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# Detour Certified Domination Number of Splitting Graphs of Certain graphs

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***Abstract***

In this paper, we introduce the concept detour certified domination number of a graph. Also detour certified domination number of a certain classes of graphs are determined and some of its general properties are studied. A set $S$ of vertices in a connected graph $G = (V, E)$ is called a detour set if every vertex not in $S$ lies on a longest path between two vertices from $S$. A set $D$ of vertices in $G$ is called a dominating set of $G$ if every vertex not in $D$ has at least one neighbour in $D$. A set $T \subseteq V(G)$ is called a certified dominating set of $G$ if $T$ is a dominating set and every vertex in $T$ has either zero or at least two neighbours in $V - T$. A set $S$ is called a detour certified dominating set of $G$ if $S$ is both detour and certified dominating set of $G$. The detour certified domination number is the minimum cardinality of a detour certified dominating set in $G$.

*Keywords:* Detour set, Dominating set, certified domination, Detour certified domination, Splitting Graph

*AMS Subject classification: 05C12, 05C69.*

## *1 Introduction*

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The order $|V|$ and size $|E|$ of $G$ are denoted by $p$ and $q$ respectively. For graph theoretic terminology we refer to west [9]. A vertex $v$ of $G$ is said to be an extreme vertex if the subgraph induced by its neighbourhood is complete. The set of all extreme vertices is denoted by $Ext(G)$. For vertices $x$ and $y$ in a connected graph $G$, the detour distance $D(x, y)$ is the length of a longest $x - y$ path in $G$. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of $G$. For $S \subseteq V$, $I_D[S] = \cup_{x,y \in S} I_D[x, y]$. A set $S$ of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. The concept of detour set in graph was introduced by G.Chartrand et.all [2,3]. For distance related terminology we refer to Buckley and Harary [1].

A set $S \subseteq V(G)$ in a graph $G$ is a dominating set of $G$ if every vertex $v$ in $V - S$, there exists a vertex $u \in S$ such that $v$ is adjacent to $u$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$ [5].

A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called an end vertex or a pendant vertex. A vertex that is adjacent to a pendant vertex is called a support vertex. A

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vertex of degree $p-1$ is called a full vertex. The concept of certified domination in graph was introduced by Magoba Dettlaff et.all [4].

**Definition 1.1** Let $G=(V,E)$ be any graph of order n. A subset $S\subseteq V(G)$ is said to be a certified dominating set of G if S is a dominating set of G and every vertex in S has either zero or at least two neighbours in $V-S$. The certified domination number denoted by $\gamma_{cer}(G)$ is the minimum cardinality of certified dominating sets in G.

For graph $G$, some subsets of $V$ are detour set but not certified dominating set in $G$. Further, some other are certified dominating set but not detour sets in $G$. Moreover, some of them that are both certified dominating and detour set in $G$.

Here we see that a detour set is not in general a certified dominating set in $G$. The converse also not true in general. This motivated to study a new domination concept of detour certified domination. Hence, we study that subsets of $V(G)$ that are both a certified dominating and a detour set in $G$. We call these sets as detour certified domination number as the minimum cardinality taken over all the detour certified dominating sets of $G$.

**Definition 1.2** Let $G=(V,E)$ be any connected graph with at least two vertices. A subset $S\subseteq V(G)$ is called a detour certified dominating set of $G$ if $S$ is both detour and certified dominating set of $G$. A detour certified dominating set $S$ is said to be a minimum detour certified dominating set of $G$ if there exists no detour certified dominating set $S'$ of $G$ such that $|S'|<|S|$. The cardinality of a minimum detour certified dominating set of $G$ is the detour certified dominating number of $G$. It is denoted by $c\gamma_d(G)$.

Any detour certified dominating set $S$ of cardinality $c\gamma_d(G)$ is called a $c\gamma_d-$ set of $G$ or $c\gamma_d(G)$ $-$ set.

## *2. Some basic results*

In this section, we call some definitions and basic results of detour set and certified dominating set of a graph which will be used throughout the paper.

**Theorem 2.1** [3] *Every end vertex of $G$ belongs to every detour set of $G$.*

**Theorem 2.2** [4] *Every support vertex of $G$ belongs to every certified dominating set of $G$.*

## *3. Detour Certified Domination number in splitting graphs.*

In this section, we define the detour certified domination number for splitting graphs some families of graphs.

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**Definition: 3.1 [8]**

The splitting graph $S'(G)$ of a connected graph G is obtained by adding a new vertex $x'$ corresponding to each vertex x of G such that $x'$ is adjacent to every vertex adjacent to x of G.

If n is the order of G, then the order of $S'(G)$ is 2n. We call the vertices $x_1, x_2, \dots, x_n$ are duplicated by $x_1', x_2', \dots, x_n'$.

**Example 3.2**

Consider the connected graph G and its corresponding splitting graph $S'(G)$ are given in Figure 3.1

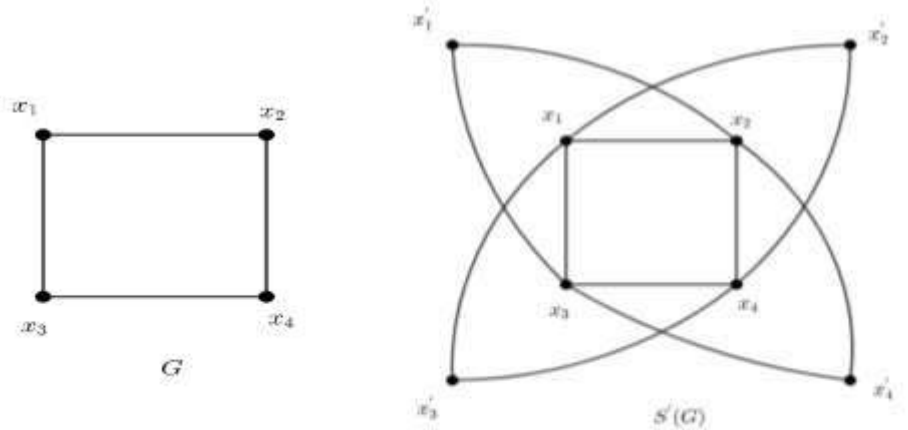


Figure 3.1: $G$ and $S'(G)$

Here $S = \{x_1, x_4\}$ or $S = \{x_2, x_3\}$ be the only minimum detour certified dominating set of G and hence, $c\gamma_d(G) = |S| = 2$.

It we set, $S = \{x_1, x_4\}$, then in $S'(G)$, $x_2'$ *and* $x_3'$ does not lie on any detour path connecting vertices from S. So that S is not a detour certified dominating set of $S'(G)$.

Now, $S_1 = S \cup \{x_2\}$. Then clearly $I_D[S_1] = V(S'(G))$ and every vertices in S has at least two neighbours in $V(S'(G)) - S_1$. So that $S_1$ is a minimum detour certified dominating set of $S'(G)$ and hence, $c\gamma_d(S'(G)) = (S_1) = 3$.

Thus, the detour certified domination number of G and $S'(G)$ are different.

**Remark 3.3**

Every end vertex of G need not belong to every detour certified dominating set of $S'(G)$.

For example, we consider the graph G given in Figure 3.2, where $S = \{x_1, x_1', x_2', x_3'\}$ is a detour certified dominating set of $S'(G)$. But $x_2$ and $x_3$ are the end vertices of G, which does not in S.

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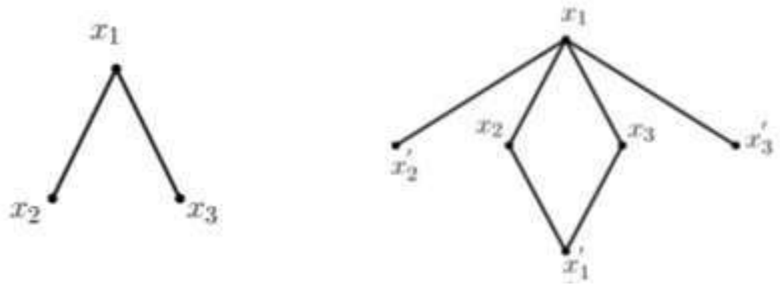


Figure 3.2: $G$ and $S'(G)$

**Observation 3.4**

For any connected graph G, $\gamma_d(G) \leq \gamma_d(S'(G))$.

**Observation 3.5**

For any connected graph G, there is no obvious relation connected $c\gamma_d(G)$ and $c\gamma_d(S'(G))$.

Example for $c\gamma_d(G) < c\gamma_d(S'(G))$.

Consider the connected graph G given in Figure 3.2. Here $S = \{x_1, x_2, x_3\}$ is the unique minimum detour certified dominating set of G and hence, $c\gamma_d(G) = 3$.

From the splitting graph, $S'(G)$, $S_1 = \{x_1, x_1', x_2', x_3'\}$ is a minimum detour certified domminating set of $S'(G)$, and hence, $c\gamma_d(S'(G)) = 4$.

In this case, $c\gamma_d(G) < c\gamma_d(S'(G))$.

Example for $c\gamma_d(G) = c\gamma_d(S'(G))$.

Consider the connected graph G given in Figure 3.3

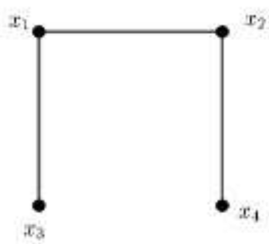


Figure 3.3: $G$

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Here $S = \{x_1, x_3, x_4, x_4\}$ is the unique minimum detour certified dominating set of G and hence, $c\gamma_d(G) = 4$. Now, we split every edge of G. A new connected graph $S'(G)$ obtained and is given in Figure 3.4.

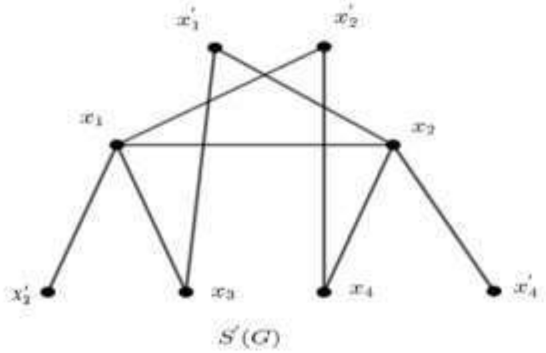


Figure 3.4

Here $S_1 = \{x_1, x_2, x_3, x_4\}$ is the unique minimum detour certified dominating set of $S'(G)$ and hence, $c\gamma_d(S'(G)) = 4$.

Thus, in this case, $c\gamma_d(G) = c\gamma_d(S'(G))$.

Example for $c\gamma_d(G) > c\gamma_d(S'(G))$.

Consider the connected graph G given in Figure 3.5

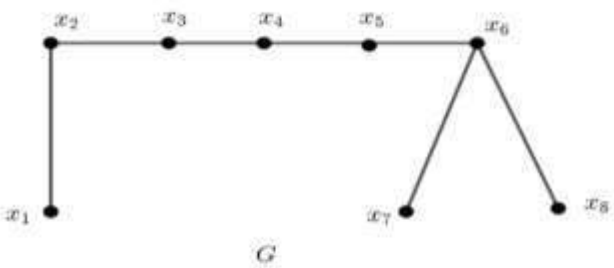


Figure 3.5

Here $S = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ is the unique minimum detour certified dominating set of G and hence, $c\gamma_d(G) = |S| = 8$.

Now, we split every edge of G. A new connected graph $S'(G)$ is obtained and is given in Figure 3.6.

Now, we split every edge of G. A new connected graph $S'(G)$ is obtained and is given in Figure 3.6.

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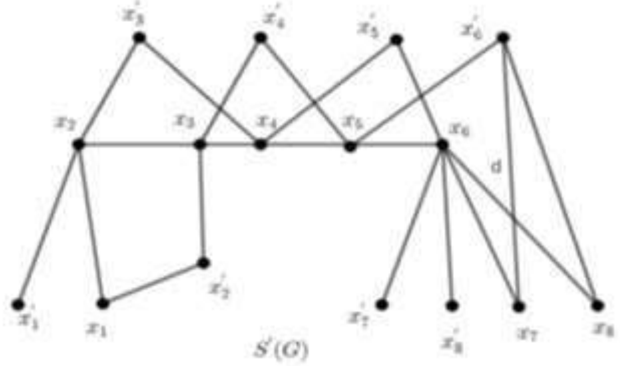


Figure 3.6

Here $S_1 = \{x_1', x_2, x_6, x_7', x_8'\}$ is the unique minimum detour certified dominating set of $S'(G)$ and hence, $c\gamma_d(S'(G)) = 5$. Thus, in this case, $c\gamma_d(G) > c\gamma_d(S'(G))$.

**Theorem 3.6**

Every support vertex of G belong to every detour certified dominating set of $S'(G)$.

**Proof**

Let G be a connected graph and let u be a support vertex of G. Then there must exists at least one pendant vertex of G which is adjacent to u in G. Take it them to v. By the definition of $S'(G)$, $v$ has a duplicated vertex $v'$, which is adjacent to $u$ in $S'(G)$. Clearly $v'$ is an pendent vertex in $S'(G)$.

Let S be a detour certified dominating set in $S'(G)$. To prove $u \in S$. Suppose $u \notin S$. then $v'$ is not dominated by any vertex of S in $S'(G)$. This shows that $v' \in S$ in $S'(G)$. Since $\deg(v') = 1$, that $v'$ has only one neighour $u$ in $S'(G)$ and also $v'$ has no adjacent vertex in S. This implies that S is not a detour certified dominating set of $S'(G)$, which is a contradiction. Hence $u \in S$ in $S'(G)$.

**Theorem 3.7**

Every duplicated vertex of an end vertex of G belong to every detour certified dominating set of $S'(G)$.

**Proof**

Let G be any connected graph and let $u$ be an end vertex of G. Then $u$ has exactly only one adjacent vertex, say $v$ in G. Let S be a detour certified dominating set of G and let $u'$ be the duplicated vertex of $x$ in $S'(G)$. Then by the definition of $S'(G)$, $u'$ is adjacent to only one vertex, that must be $v$ in $S'(G)$. This implies that the neighbour is $u'$ is a support vertex of $S'(G)$. It follows that $u'$ is an end vertex of $S'(G)$. Therefore, by Theorem 2.1, $u' \in S$.

**Theorem 3.8**

For integer $p \geq 2$, $c\gamma_d\,(S'(K_{1,p-1})) = p + 1$.

**Proof**

Let $u_1$. $u_2$, ... $u_{p-1}$ be the end vertices and $u$ be the central vertex of the star graph

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$K_{1,p-1}$. Let $u', u_1', u_2', \ldots, u_{p-1}'$ be the corresponding duplicated vertices of $u, u_1. u_2, \ldots u_{p-1}$, respectively to form $S'(K_{1,p-1})$ and is given in Figure 3.7.

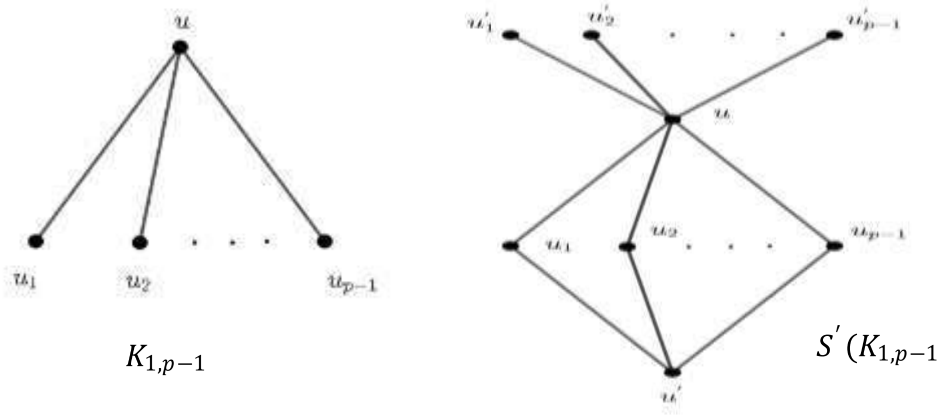


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Figure 3.7

Here, $V\,(S'(K_{1,p-1})) = \{u, u_1. u_2, \ldots u_{p-1}, u_1', u_2', \ldots, u_{p-1}'\}$ and $|V\,(S'(K_{1,p-1}))| = 2p$. Since, each $u_1', u_2', \ldots, u_{p-1}'$ are the end vertices of $S'(K_{1,p-1})$, by Theorem 3.2.9, $S = \{u_1', u_2', \ldots, u_{p-1}'\}$ is a subset of a minimum detour certified dominating set of $S'(K_{1,p-1})$. Also , $u$ be a support vertex of $S'(K_{1,p-1})$ and hence by Theorem 3.6, $S_1 = S \cup \{u\}$ is a subset of minimum detour certified dominating set of $S'(K_{1,p-1})$. Since $u'$ is not dominated by any vertices of $S_1$, that $S_1$ is not itself a detour certified dominating set of $S'(K_{1,p-1})$. Now it is clear that $S_2 = S_1 \cup \{u'\}$ is a certified dominating set of $S'(K_{1,p-1})$. Moreover, every vertices in $S'(K_{1,p-1})$ lies on a detour path between vertices from $S_2$. Therefore, that $S_2$ is a detour certified dominating set of $S'(K_{1,p-1})$. Since, $c\gamma_d\,(S'(K_{1,p-1})) \geq p + 1$, we conclude that $S_2$ is a minimum detour certified dominating set of $S'(K_{1,p-1})$ and hence,
$c\gamma_d\,(S'(K_{1,p-1})) = p + 1$.

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**Example 3.9**

For the star graph $S'(K_{1,7})$, given in Figure 3.8, by Theorem 3.8, we conclude $S = \{u_1', u_2', u_3', u_4', u_5', u_6', u_7', u, u'\}$ is a minimum detour certified dominating set of $S'(K_{1,7})$ and hence, $c\gamma_d(S'(K_{1,7})) = 9$.

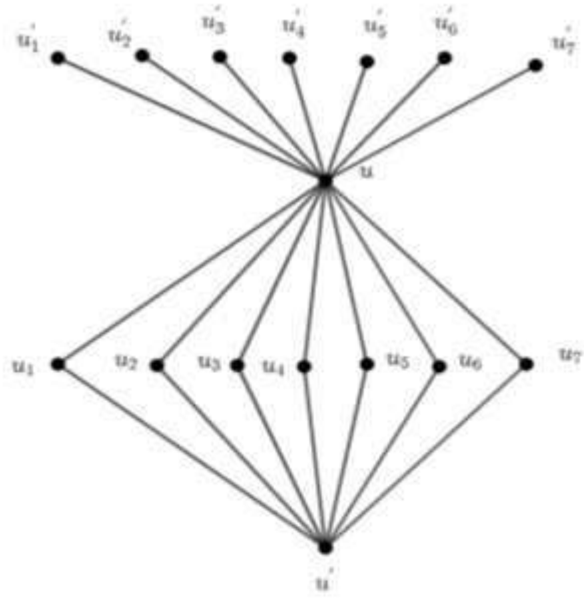


Figure 3.8 **Corollary 3.10**

For integer $n \geq 2$, $c\gamma_d(S'(K_{1,p-1})) = c\gamma_d(K_{1,p-1}) + 1$.

**Theorem 3.11**

For the Bistar graph $B_{m,n}$ $(m, n \geq 1)$, $c\gamma_d(S'(B_{m,n})) = m + n + 2$.

**Proof**

Consider the bistar graph $B_{m,n}$ with $mn, \geq 1$. Let $u$ and $v$ be the central vertices of $B_{m,n}$ and let $u_1, u_2, \dots, u_m$ and $v_1, v_2, \dots, v_n$ be the end vertices adjacent to $u$ and $v$ in $B_{m,n}$respectively. Thus, $V(B_{m,n}) = \{u_1v, u_i, v_j; 1 \leq i \leq m; 1 \leq j \leq n\}$. Let $u', u_1', u_2', \dots, u_m', v_1', v_2', \dots, v_n', v'$ be the corresponding duplicated vertices of $u, u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n, v$, respectively, which are added to obtained the new graph $S'(B_{m,n})$ and is given in Figure 3.9.

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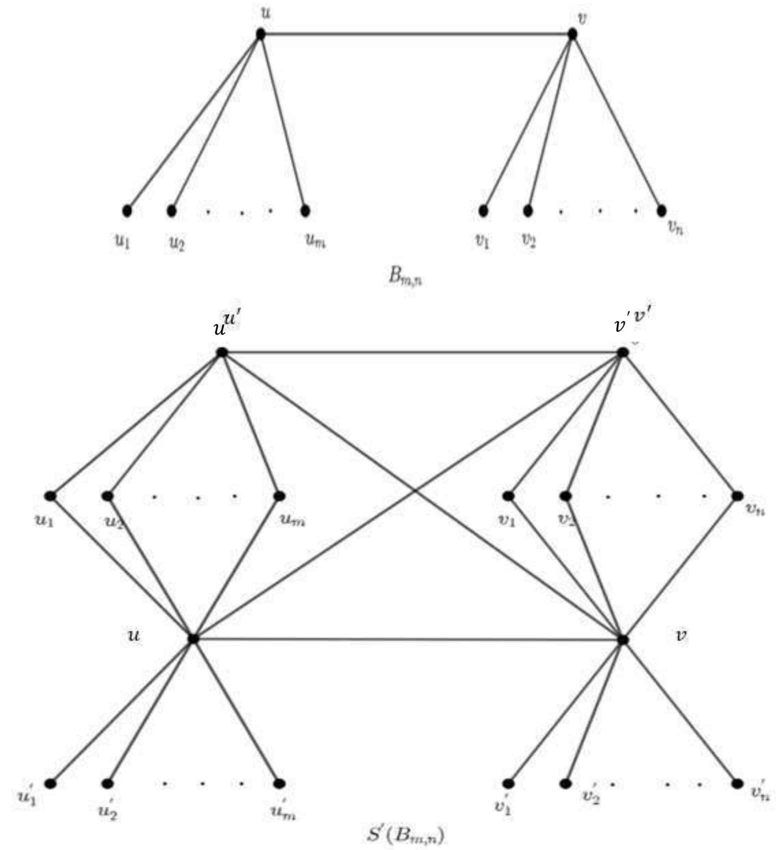


Figure 3.9

Then $V\ (S'(B_{m,n})) = \{u, v, u_i, v_j, u_i', v_j', u', v';\ 1 \le i \le m, 1 \le j \le n\}$ and hence

$|V\ (S'(B_{m,n}))| = 2(m + n + 2)$. Now, $u_1', u_2', \ldots, u_m', v_1', v_2', \ldots, v_n'$ are the end vertices of $S'(B_{m,n})$. Also that $u$ and $v$ are the support vertices of $S'(B_{m,n})$. Let S be a minimum detour certified dominating set of $S'(B_{m,n})$. Then by Theorem 2.1 and Theorem 2.2,

$\{u, v, u_1\ \ {}_2', u', \ldots, u_m', v_1\ \ {}_2', v'u, \ldots, v_n'\} \subseteq S$. Now, there is a longest path between $u_i'$ and $v_j'$ for

$1 \le i \le m, 1 \le j \le n$, which includes the remaining vertices $u_i, v_j, u'$ *and* $v'$. Also, it can be easily verified that $N^{[S]} = V\left(S'(B_{m,n})\right)$. Since, $m, n \ge 1$ that $S =$

$\{u, v, u_1', u_2', \ldots, u_m', v_1\ \ {}_2', v', \ldots, v_n'\}$ is itself a minimum detour certified dominating set of $S'(B_{m,n})$ and hence, $c\gamma_d\left(S'(B_{m,n})\right) = |S| = m + n + 2$.

**Example 3.12**

Consider the bistar graph $S'(B_{3,4})$ which is given in Figure 3.10.

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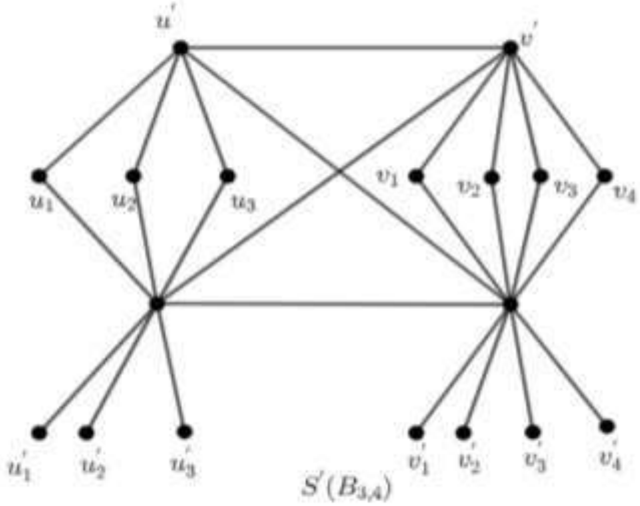



Figure 3.10

By Theorem 3.11, $S = \{u, v, u_1', u_2', u_3', v_1', v_2', v_3', v_4'\}$ is a minimum detour certified dominating set of $S'(B_{3,4})$ and hence, $c\gamma_d(S'(B_{3,4})) = |S| = 9$.

**Corollary 3.13**

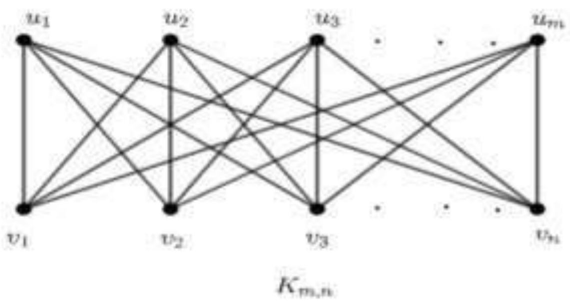
For integers, $m, n \geq 1$, $c\gamma_d(S'(B_{3,4})) = c\gamma_d(B_{m,n})$.

**Theorem 3.14**

For the complete bipartite graph $K_{m,n}$, $(m, n \geq 2)$, $c\gamma_d(S'(B_{m,n})) = 2$.

**Proof**

Consider the complete bipartite graph $K_{m,n}$ with $m, n \geq 2$. Let the vertex set be $V(K_{m,n}) = \{u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Here $u_1, u_2, \ldots, u_m$ are adjacent with $v_1, v_2, \ldots, v_n$ respectively. Let $u_1', u_2', \ldots, u_m', v_1', v_2', \ldots, v_n'$ be the corresponding duplicated vertices of $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$, respectively which are added to obtained $S'(K_{m,n})$. This graph is given in Figure 3.11. Then $V(S'(B_{m,n})) = \{u_i, v_j, u_i', v_j'; 1 \leq i \leq m, 1 \leq j \leq n\}$ and hence, $|V(S'(B_{m,n}))| = 2(m + n)$.



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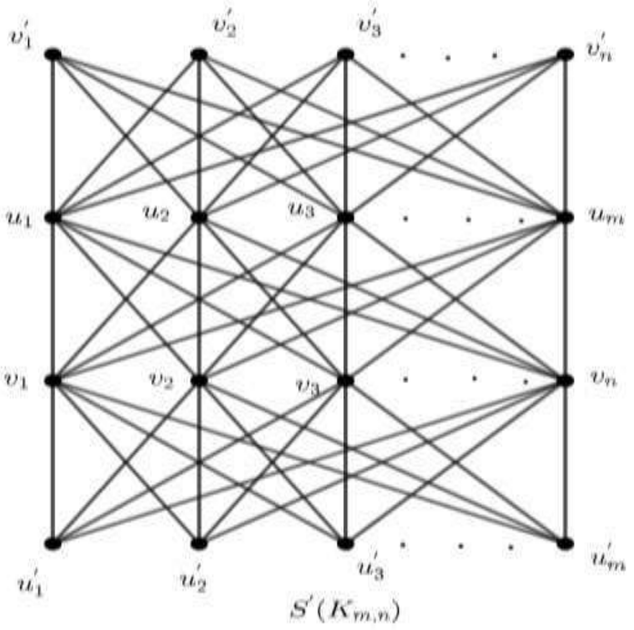


Figure 3.11

In $S'(K_{m,n})$, that $u_i$ is adjacent to every $v_j$ and $v_j'$; $v_j$ is adjacent with each $u_i$ and $u_i'$ is adjacent with each $v_j$ and $v_j'$ is adjacent with each $u_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Clearly $u_1$ and $v_n$ dominates every other vertices in $S'(K_{m,n})$. Since $m, n \geq 1$, we take $S = \{u_1, v_n\}$, then S is a certified dominating set of $S'(K_{m,n})$. Also every vertices in $S'(K_{m,n})$ lies on the longest path connecting vertices from S. So that S itself is a detour certified dominating set of $S'(K_{m,n})$ and thus, $c\gamma_d\,(S'(K_{m,n})) \leq |S| = 2$. Hence, by Theorem 3.2.12, we conclude that $c\gamma_d\,(S'(K_{m,n})) = 2$.

**Example 3.15**

Consider the connected graph $S'(K_{5,5})$ which is given in Figure 3.12. By Theorem 3.14, $S = \{u_1, v_5\}$ is a minimum detour certified dominating set of $S'(K_{5,5})$ and hence, $c\gamma_d$ $(S'(K_{5,5})) = |S| = 2$.

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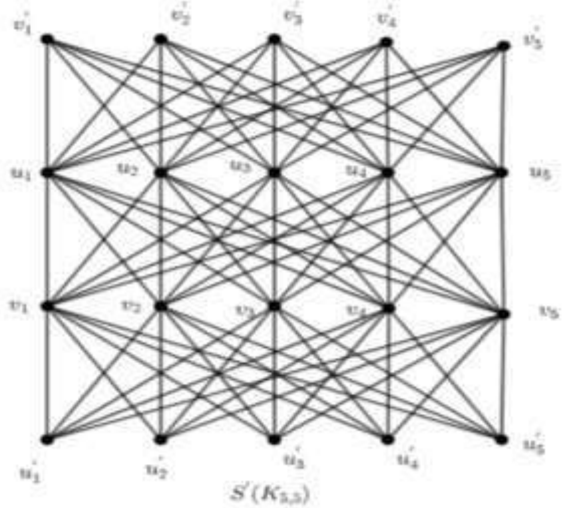


Figure 3.12

**Corollary 3.16**

For the complete bipartite graph $K_{m,n}$, $(m, n \geq 2)$, $c\gamma_d(S'(K_{m,n})) = c\gamma_d(K_{m,n})$.

**Theorem 3.17**

For integer $p \geq 4$, $c\gamma_d(S'(P_p)) = \begin{cases} \frac{p+4}{2} & \text{if } p \equiv 0 (mod 4) \\ \frac{p+5}{5} & \text{if } p \equiv 1 \text{ or } 3 (mod\ 4). \\ \frac{p+6}{2} & \text{if } p \equiv 2 (mod\ 4) \end{cases}$

**Proof**

Let $p \geq 4$ and $v_1, v_2, \ldots, v_p$ be the vertices of the path graph $P_p$, which are duplicated by the vertices $v_1', v_2', \ldots, v_p'$ respectively in $S'(P_p)$ and the graph is shown in Figure 3.13.

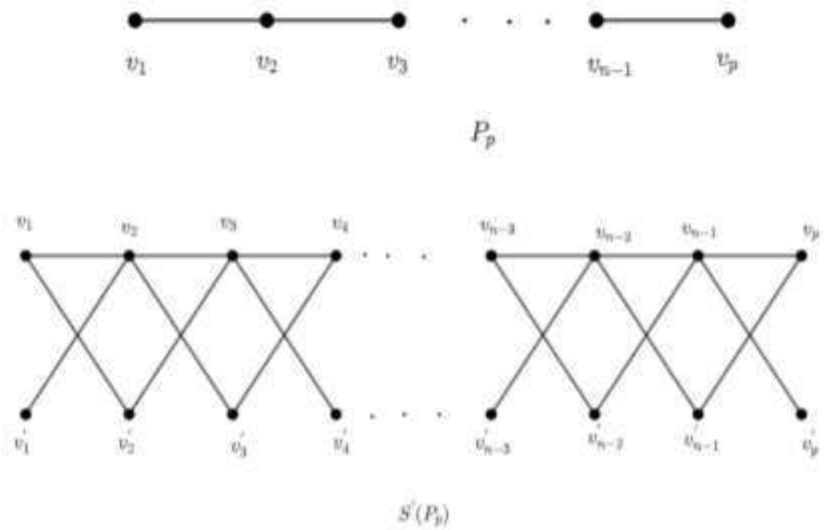


Figure 3.13

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Then $V(S'(P_p)) = \{v_1, v_2, v_3, \dots, v_n, v_1', v_2', \dots, v_n'\}$ and so $|V(S'(P_p))| = 2n$. Since $v_1'$ and $v_n'$ are the only end vertices of $S'(P_p)$, by Theorem 3.2.9, $v_1'$ and $v_p'$ are in every detour certified dominating set of $S'(P_p)$. Also, $v_2$ and $v_{p-1}$ are the support vertices of $S'(P_n)$. Therefore, by Theorem 3.2.10, $v_2$ and $v_{p-1}$ are in every detour certified dominating set of $S'(P_p)$. Let $W = \{v_1', v_2, v_{p-1}, v_p'\}$. So W contained in every detour certified dominating set of $S'(P_p)$. We consider four cases.

**Case (i)** $p \equiv 0 (mod\ 4)$

For $p = 4$, it is clear that W itself is a detour certified dominating set of minimum cardinality and hence $c\gamma_d(S'(P_4)) = 4$. Now, let $p = 4k$, where $k = 2,3,4, \dots$. Here $v_2$ dominates $v_1, v_3, v_1'$ and $v_3'$. Therefore to certified dominates $v_2'$, we must select the vertex $v_3$. That is the neighbour of $v_2$ which dominates $v_4, v_4'$. To dominate $v_5'$, we need to take the vertex $v_6$. To dominate $v_6'$ we need to take the neighbour of $v_6$, that is $v_7$. Consider a group of every four vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, \dots, v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}$, the vertices $v_2, v_3, v_6, v_7, v_{10}, v_{11}, \dots, v_{p-6}, v_{p-5}, v_{p-2}, v_{p-1}$ forms a pair that dominates all the vertices of $S'(P_p)$. If we include $v_1'$ and $v_p'$, then clearly every vertices lies in the detour path and certified dominates so $S = W \cup \{v_3, v_6, v_7, \dots, v_{p-6}, v_{p-5}, v_{p-2}\}$ is a minimum detour certified dominating set of $S'(P_{4k})$ and so $c\gamma_d(S'(P_{4k})) = |S| = \frac{p+4}{2}$.

**Case (ii)** $p \equiv 1 (mod\ 4)$

For $p = 5$, it is clear that $W \cup \{v_3\}$ is a minimum detour certified dominating set of $S'(P_5)$ and so $c\gamma_d(S'(P_5)) = |W \cup \{v_3\}| = 4 + 1 = 5 = \frac{5+5}{2}$. Now let $p = 4k + 1$, where $k = 2,3,4, \dots$. In this case, we have to add some pair of vertices $v_6\ _7, v, \dots, v_{p-7,}\ _{p-6}\ _{p-3}\ _{p-2}v, v, v$ and $v_3$ to W to form a detour certified dominating set of $S'(P_p)$. Therefore $S = W \cup \{v_3\ _6\ _7, v, v, \dots, v_{p-7}, v_{p-6}, v_{p-3}, v_{p-2}\}$ is a minimum detour certified dominating set of $S'(P_p)$ and so $c\gamma_d(S'(P_p)) = |S| = \frac{p+5}{2}$.

**Case (iii)** $p \equiv 2 (mod\ 4)$

For $p = 6$, it is clear that $W \cup \{v_3, v_4\}$ is a minimum detour certified dominating set of $S'(P_6)$ and so $c\gamma_d(S'(P_6)) = |W \cup \{v_3, v_4\}| = 4 + 2 = 6 = \frac{6+6}{2}$. Now let $p = 4k + 2$, where $k = 2,3,4, \dots$. In this case, we have to add some pair of vertices $v_6\ _7, v, \dots, v_{p-8,}\ _{p-7}\ _{p-4}\ _{p-3}\ _{p-2}v, v, v, v$ and $v_3$ to W to form a detour certified dominating set of $S'(P_p)$. Therefore $S = W \cup \{v_3\ _6\ _7, v, v, \dots, v_{p-8}, v_{p-7}, v_{p-4}, v_{p-3}, v_{p-2}\}$ is a minimum detour certified dominating set of $S'(P_p)$ and so $c\gamma_d(S'(P_p)) = |S| = \frac{p+6}{2}$.

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**Case (iv)** $p \equiv 3(mod\ 4)$

For $p = 7$, it is clear that $W \cup \{v_3, v_5\}$ is a minimum detour certified dominating set of $S'(P_7)$ and so $c\gamma_d(S'(P_7)) = |W \cup \{v_3, v_5\}| = 4 + 2 = 6 = \frac{7+5}{2}$. Now let $p = 4k + 3$, where $k = 2,3,4, \dots$. In this case, we have to add some pair of vertices $v_6$ $_7$, $v$, ..., $v_{p-5}$, $_{p-4}$ $v$ , $v_{p-2}$, $v_3$ to W to form a detour certified dominating set of $S'(P_P)$. Therefore $S = W \cup \{v_3, v_6, v_7, \dots, v_{p-5}, v_{p-4}, v_{p-2}\}$ is a minimum detour certified dominating set of $S'(P_p)$ and so $c\gamma_d$ $\frac{+5}{2}$. $(S'(P_p)) = |S| = {}^p$

$\frac{p+4}{2}$ $if\ p \equiv 0(mod 4)$

$\frac{p+5}{5}$

Thus, $c\gamma_d(S'(P_p)) =$ $\frac{p+6}{}$ $\equiv 2$ $if\ p \equiv 1\ or\ 3(mod\ 4)$.

$if\ p(mod\ 4)$

$\{$ 2

**Example 3.18**

Consider the graph $S'(P_6)$ given in Figure 3.14.

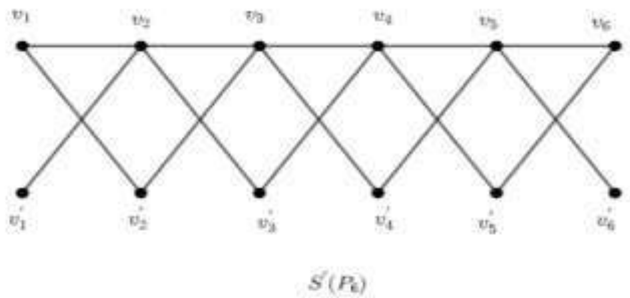


Figure 3.14

By Theorem 3.17, $S = \{v_1', v_2, v_3, v_4, v_5, v_6'\}$ is a minimum detour certified dominating set of $S'(P_6)$ and hence, $c\gamma_d(S'(P_6)) = 6$.

Consider the graph $S'(P_7)$ given in Figure 3.15.

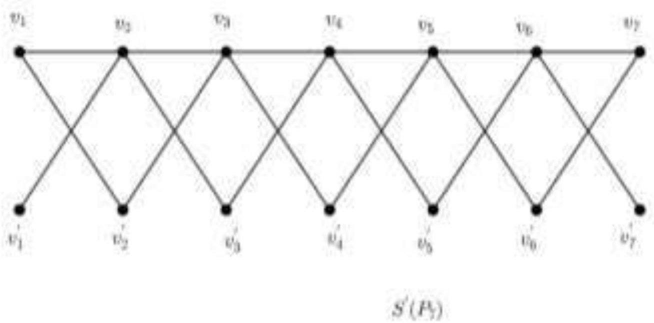


Figure 3.15

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Consider the graph $S'(P_8)$ given in Figure 3.16

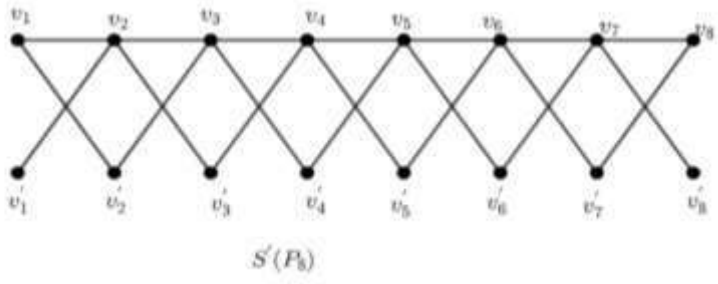


Figure 3.16

By Theorem 3.17, $S = \{v_1', v_2, v_3, v_6, v_7, v_8'\}$ is a minimum detour certified dominating set of $S'(P_8)$ and hence, $c\gamma(S'(P_8)) = 6$.

**Theorem 3.19**

For integer $p \geq 4$, $c\gamma_d(S'(C_p)) = \begin{cases} \frac{p}{2} & if\ p \equiv 0(mod 4) \\ \frac{p+1}{5} & if\ p \equiv 1\ or\ 3(mod\ 4). \\ \frac{p+2}{2} & if\ p \equiv 2(mod\ 4) \end{cases}$

**Proof**

Let $p \geq 4$ and $v_1, v_2, \dots, v_p$ be the vertices of the cycle graph $C_p$ which are duplicated by the vertices $v_1', v_2', \dots, v_p'$ respectively in $S'(C_p)$ and is shown in Figure 3.17. Then $V(S'(C_p)) = \{v_1, v_2, \dots, v_p, v_1', v_2', \dots, v_p'\}$ and so $|V(S'(C_p))| = 2p$.

Let S be a detour certified dominating set of $S'(C_p)$. To construct S, we consider four cases.

**Case (i)** $p \equiv 0(mod\ 4)$

We consider the splitting graphs $S'(C_{4k})$ where $k = 1,2,3, \dots$. Here $N[v_1] = \{v_1, v_2, v_{2'}, v_p, v_{p'}\}$; $N[v_5] = \{v_4, v_5, v_{4'}, v_6, v_{6'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, v_{x'-1}, v_{x'+1}\}$ where $x \equiv 1(mod\ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_3', v_1'\}$; $N[v_6] = \{v_5, v_6, v_7, v_7', v_5'\}$; $N[v_y] = \{v_{y-1}, v_{y'-1}, v_y, v_{y+1}, v_{y'+1}\}$, where $y \equiv 2(mod 4)$. It is clear that $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup \dots \cup N[v_x] \cup N[v_y] \cup \dots \cup N[v_{p-3}] \cup N[v_{p-2}] = V(C_p)$. Hence we obtain a certified dominating set $S = \{v_1, v_2, v_5, v_6, v_x, v_y, v_{p-3}, v_{p-2}\}$ of $S'(C_p)$, where $x \equiv 1(mod\ 4)$ and $y \equiv 2(mod\ 4)$. Also every vertices in $V(S'(C_p))$ lies in the longest path between vertices $x_1$ and $x_2$. Therefore, that S is a detour set of $S'(C_p)$. Thus S is a minimum detour certified dominating set of $S'(C_n)$ and hence, $c\gamma_d\left(S'(C_p)\right) = |S| = \frac{p}{2}$.

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**Case (ii)** $p \equiv 1(mod\ 4)$

We consider the splitting graphs $S'(C_{4k+1})$, where $k = 1,2,3, ....$ Here $N[v_1] = \{v_1, v_2, v_{2'}, v_p, v_{p'}\}$; $N[v_5] = \{v_4, v_5, v_6, v_{6'}, v_4\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, v_{x'+1}, v_{x'-1}\}$ where $x \equiv 1(mod\ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_3', v_1'\}$; $N[v_6] = \{v_5, v_6, v_7, v_7', v_6'\}$; $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, v_y'{}_{+1}, v_y'{}_{-1}\}$, where $y \equiv 2(mod4)$. It is clear that $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup ... \cup N[v_x] \cup N[v_y] \cup ... \cup N[v_{p-4}] \cup N[v_{p-3}] \cup N[v_{p-2}] = V\ (S'(C_p))$. Hence we obtain a certified dominating set $S = \{v_1, v_2, v_5, v_6, ... , v_x, v_y, ... , v_{p-4}, v_{p-3}, v_{p-2}\}$ of $S'(C_p)$, where $x \equiv 1(mod\ 4)$ and $y \equiv 2(mod\ 4)$. Also every vertices in $V\ (S'(C_p))$ lies on the longest path between vertices $x_1$, $x_2$ which is in S. So that S is a detour set of $S'(C_p)$. Thus S is a minimum detour certified dominating set of $S'(C_p)$ and hence, $c\gamma_d\,(S'(C_p)) = |S| = \frac{p+1}{2}$.

**Case (iii)** $p \equiv 2(mod\ 4)$

We consider the splitting graphs $S'(C_{4k+2})$, where $k = 1,2,3, ....$ Here $N[v_1] = \{v_1, v_2, v_{2'}, v_p, v_{p'}\}$; $N[v_5] = \{v_4, v_5, v_6, v_{6'}, v_{4'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, v_{x'+1}, v_{x'-1}\}$ where $x \equiv 1(mod\ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_3', v_1'\}$; $N[v_6] = \{v_5, v_6, v_7, v_7', v_6'\}$; $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, v_y'{}_{+1}, v_y'{}_{-1}\}$, where $y \equiv 2(mod4)$. It is clear that $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup ... \cup N[v_x] \cup N[v_y] \cup ... \cup N[v_{p-5}] \cup N[v_{p-4}] \cup N[v_{p-3}] \cup N[v_{p-2}] = V\ (S'(C_p))$. Hence we obtain a certified dominating set $S = \{v_1, v_2, v_5, v_6, ... , v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}\}$ of $S'(C_p)$, where $x \equiv 1(mod\ 4)$ and $y \equiv 2(mod\ 4)$. Also every vertices in $V\ (S'(C_p))$ lies on the longest path connecting pair of vertices $x_1$, $x_2$ which is in S. So that S is a detour certified dominating set of $S'(C_p)$ such that S is a minimum cardinality. Therefore $c\gamma_d\,(S'(C_p)) = |S| = \frac{p+2}{2}$.

**Case (iv)** $p \equiv 3(mod\ 4)$

We consider the splitting graphs $S'(C_{4k+3})$, where $k = 1,2,3, ....$ Here $N[v_1] = \{v_1, v_2, v_2', v_p, v_p'\}$; $N[v_5] = \{v_4, v_5, v_6, v_6', v_4'\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, v'_{x+1}, v'_{x-1}\}$ where $x \equiv 1(mod\ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_3', v_1'\}$; $N[v_6] = \{v_5, v_6, v_7, v_7', v_6'\}$; $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, v'_{y+1}, v'_{y-1}\}$, where $y \equiv 2(mod4)$. It is clear that $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup ... \cup N[v_x] \cup N[v_y] \cup ... \cup N[v_{p-3}] \cup N[v_{p-2}] = V\left(S'(C_p)\right)$. Hence we obtain a

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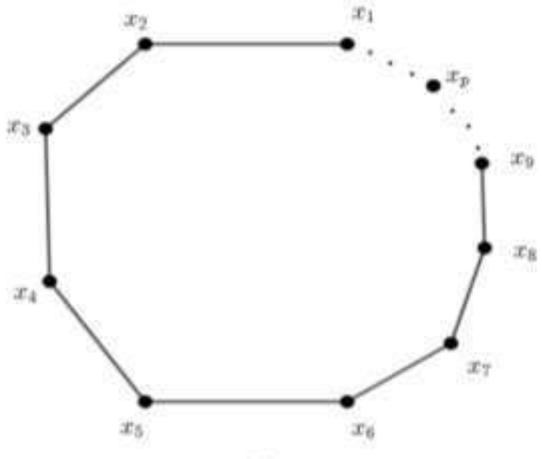
certified dominating set $S = \{v_1, v_2, v_5, v_6, \ldots, v_{p-3}, v_{p-2}\}$ of $S'(C_p)$, where $x \equiv 1 (mod\ 4)$ and $y \equiv 2 (mod\ 4)$. Also every vertices in $V\left(S'(C_p)\right)$ lies on the longest path connecting pair of vertices $x_1$, $x_2$ which is in S. So that S is a minimum detour certified dominating set of $S'(C_p)$ and so $c\gamma_d\left(S\ (C_p)\right) = |S| = \frac{p+1}{2}$.

$$\text{Thus, } c\gamma_d\left(S'(C_p)\right) = \begin{cases} \frac{p}{2} & \text{if } p \equiv 0 (mod 4) \\ \frac{p+1}{2} & \text{if } p \equiv 1 \text{ or } 3 (mod\ 4). \\ \frac{p+2}{2} & \text{if } p \equiv 2 (mod\ 4) \end{cases}$$



$C_p$

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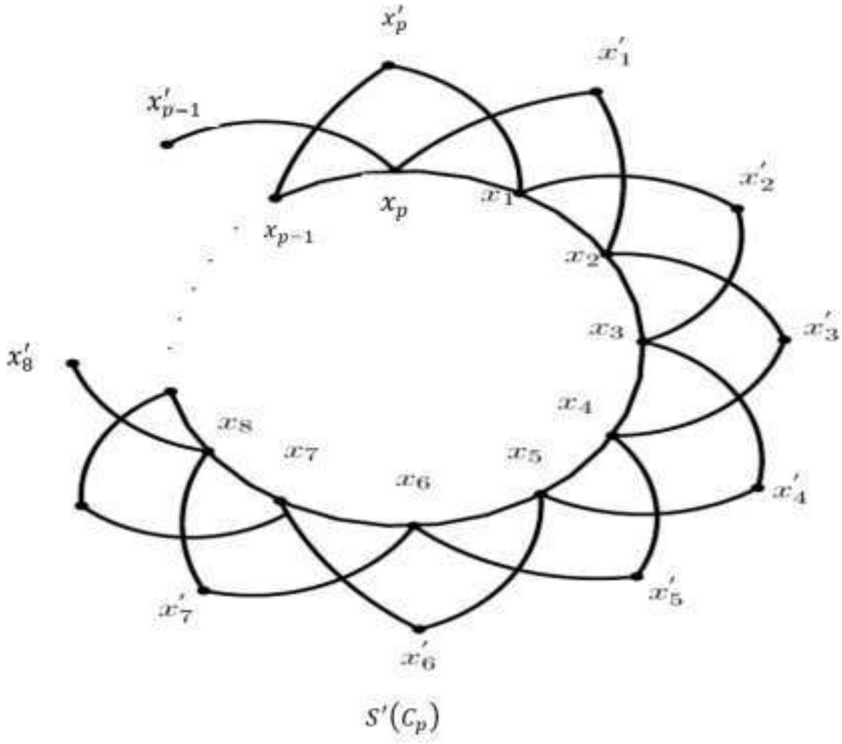


Figure 3.17

**Example 3.20**

Consider the cycle graph $S'(C_{12})$ given in Figure 3.18.

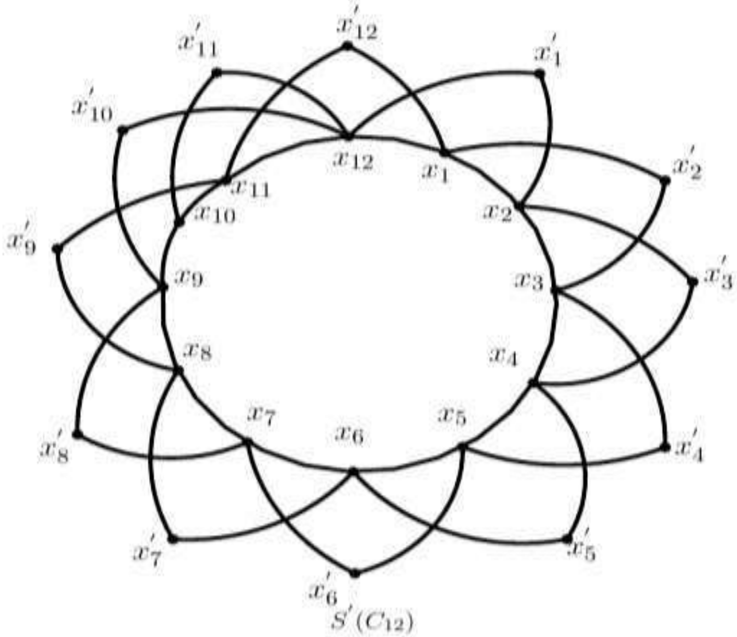


Figure 3.18.

By Theorem 3.19, $S = \{x_1, x_2, x_5, x_6, x_9, x_{10}\}$ is a minimum detour certified dominating set of $S'(C_{12})$ and hence, $c\gamma_d(S'(C_{12})) = |S| = 6$.

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