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CO-EVEN DOMINATING SETS AND CO-EVEN DOMINATION POLYNOMIALS OF SPIDER GRAPH Sp_n

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ABSTRACT: Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a simple graph. A set $D \subseteq V$ is a dominating set of G, if for every vertex in $V \setminus D$ is adjacent to exactly one vertex in D. A subset D of V is called a co-even dominating set if $deg(v)$ is even for all $v \in V - D$. The set D is called a minimal co-even dominating set if it has no proper. The minimal co-even dominating sets of a graph G is denoted by MCEDS(G). The co-even domination number is denoted by $\gamma_{coe}(G)$. Let G be a graph with n vertices and $D_{coe}(G, i)$ denote the family of all co-even dominating sets of G with cardinality i. In this paper, we construct the co-even dominating sets and domination polynomials of Spider Sp_n .

Keywords : Co-even dominating set, co-even domination polynomial, Recurrence relation Spider Sp_n.

Mathematics Subject Classification : 05C31, 05C761

INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by m and n respectively. For basic definitions and terminologies we refer to Buckley et. Harary, 1990. Two vertices $u \; and \; v$ are said to be adjacent if uv is an edge of G. The open neighborhood of a vertex v in a graph G is defined as $NG(v) = N(v) \cup \{v\}$. Domination problems came from chess. In the 1850's several chess players were interested in the minimum number of queens such that every square on the chess board either contains a queen or is attacked by a queen. The concept of domination in graph theory was introduced by Claude Berge in 1958 and Oystein Ore in 1962. A subset $S \subseteq V(G)$ is called a dominating set if every vertex $v \in V(G) \setminus D$ is adjacent to a vertex $u \in S$. The domination number $\gamma(G)$ of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as $\gamma - set$ of G in Alikhani et Peng,2009. A dominating set D is called a co-even dominating set if $deg(v)$ is even number for all $v \in V - D \; and \; \gamma_{coe}(G)$ is the minimum cardinality of a co-even dominating set of G. The concept of co-even domination number of a graph is defined by M.M. Shalaan and A.A. Omran[5]. In this paper we introduce the

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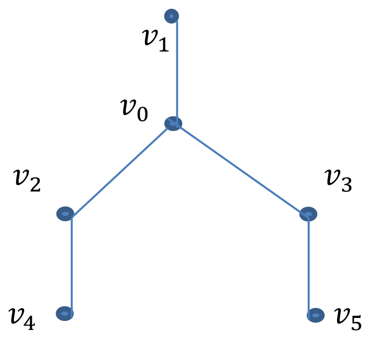
concepts of co-even dominating sets and domination polynomials of Spider Sp_n. A Spider graph, spider tree, or simply "spider" is a tree with one vertex of degree at most 2 and the vertex is called the branch vertex[8]. Let $\gamma_{coe}(G)$ be the domination number of G. Let $D_{coe}(G,i)$ denote the family of all co-even dominating sets of G with cardinality i. Let $d_{coe}(G,i) = | D_{coe}(G,i) |$. Then the co-even domination polynomial of is $D_{coe}(G,x) = \sum_{i=\gamma coe(G)}^{n} d_{coe}(G,i)x^i$.

Example 1.1

Consider the graph Sp_6,



From the figure, let $V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ and $D = \{v_0, v_1, v_4, v_5\}$ is a co-even dominating set. For $V - D = \{v_2, v_3\}$ the vertex $v_2 \in V - D$ has even degree, the vertex $v_3 \in V - D$ has even degree. So the set $\{v_0, v_1, v_4, v_5\}$ is a co-even dominating set.

In the next section, we construct the families of co-even dominating sets of Sp_n with cardinality i by the families of co-even dominating sets of Sp_{n-1}, Sp_{n-2} with cardinality i and $i - 1$. We investigate the co-even domination polynomial of spider Sp_n in section 3.

2. Co-even dominating sets of Sp_n

We denote the family of co-even dominating sets of Spider Sp_n with cardinality i by $D_{coe}(Sp_n, i)$. The following lemmas and theorems are necessary to support the primary findings of this article.

MAIN RESULTS

Lemma 2.1. Let Sp_n be the Spider graph with n vertices. Then $D_{coe}(Sp_n, n) = V(P_n)$.

Proof

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Here $i = n$. Therefore the co-even domination sets contain all the vertices of the Spider Sp_n. (Since the whole vertex set always dominates.) Hence there is only one set available in $D_{coe}(Sp_n, n)$ that is nothing but vertex set of Sp_n, $V(Sp_n)$.. Thus $D_{coe}(Sp_n, n) = V(Sp_n)$.

Proposition 2.2[5]. Let $\boldsymbol{G} = (\mathrm{n}, \mathrm{m})$ be a graph and D is a co-even dominating set, then

i) All vertices of odd or zero degrees belong to every co-even dominating set.
ii) $\deg(v) \geq 2, for\ all\ v \in V - D$
iii) If G is r-regular graph then $\gamma_{coe}(G) = \begin{cases} n & if\ r\ is\ odd \\ \gamma(G) & if\ r\ is\ even \end{cases}$
iv) $\gamma(G) \leq \gamma_{coe}(G)$

Lemma 2.3. If Sp_n be the Spider graph with n vertices. Then $\gamma_{coe}(Sp_n) = \begin{cases} 4, & 5 \leq n \leq 8 \\ \left\lceil \frac{n}{3} \right\rceil + 2, & n \geq 9 \end{cases}$

Proof

Let Sp_n be the Spider graph with n vertices, then the co-even domination number of spider graph can be discussed as follows,

Case 1. If $5 \leq n \leq 8$

If n lies between [5,8] then every vertex of Sp_n has an even degree except the end vertex and the branch vertex. Thus according to Proposition 2.2, all vertices of odd degree belong to co-even dominating set, and these vertices dominate the other vertices. Therefore $\gamma_{coe}(Sp_n) = 4$.

Case 2. For $n \geq 9$

every vertex of Sp_n has an even degree except the end vertex and the branch vertex. Thus according to Proposition 2.2, all vertices of odd degree belongs to the co-even dominating set and the vertices of even degree which doesnot satisfies the co-even domination condition also belongs to the co-even dominating set.

Therefore $\gamma_{coe}(Sp_n) = \begin{cases} 4, & 5 \leq n \leq 8 \\ \left\lceil \frac{n}{3} \right\rceil + 2, & n \geq 9 \end{cases}$

Lemma 2.4. If $Y \in D_{coe}(Sp_{n-1}, i-1), then\ Y \cup \{n\} \in D_{coe}(Sp_n, i)$.

Proof

If $Y \in D_{coe}(Sp_{n-1}, i-1)$, then at least one vertex labelled n−1 or n−2 is in Y. If $n-1 \in Y$ then $Y \cup \{n\}$ co-evenly dominates P_n. Thus $Y \cup \{n\} \in D_{coe}(Sp_n, i)$. If $n-2 \in Y$, then $Y \cup \{n-1\}, Y \cup \{n\} \in D_{coe}(Sp_n, i)$. Hence in all cases $Y \cup \{n\}$ co-evenly dominates Sp_n if $\mathrm{Y} \in D_{coe}(Sp_{n-1}, \mathrm{i}-1) - \{\mathrm{n}\}$. Therefore $Y \cup \{n\} \in D_{coe}(Sp_n, i)$.

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Lemma 2.5. If Sp_n be the Spider graph with n vertices then the co-even dominating set must contains the branch vertex and the pendant vertices.

Lemma 2.6. If Sp_n be the spider graph with n vertices and $D_{coe}(Sp_n, i)$ be the family of all co-even dominating sets with cardinality i. Then $D_{coe}(Sp_n, i) \neq \emptyset\ iff\ \gamma_{coe}(Sp_n) \leq i \leq n$. Also $D_{coe}(Sp_n, i) = \emptyset\ iff\ i < \gamma_{coe}(Sp_n)\ or\ i > n$.

Proof

By the definition of co-even domination number there is at least one co-even dominating set in (Sp_n, i) say D where $i = \gamma_{coe}(Sp_n)$. All the supersets are again co-even dominating sets therefore, $D_{coe}(Sp_n, i) \neq \emptyset\ if\ \gamma_{coe}(Sp_n) \leq i \leq n$. Also, clearly by the definition of co-even domination number $D_{coe}(Sp_n, i) = \emptyset\ if\ i < \gamma_{coe}(Sp_n)\ or\ i > n$.

Lemma 2.7. Let Sp_n be the spider graph with n vertices then for every $n \geq 7$

i) If $D_{coe}(Sp_{n-1}, i) = \emptyset\ then\ D_{coe}(Sp_{n-2}, i-1) = \emptyset$.

ii) $If\ D_{coe}(Sp_{n-1}, i) \neq \emptyset\ then\ D_{coe}(Sp_{n-2}, i-1) \neq \emptyset$.

Proof.

i) Since $D_{coe}(Sp_{n-1}, i) = \emptyset$,

by lemma 2.6, $i < 2 + \left\lceil \frac{n-1}{3} \right\rceil\ or\ i > n-1$

which implies , $i - 1 < 1 + \left\lceil \frac{n-1}{3} \right\rceil\ or\ i - 1 > n - 2$ (2.1)

Suppose $D_{coe}(Sp_{n-2}, i-1) \neq \emptyset$ by lemma 2.6, $2 + \left\lceil \frac{n-2}{3} \right\rceil \leq i - 1 \leq n - 2$

If $i \leq n-1$ from (2.1) and lemma 2.6 we get $(Sp_{n-1}, i) \neq \emptyset$ which is a contradiction.

Therefore $D_{coe}(Sp_{n-2}, i-1) = \emptyset$.

i) Since $D_{coe}(Sp_{n-1}, i) \neq \emptyset$,

by lemma 2.6, $2 + \left\lceil \frac{n-1}{3} \right\rceil \leq i \leq n-1$

which implies $1 + \left\lceil \frac{n-1}{3} \right\rceil \leq i - 1 \leq n - 2$ (2.2)

Suppose $D_{coe}(Sp_{n-2}, i-1) = \emptyset$, by lemma 2.6 $i - 1 < 2 + \left\lceil \frac{n-2}{3} \right\rceil\ or\ i - 1 > n - 2$

Which implies $i > n - 1$

If $i > n - 1\ or\ i - 1 > n - 2$ from (2.2) and lemma 2.6, $D_{coe}(Sp_{n-1}, i) = \emptyset$ which is a contradiction.

Therefore $D_{coe}(Sp_{n-2}, i-1) \neq \emptyset$.

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Lemma 2.8. If $D_{coe}(Sp_n, i) \neq \emptyset$ then

i) For all $7 \leq n \leq 10$ $D_{coe}(Sp_{n-1}, i) = \emptyset$ $then$ $D_{coe}(Sp_{n-2}, i-1) \neq \emptyset$ $if\ and\ only\ if$ $n = i + k$ and $i = m$ $for\ some$ $m = 4$ and $k \in [3,4]$.

ii) $D_{coe}(Sp_{n-1}, i) \neq \emptyset$ $then$ $D_{coe}(Sp_{n-2}, i-1) \neq \emptyset$ $if\ and\ only\ if$ $\left\lceil \frac{n}{3} \right\rceil + 2 \leq i \leq n-1$.

Proof.

i) ⇒)Since $D_{coe}(Sp_{n-2}, i-1) = \emptyset$

by lemma 2.3, $i - 1 < 2 + \left\lceil \frac{n-2}{3} \right\rceil$ or $i - 1 > n - 2$

If $i > n - 1$, by lemma 2.3 $D_{coe}(Sp_n, i) = \emptyset$ which is not possible . So we have $i < 3 + \left\lceil \frac{n-2}{3} \right\rceil$ and since $D_{coe}(Sp_n, i) \neq \emptyset$ by lemma 2.3 $2 + \left\lceil \frac{n}{3} \right\rceil \leq i \leq n$, together we have $2 + \left\lceil \frac{n}{3} \right\rceil \leq i \leq 2 + \left\lceil \frac{n-2}{3} \right\rceil$ which gives us $n = i + k$ and $i = m$.

⇐) If $n = i + k$ and $i = m$ for some $m = 4$ and $k \in [3,4]$ by lemma 2.3 we have D_{coe} $(Sp_{n-1}, i) = \emptyset$ $then$ $D_{coe}(Sp_{n-2}, i-1) \neq \emptyset$.

ii) ⇒) Since $D_{coe}(Sp_{n-2}, i-1) \neq \emptyset$ and $D_{coe}(Sp_{n-1}, i) \neq \emptyset$ by applying lemma 2.3 we have $2 + \left\lceil \frac{n-2}{3} \right\rceil \leq i - 1 \leq n - 2$ and $2 + \left\lceil \frac{n-1}{3} \right\rceil \leq i \leq n - 1$. Also

since $D_{coe}(Sp_n, i) \neq \emptyset$ by lemma 2.3 $2 + \left\lceil \frac{n}{3} \right\rceil \leq i \leq n$ and hence $2 + \left\lceil \frac{n}{3} \right\rceil \leq i \leq n - 1$.

⇐) If $2 + \left\lceil \frac{n}{3} \right\rceil \leq i \leq n - 1$ by lemma 2.3 we have the result.

The following theorem constructs the families of Co-even dominating sets of Sp_n

Theorem 2.9 $For\ every\ n \geq 7,$

$$i)\ If\ \ D_{coe}(Sp_{n-1}, i) = \emptyset\ \ and\ D_{coe}(Sp_{n-2}, i-1) \neq \emptyset\ \ then\ \ D_{coe}(Sp_n, i) = \{v_0, v_1, v_{n-2}, v_{n-1}\} \cup \{v_3, v_6, \dots, v_{n-5}\}.$$

$$ii)\ If\ D_{coe}(Sp_{n-1}, i) \neq \emptyset\ and\ D_{coe}(Sp_{n-2}, i-1) \neq \emptyset\ , then\ D_{coe}(Sp_n, i) = \{X_1 \cup \{v_{n-2}\} | X_1 \in\ D_{coe}(Sp_{n-2}, i-1)\} \cup \{X_2 \cup \{v_{n-1}\} | X_2 \in\ D_{coe}(Sp_{n-1}, i)\}$$

Proof

Let Sp_n be the spider graph with n vertices. Let $V(Sp_n) = \{v_0, v_1, \dots, v_{n-1}\}$ where v_0 is the branch vertex.

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i) Since $D_{coe}(Sp_{n-1},i)=\emptyset$ and $D_{coe}(Sp_{n-2},i-1)\neq\emptyset$ by lemma 2.8 (i) $n=i+k$ and $i=m$ for $some$ $m=4$ and $k\in[3,4]$. Therefore

$$D_{coe}(Sp_n,i)=\{v_0,v_1,v_{n-2},v_{n-1}\}\cup\{v_3,v_6,\dots,v_{n-5}\}.$$

ii) We have $D_{coe}(Sp_{n-1},i)\neq\emptyset$ and $D_{coe}(Sp_{n-2},i-1)\neq\emptyset$, we denote $\{X_1\cup\{v_{n-2}\}|X_1\in D_{coe}(Sp_{n-2},i-1)\}$ simply by Y_1. Let $X_1\in D_{coe}(Sp_{n-2},i-1)$. At least one of the elements are labelled with v_{n-2} or v_{n-3}. If v_{n-2} or v_{n-3} is in X_1 then $X_1\cup\{v_{n-2}\}\in D_{coe}(Sp_n,i)$. Next we denote $\{X_2\cup\{v_{n-1}\}|X_2\in D_{coe}(Sp_{n-1},i)\}$ simply by Y_2. Suppose $X_2\in D_{coe}(Sp_{n-1},i)$

At least one of the elements are labelled with v_{n-2} or v_{n-3} is in X_2, then $X_2\cup\{v_n\}\in D_{coe}(Sp_n,i)$. Hence $Y_1\cup Y_2\subseteq D_{coe}(Sp_n,i)$.

Conversely suppose $Y\in D_{coe}(Sp_n,i)$, $here$ the elements ends with v_n or v_{n-1}. If v_{n-1} is in Y, then $Y=X_1\cup\{v_{n-1}\}$, for some $X_1\in D_{coe}(Sp_{n-2},i-1)$. If v_n is in Y, then $Y=X_2\cup\{v_n\}$ for some $X_2\in D_{coe}(Sp_{n-1},i)$. Hence $Y\subseteq Y_1\cup Y_2$.

By theorem 2.9 we have the following theorem for $|D_{coe}(Sp_n,i)|$

Theorem 2.10 If $D_{coe}(Sp_n,i)$ be the family of all co-even dominating sets with cardinality i and let $d_{coe}(Sp_n,i)=|D_{coe}(Sp_n,i)|$ for $every$ $n\geq 7. Then,$

$|D_{coe}(Sp_n,i)|=|D_{coe}(Sp_{n-1},i)|+|D_{coe}(Sp_{n-2},i-1)|$.

Proof

The proof follows from theorem 2.9

3. Co-even Domination Polynomials of Spider Graph Sp_n

In this section we introduce and investigate the co-even domination polynomials of spider graph Sp_n.

Definition 1. Let $D_{coe}(Sp_n,i)$ denote the family of all co-even dominating sets of Sp_n with cardinality i. Let $d_{coe}(Sp_n,i)=|D_{coe}(Sp_n,i)|$. Then the co-even domination polynomial of is $D_{coe}(Sp_n,x)=\sum_{i=\gamma coe(G)}^{n}d_{coe}(Sp_n,i)x^i$.

By definition 1 and theorem 2.10 we have the following theorem.

Theorem 3.1. For every $n\geq 7$, $D_{coe}(Sp_n,x)=x[D_{coe}(Sp_{n-1},x)+D_{coe}(Sp_{n-2},x)]$ with initial values $D_{coe}(Sp_5,x)=x^5+x^4$ and $D_{coe}(Sp_6,x)=x^6+2x^5+x^4$.

Using theorem 2.10 we obtain some interesting relationship between the coefficients of $D_{coe}(Sp_n,x)$

Theorem 3.2. The following properties hold for the coefficients of $D_{coe}(Sp_n,i)$:

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i) $For\ every\ n \geq 5,\ d_{coe}(Sp_n, n) = 1.$

ii) $For\ every\ n \geq 5,\ d_{coe}(Sp_n, n+1) = 0.$

iii) $For\ every\ 5 \leq n \leq 8,\ d_{coe}(Sp_n, 4) = 1$

iv) $For\ every\ 5 \leq n \leq 8,\ d_{coe}(Sp_n, 5) = n - 4.$

Proof

i) There is only one unique set of size n. Hence $d_{coe}(Sp_n, n) = |\, D_{coe}(Sp_n, n)| = 1.$

ii) By the definition of co-even domination and from the table we can write d_{coe} $(Sp_n, n+1) = 0.$

iii) From the table $d_{coe}(Sp_n, 4) = 1.$

iv) From the table $d_{coe}(Sp_n, 5) = n - 4.$

Table 1 $D_{coe}(Sp_n, i)$, the number of co-even dominating sets of Sp_n with cardinality i

n / i	4	5	6	7	8	9	10	11	12	13
5	1	1								
6	1	2	1							
7	1	3	4	1						
8	1	4	6	4	1					
9	0	3	9	8	8	1				
10	0	2	13	14	12	7	1			
11	0	1	16	25	20	15	6	1		

CONCLUSION

In this paper we analyse and obtain the co-even domination polynomial and co-even dominating sets of Spider graph Sp_n.

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