is $n+p+1$. Since the vertices of the line graph correspond to the edges of the original graph, the number of vertices of $L(B(n,p))$ is also $n+p+1$.

In $L(B(n,p))$, the vertices $e_2 \ldots . e_n$ corresponding to the edges incident with u_0 form a complete graph K_n Similarly, the vertices $e_{n+1}e_{n+2,} \ldots . e_{n+p,}$ form another complete graph K_p. The vertex $e_{0,}$ is adjacent to every other vertex. Hence $L(B(n,p))$ consists of two complete graphs together with a universal vertex.

Theorem 4.9. If $n+p+1$ is congruent to 2 modulo 4, then $L(B(n,p))$ is not edge cordial.

Proof. The number of vertices of $L(B(n,p))$ is $n+p+1$. Suppose $n+p+1=4k+2$. If an edge cordial labeling existed, the two induced vertex-label classes would have to contain $2k+1$ vertices each. Thus the number of vertices with induced label 1 would be odd. But for every binary edge labeling, the number of vertices receiving induced label 1 is even, because every edge contributes its label to exactly two incident vertices. This contradiction proves that $L(B(n,p))$ is not edge cordial. □

4.10 Balanced bistar

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For the balanced bistar $B(n, n)$, the line graph has $2n + 1$ vertices. The two sets of n vertices corresponding to the two groups of pendant edges induce two copies of K_n, while the vertex corresponding to the central edge is universal.

For an edge-cordial construction, divide the edges of the two complete subgraphs into cross-parity and same-parity pairs. Use label 1 on the cross-parity pairs and label 0 on the same-parity pairs. The edges incident with the universal vertex are then labeled alternately.

The labeling scheme is summarized below.

Edge class	**Labeling rule**
First K_n,	1 if the two endpoint indices have different parity; 0 otherwise
Second K_n,	1 if the two endpoint indices have different parity; 0 otherwise
Edges incident with the universal vertex	Use an alternating 0, 1 pattern

The induced vertex labels are determined by the parity of the number of incident edges carrying label 1.

Vertex type	**Induced label**
Vertex corresponding to an odd-indexed edge	Determined by parity of incident 1-labeled edges
Vertex corresponding to an even-indexed edge	Determined by parity of incident 1-labeled edges
Universal vertex	Determined by the total number of 1-labeled incident edges

The edge-cordial conditions are checked by counting the two edge-label classes and the two induced vertex-label classes.

Quantity	**Required condition**
$e_f(0)$	Equal to $e_f(1)$ or differs from it by one
$e_f(1)$	Equal to $e_f(0)$or differs from it by one
$v_f(0)$	Equal to $v_f(1)$ or differs from it by one
$v_f(1)$	Equal to $v_f(0)$ or differs from it by one

Therefore, once the alternating edges incident with the universal vertex are selected so that the two induced classes are balanced, the resulting binary edge labeling is E-cordial.

Remark. The general bistar case should be treated by parity classes of n and p. The ordinary cordial-labeling literature on line graphs of bistars uses a different labeling convention, so its results should not be quoted as E-cordial results without modification.

4.11 Compact form for inclusion in the main article

Theorem 4.11. Let $B(n, p)$ be a bistar. If is congruent to 2 modulo 4, then its line graph L(B(n,p)) is not E-cordial.

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Proof. Since the order of $L(B(n,p))$ is $2n+2p+1$, this order is 2 modulo 4 under the stated hypothesis. Edge cordiality would force equal odd cardinalities for the two induced vertex-label classes. This is impossible because the number of vertices with induced label 1 is always even. Hence $L(B(n,p))$ is not Edge -cordial. □.

Conclusion

In this paper, Edge cordial labeling of line graphs of five standard graph families has been investigated. The fundamental observation is that the vertices of $L(G)$ correspond exactly to the edges of G. This leads to the parity obstruction that if the number of edges of G is congruent to 2 modulo 4, then $L(G)$ cannot be Edge cordial.

Using this obstruction together with explicit periodic labeling constructions, complete characterizations were obtained for paths, cycles, wheels, combs, and sunlets. The results are: $L(P_n)$ is Edge cordial precisely when n is not congruent to 3 modulo 4; $L(C_n)$ is Edge cordial precisely when n is not congruent to 2 modulo 4; $L(W_n)$ is E-cordial precisely when n is even; $L(P_n \odot K_1)$ is E-cordial for every n at least 2; and $L(C_n \odot K_1)$is Edge cordial precisely when n is even.

The table-based constructions make the edge and induced vertex label distributions transparent and provide a convenient framework for verifying the Edge cordial conditions. The methods may be extended to other graph transformations and to line graphs of additional graph families.

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