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A Study on Coupling Index of Graphs

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Abstract

A range of fields, including chemistry, economy, gadgets, business analysis, social sciences, and medical, will always benefit from different topological indices. The computation of the graph coupling index and coupling distance characteristics are the focus of this study. Various methods for computing coupling index been illustrated. This study derives an expression for the coupling index of graphs regarding coupling diameters. Relation between coupling radius, eccentricity and diameter along with lower and upper bounds for coupling index has also been demonstrated. On coupling index, several related theorems have been developed. The Coupling strength of graphs has further been suggested.

Keywords: Geodesic, coupling distance, coupling eccentricity, coupling index.

AMS Classification: 05C12

1. Introduction

Only non-trivial, finite simple and connected graphs are taken into consideration in this research. Simply consult Harary [1] for definitions of ambiguous terms. The distance notion has grown in significance as networks in graph theory have progressed substantially. The graph theory distance idea has been illustrated to be helpful in designing software. Some indispensable graph theory parameters, including eccentricity, radius, diameter, even metric dimension, have been established using this distance conception. The aforementioned parameters are connected, and the radius along with diameter was determined under certain constraints [1]. The length of the shortest $x - y$ path in a graph is the geodesic distance, or $\mathrm{d(x, y)}$, between any two vertices. In a graph, x and y are neighboring if and only if $\mathrm{d(x, y)}$ equals unity, whereas x and y are not neighbors if and only if $\mathrm{d(x, y)} \geq 2$. Detour distance outcomes varied and were compared to geodesic distance [2–4].

One of the first and most extensively researched distance-based molecular structure descriptors is the Wiener index [5–6]. There have been several reports of its chemical uses [7-8]. The Wiener index's mathematical features are very well understood (see publications [9–11], as well as [12–17] and the references referenced therein). In [18], several limits in the sense of diameter together with radius were studied.

The total of the diversion and the geodesic distance is the round distance. Circular distance reaches metric over all of the graph's vertices. Results on the relationships between the diameters and radius of the graphs analyzed [19].

In this research, we create the coupling index for graphs along with investigate its characteristics. The length of the shortest path between any two vertices is added to the total number of vertices on the path to get the coupling distance among any two vertices. Theorems about the relationship between coupling distance parameters, such as coupling radius and coupling diameter are formulated and shown.

Definition:

The coupling distance between any two vertices in a graph is defined [20] by

$$\mathfrak{C}\mathrm{d(s,t)} = \begin{cases} \mathrm{g(s,t)} + |\mathrm{P(s,t)}| & \text{if } \mathrm{s} \neq \mathrm{t} \\ 0 & \text{if } \mathrm{s} = \mathrm{t} \end{cases}$$

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Here $g(s,t)$ indicates the geodesic distance of s and t whereas $|P(s,t)|$ indicates the cardinality of vertices lies the path P of $g(s,t)$. It is obvious that the coupling distance is always odd.

Definition:

The coupling index of a graph is defined as

$$\mathfrak{C}(G) = \frac{1}{2}\Sigma_{\{s,t\}\in V(G)}\mathfrak{C}d(s,t) = \Sigma_{\{s,t\}\subseteq V(G)}\mathfrak{C}d(s,t)$$

Here $\mathfrak{C}d(s,t)$ is the coupling distance of s and t. The notations $\mathfrak{C}r$ and $\mathfrak{C}d$ ($i.e, \mathfrak{C}diam$) refers for Coupling radius and diameter, duly.

If $\mathfrak{C}d(s|G)$ is defined as a distance number of s, i.e. $\mathfrak{C}d(s|G) = \sum_{t\in V(G)} \mathfrak{C}d(s,t)$, then

$$\mathfrak{C}(G) = \frac{1}{2}\Sigma_{s\in V(G)}\mathfrak{C}d(s|G)$$

2. Methods of Computing Coupling Index

There are many methods for finding the coupling index of a graph G with restrictions on the graphs. We shall dictate one by one.

2.1 Matrix Method

In this method, the coupling index is computed from coupling distance matrix $\mathfrak{C}$ of the graph. The coupling matrix is defined as

$$\mathfrak{C} = \begin{cases} \mathfrak{C}d(s,t) & \text{if } s \neq t \\ 0 & \text{otherwise} \end{cases}$$

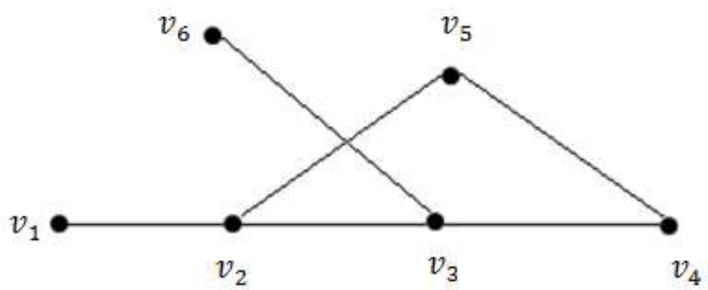

Figure 2.1

The coupling matrix of Fig. 2.1 is,

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$$\mathfrak{C} = \begin{pmatrix} 0 & 3 & 5 & 7 & 5 & 7 \\ 3 & 0 & 3 & 5 & 3 & 5 \\ 5 & 3 & 0 & 3 & 5 & 3 \\ 7 & 5 & 3 & 0 & 3 & 5 \\ 5 & 3 & 5 & 3 & 0 & 7 \\ 7 & 5 & 3 & 5 & 7 & 0 \end{pmatrix}$$

The Coupling index of Fig. 2.1 is

$$\mathfrak{C}(G) = \frac{1}{2}\Sigma_{\{s,t\}\in V(G)}\mathfrak{C}d(s,t) = 69$$

The generalized Coupling polarity index $\mathfrak{C}_k(G)$ of a graph G is the number of unordered pair of vertices $\{s,t\}$ of G such that the distance $\mathfrak{C}d(s,t)$ between s and t is k.

$$\mathfrak{C}_k(G) = |\{\{s,t\}; \mathfrak{C}d(s,t) = k, s,t \in V(G)\}|$$

For G with diameter $\mathfrak{C}diam = d$,

$$\mathfrak{C}(G) = \Sigma_{\{s,t\}\subseteq V(G)}\mathfrak{C}d(s,t) = \Sigma_{1\le k\le d}\, k\mathfrak{C}_k(G)$$

2.2 Generalized Coupling Polarity Index:

The Coupling index is computed using generalized Coupling polarity index.

$\mathfrak{C}_1(G) = 0, \mathfrak{C}_2(G) = 0, \mathfrak{C}_4(G) = 0, \mathfrak{C}_6(G) = 0,$

$$\mathfrak{C}_3(G) = |\{(v_1,v_2),(v_2,v_3),(v_2,v_5),(v_3,v_4),(v_3,v_6),(v_4,v_5)\}| = 6$$

$$\mathfrak{C}_5(G) = |\{(v_1,v_3),(v_1,v_5),(v_2,v_4),(v_2,v_6),(v_3,v_5),(v_4,v_6)\}| = 6$$

$$\mathfrak{C}_7(G) = |\{(v_1,v_4),(v_1,v_6),(v_5,v_6)\}| = 3$$

$$\mathfrak{C}(G) = \Sigma_{1\le k\le 7}\, k\mathfrak{C}_k(G)$$

$$\mathfrak{C}(G) = 3.\mathfrak{C}_3(G) + 5.\mathfrak{C}_5(G) + 7.\mathfrak{C}_7(G) = 18 + 30 + 21 = 69$$

2.3 Transmission Method

The Coupling index using transmission method for Fig.2.1:

The transmission $T_G(s)$ of a vertex $s \in V(G)$ is

$$T_G(s) = \Sigma_{t\in V(G)}\mathfrak{C}d(s,t)$$

$$T_G(v_1) = \mathfrak{C}d(v_1,v_2) + \mathfrak{C}d(v_1,v_3) + \mathfrak{C}d(v_1,v_4) + \mathfrak{C}d(v_1,v_5) + \mathfrak{C}d(v_1,v_6) = 27$$

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As well as ,

$T_G(v_2) = 19; T_G(v_3) = 19; T_G(v_4) = 23; T_G(5) = 23; T_G(v_6) = 27$

2.4 Result: For a graph with diameter $T_G(s)$, transmission of $s \in G$,

$$\mathfrak{C}(G) = \frac{1}{2}\Sigma_{s\in V(G)} T_G(s)$$

For Fig. 2.1, $\mathfrak{C}(G) = \frac{1}{2}[27 + 19 + 19 + 23 + 23 + 27] = 69$.

3. Main Results

3.1 Theorem: For any graph G, $3 \leq \mathfrak{C}d(s,t) \leq 2p + 1$

Proof: From [20], the lower bound is tight for complete graphs as every vertex are adjacent to each other. $\mathfrak{C}d(s,t) \geq 3$. Also from [20], if $d(s,t) = t$, $t = diam(G)$, then $\mathfrak{C}d(s,t) = 2t + 1$. The upper bound is tight for path P_p.

3.2 Theorem: Let G be connected having $p \geq 3$. Then $3(p-1)^2 - \frac{p^2-3p+2}{2} \leq \mathfrak{C}(G) \leq \frac{dp(p-1)}{2}$ with left equality if and only if $G = S_n$ and with right equality if and only if $G = K_n$.

Proof: Let $u,t \in V(G)$, then $\mathfrak{C}d(u,v) \leq d$, $d = \mathfrak{C}e(G)$. This equality hold if and only if G has a u-v path so that $d(u,v) = \frac{d-1}{2}$. Thus

$$\mathfrak{C}(G) \leq d\left(\frac{p(p-1)}{2}\right) = \frac{dp(p-1)}{2}$$

Such equality is tight if and only if $d(u,v) = \frac{d-1}{2}$ for all $u,t \in V(G)$. i.e., $G = K_n$.

Whereas, for any spanning tree T of G, $\mathfrak{C}d(u,v|G) \geq 2d(u,v|T) + 1$ so that $\mathfrak{C}(G) \geq \mathfrak{C}(T)$ with equality if and only if $G = T$.

And that $\mathfrak{C}(T) \geq 3(p-1)^2 - \frac{p^2-3p+2}{2}$ with equality if and only if $T = S_n$. Thus $\mathfrak{C}(G) \geq 3(p-1)^2 - \frac{p^2-3p+2}{2}$ with equality if and only if $G = T = S_n$.

3.3 Theorem: $\mathfrak{C}d(s,t) = \mathfrak{C}d(s,w) + \mathfrak{C}d(w,t) - 1$, where $s,t,w \in V(G)$, G is connected.

Proof: Let $V = \{s = x_1, x_2, \dots w = x_k, x_{k+1} \dots x_p = t\}$. Let $s - w$ and $w - t$ be geodesic paths.

Therefore,

$$\mathfrak{C}d(s,t) = g(s,t) + |P(s,t)| = p - 1 + p - 1 + 1 = 2p - 1$$

Now, $\mathfrak{C}d(s,w) = g(s,w) + |P(s,w)| = k - 1 + k - 1 + 1 = 2k - 1$ and

$$\mathfrak{C}d(w,t) = g(w,t) + |P(w,t)| = p - k + p - k + 1 = 2(p - k) + 1$$

Consider $\mathfrak{C}d(s,w) + \mathfrak{C}d(w,t) = 2k - 1 + 2(p - k) + 1 = 2k + 2p - 2k = 2p$.

Thus $\mathfrak{C}d(s,t) = \mathfrak{C}d(s,w) + \mathfrak{C}d(w,t) - 1$ or $\mathfrak{C}d(s,t) + 1 = \mathfrak{C}d(s,w) + \mathfrak{C}d(w,t)$.

3.4 Theorem: Let $o(G) = a + b$, such that b are pendent vertices. Then

$$\mathfrak{C}d(s,t) \leq \begin{cases} 2(a-b)+1 & \text{if} \quad s,t \in Y \\ 2(a-b)+5 & \text{if} \quad s,t \in X \\ 2(a-b)+3 & \text{if} \ s \in X, t \in Y \end{cases}$$

Proof: Let X and Y be the set of pendent and non-pendent vertices of G.

Case i: Suppose $s,t \in Y$. Then any $s - t$ path doesnot contain any vertices from X as well as the pendent edges. Therefore $d(s,t) \leq a - b$. So that $\mathfrak{C}d(s,t) \leq 2(a - b) + 1$.

Case ii: Suppose $s,t \in X$. Then any $s - t$ geodesic path only contains vertices of Y. Therefore $d(s,t) \leq a - b + 2$. Thus $\mathfrak{C}d(s,t) \leq 2(a - b + 2) + 1 = 2(a - b) + 5$.

Case iii: Suppose $s \in X, t \in Y$. Then any $s - t$ path will originate in s. Such paths will definitely pass the pendent edge corresponding to s and not pass the remaining $b - 1$ vertices of Y and their corresponding edges. Thus, geodesic distance with maximum edges is $a - b + 1$. Hence $\mathfrak{C}d(s,t) \leq 2(a - b + 1) + 1 = 2(a - b) + 3$.

3.5 Theorem: $\mathfrak{C}r(G) \leq \mathfrak{C}d(G) \leq 2\mathfrak{C}r(G) - 1$

Proof: It is obvious that $\mathfrak{C}r(G) \leq \mathfrak{C}d(G)$. Let $s,t \in V(G)$, and $\mathfrak{C}d(s,t) = \rho$, the diameter. Let $w \in G$ such that $\mathfrak{C}e(w) = \mathfrak{C}r(G)$.

Now,

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$$\rho = \mathfrak{C}d(s,t) \leq \mathfrak{C}d(s,w) + \mathfrak{C}d(w,t) - 1 = \mathfrak{C}e(w) + \mathfrak{C}e(w) - 1 = 2\mathfrak{C}e(w) - 1$$

$$= 2\mathfrak{C}r(G) - 1$$

Therefore $\mathfrak{C}d(G) \leq 2\mathfrak{C}r(G) - 1$. Hence $\mathfrak{C}r(G) \leq \mathfrak{C}d(G) \leq 2\mathfrak{C}r(G) - 1$.

3.6 Remark: In the above theorem, Upper bound is tight for P_p whereas lower bound is tight for K_p.

3.7 Theorem: $\mathfrak{C}e(s) - \mathfrak{C}e(t) \leq \mathfrak{C}d(s,t) - 1$ for $s, t \in G$.

Proof: Let $s, t \in V(G)$. Let $w \in V(G)$ such that $\mathfrak{C}d(s,w) = \mathfrak{C}e(s)$. Now,

$$\mathfrak{C}e(s) = \mathfrak{C}d(s,w) \leq \mathfrak{C}d(s,t) + \mathfrak{C}d(t,w) - 1$$

$$\leq \mathfrak{C}d(s,t) + \mathfrak{C}e(t) - 1$$

$$\therefore \mathfrak{C}e(s) \leq \mathfrak{C}d(s,t) + \mathfrak{C}e(t) - 1$$

$$\Longrightarrow \mathfrak{C}e(s) - \mathfrak{C}e(t) \leq \mathfrak{C}d(s,t) - 1$$

3.8 Theorem: Suppose $o(G) = p$ and $s(G) = q$, here $s(G)$ refers for size. Then $\mathfrak{C}(G) = \frac{5p(p-1)-4q}{2}$ if and only if $\mathfrak{C}diam(G) \leq 5$.

Proof: Let $G = (p,q)$ and $\mathfrak{C}diam(G) \leq 2$. Create $M = \{s \in V; \mathfrak{C}e(s) = 3\}$ and $N = \{s \in V; \mathfrak{C}e(s) = 5\}$. So that $|M| + |N| = p$.

Suppose $s \in M$. Then $\mathfrak{C}d(s|G) = 3(p-1)$.

Suppose $s \in N$. Then take $N_1 = \{t \in V: \mathfrak{C}d(s,t) = 3\}$ and $N_2 = \{t \in V: \mathfrak{C}d(s,t) = 5\}$.

Then, $\mathfrak{C}d(s|G) = 3|N_1| + 5|N_2|$

$$= 3(|N_1| + |N_2|) + 2|N_2|$$

$$= 3(p-1) + 2|N_2|$$

$$= 3(p-1) + 2(p-1-|N_1|)$$

$$= 5(p-1) - 2|N_1|$$

$$\mathfrak{C}d(s|G) = 5(p-1) - 2\deg(s)$$

Now, $\mathfrak{C}(G) = \frac{1}{2}\sum_{s \in V} \mathfrak{C}d(s|G)$

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$$=\frac{1}{2}\left(\sum_{s\in M}\mathfrak{C}d(s|G)+\sum_{s\in N}\mathfrak{C}d(s|G)\right)$$

$$=\frac{1}{2}[3(p-1)|M|+(5(p-1)-2\deg(s))\,|N|]$$

$$=\frac{1}{2}\left[3(p-1)|M|+5(p-1)|N|-2\sum_{s\in N}\deg(s)\right]$$

$$=\frac{1}{2}\left[3(p-1)(|M|+|N|)+2(p-1)|N|-2\sum_{s\in N}\deg(s)\right]$$

$$=\frac{1}{2}\left[3p(p-1)+2(p-1)(p-|M|)-2\sum_{s\in N}\deg(s)\right]$$

$$=\frac{1}{2}\left[3p(p-1)+2p(p-1)-2(p-1)|M|-2\sum_{s\in N}\deg(s)\right]$$

$$=\frac{1}{2}\left[5p(p-1)-2\sum_{s\in M}\deg(s)-2\sum_{s\in N}\deg(s)\right]$$

$$=\frac{1}{2}\left[5p(p-1)-2\left(\sum_{s\in M}\deg(s)+\sum_{s\in N}\deg(s)\right)\right]$$

$$=\frac{1}{2}\left[5p(p-1)-2\sum_{s\in V}\deg(s)\right]$$

$$=\frac{1}{2}[5p(p-1)-2(2q)\,]=\frac{1}{2}[5p(p-1)-4q\,]$$

Thus $\mathfrak{C}(G)=\frac{5p(p-1)-4q}{2}$.

Conversely, Assume $\mathfrak{C}(G)=\frac{5p(p-1)-4q}{2}$. Now, to prove $\mathfrak{C}\mathrm{diam}(G)\le 5$. On contrary, let $\mathfrak{C}\mathrm{diam}(G)\ge 7$. Take an arbitrary $s\in V(G)$. Now, define $M=\{s\in V;\mathfrak{C}e(s)=5\}$ and $N=\{s\in V;\mathfrak{C}e(s)\ge 7\}$, sothat $|M|+|N|=p$.

Suppose $s\in M$. Then $\mathfrak{C}d(s|G)=5(p-1)-2\deg(s)$, as before.

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Suppose $s \in N$. Then take $N_1 = \{t \in V: \mathfrak{C}d(s,t) = 3\}$, $N_2 = \{t \in V: \mathfrak{C}d(s,t) = 5\}$ and $N_3 = \{t \in V: \mathfrak{C}d(s,t) \geq 7\}$. Clearly $|N_1| + |N_2| + |N_3| = p - 1$.

Then, $\mathfrak{C}d(s|G) \geq 3|N_1| + 5|N_2| + 7|N_3|$

$$= 3(|N_1| + |N_2| + |N_3|) + 2|N_2| + 4|N_3|$$

$$= 3(p-1) + 2(|N_2| + |N_3|) + 2|N_3|$$

$$= 3(p-1) + 2(p-1-|N_1|) + 2|N_3|$$

$$= 5(p-1) - 2|N_1| + 2|N_3|$$

$$= 5(p-1) - 2\deg(s) + 2|N_3|$$

$$\geq 5(p-1) - 2\deg(s) + 2 \qquad (\because\ |N_3| \geq 1)$$

$$\Rightarrow \mathfrak{C}d(s|G) \geq 5p - 2\deg(s) - 3$$

Now, $\mathfrak{C}(G) = \frac{1}{2}\left[\sum_{s\in V}\mathfrak{C}d(s|G)\right]$

$$= \frac{1}{2}\left[\sum_{s\in M}\mathfrak{C}d(s|G) + \sum_{s\in N}\mathfrak{C}d(s|G)\right]$$

$$\geq \frac{1}{2}[(5(p-1) - 2\deg(s))|M| + (5p - 2\deg(s) - 3)|N|]$$

$$= \frac{1}{2}[5(p-1)|M| - 2\deg(s)\,|M| + (5(p-1) - 2\deg(s) + 2)|N|]$$

$$= \frac{1}{2}[5(p-1)|M| - 2\deg(s)\,|M| + 5(p-1)|N| - 2\deg(s)\,|N| + 2|N|]$$

$$= \frac{1}{2}[5(p-1)(|M| + |N|) - 2\deg(s)\,|M| - 2\deg(s)\,|N| + 2|N|]$$

$$= \frac{1}{2}\left[5p(p-1) - 2\left(\sum_{s\in M}\deg(s) + \sum_{s\in N}\deg(s)\right) + 2|N|\right]$$

$$= \frac{1}{2}\left[5p(p-1) - 2\sum_{s\in V}\deg(s) + 2|N|\right]$$

$$= \frac{1}{2}[5p(p-1) - 4q + 2|N|]$$

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$$= \frac{5p(p-1)-4q}{2} + |N|$$

$\geq \frac{5p(p-1)-4q}{2} + 2$ as $|N| \geq 2$, Since N has minimum 2 vertices with coupling eccentricity ≥ 7 as $\mathfrak{C}\mathrm{diam}(G) \geq 7$.

$$\therefore\ \mathfrak{C}(G) \geq \frac{5p(p-1)-4q}{2} + 2$$

This contradict the hypothesis of $\mathfrak{C}(G) = \frac{5p(p-1)-4q}{2}$. Hence $\mathfrak{C}\mathrm{diam}(G) \leq 5$. Proof attained.

3.9 Theorem: For $G = (p, q)$ with $\mathfrak{C}\mathrm{diam}(G) \geq 7$, $\mathfrak{C}(G) \geq \frac{5p(p-1)-4q}{2} + 2$ holds. Bound is tight if only two vertices of G has $\mathfrak{C}$e three whereas rest are of $\mathfrak{C}$e '2'.

Proof: Proof is similar to previous theorem.

3.10 Theorem: For $K_{p,q}$, $\mathfrak{C}r(K_{p,q}) = 3$ and $\mathfrak{C}d(K_{p,q}) = 5$.

Proof: Let A_1 and A_2 be the vertex sets of $K_{p,q}$. Let $n = p + q$.

Case i. Suppose $s, t \in A_1$. Then $s - w - t$ is a $s - t$ geodesic path for any $w \in A_2$. Therefore $d(s, t) = 2$. Implies that $\mathfrak{C}d(s, t) = 5$.

Case ii. Suppose $a, b \in A_2$. Then $a - s - b$ is a $a - b$ geodesic path for any $s \in A_1$. Therefore $d(a, b) = 2$. This implies that $\mathfrak{C}d(a, b) = 5$.

Case iii. Suppose $s \in A_1, a \in A_2$. Then for any such a and s $d(a, s) = 1$. Therefore $\mathfrak{C}d(a, s) = 3$.

From cases (i-iii), $\mathfrak{C}e(s) = 5$ and $\mathfrak{C}e(a) = 3$. Thus $\min\{\mathfrak{C}e(s), \mathfrak{C}e(a)\} = \mathfrak{C}r(K_{p,q}) = 3$ and $\max\{\mathfrak{C}e(s), \mathfrak{C}e(a)\} = \mathfrak{C}d(K_{p,q}) = 5$.

3.11 Result: From theorem 3.8, it is easy that $\mathfrak{C}(K_{p,q}) = \frac{5n(n-1)-4\left(\frac{n^2}{2}\right)}{2}$, where $n = p + q$.

3.12 Theorem: For S_p^0, with $p > 3$, $\mathfrak{C}r(S_p^0) = 3$ and $\mathfrak{C}d(S_p^0) = 7$.

Proof: Let A_1 and A_2 be the vertex sets of S_p^0.

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Case i. Suppose $s,t \in A_1$. Then $s - w - t$ is a $s - t$ geodesic path for any $w \in A_2$. Therefore $d(s,t) = 2$. Implies that $\mathfrak{C}d(s,t) = 5$.

Case ii. Suppose $a,b \in A_2$. Then $a - s - b$ is a $a - b$ geodesic path for any $s \in A_1$. Therefore $d(a,b) = 2$. Implies that $\mathfrak{C}d(a,b) = 5$.

Case iii. Suppose $s \in A_1, a \in A_2$.

Subcase (i): If a adjacent to s $d(a,s) = 1 \Longrightarrow \mathfrak{C}d(a,s) = 3$.

Subcase (ii): If a not adjacent to s, then $a - t - b - s$ is a geodesic path for any such a and s. Thus $d(a,s) = 3 \Longrightarrow \mathfrak{C}d(a,s) = 7$.

From all cases, $\min\{\mathfrak{C}e(s), \mathfrak{C}e(a)\} = \mathfrak{C}r\left(S_p^0\right) = 3$ and $\max\{\mathfrak{C}e(s), \mathfrak{C}e(a)\} = \mathfrak{C}d\left(S_p^0\right) = 7$.

3. 13 Theorem: For $\overline{pP_2}$, with $p \geq 2, \mathfrak{C}r(\overline{pP_2}) = 3$ and $\mathfrak{C}d(\overline{pP_2}) = 5$.

Proof: For $\overline{pP_2}$, $o(\overline{pP_2}) = 2p$.

Case i. If $s,t \in E(G)$, then $\mathfrak{C}d(s,t) = 3$.

Case ii. If $s,t \notin E(G)$. Since $\overline{pP_2}$ is a $(2p - 2)$-regular graph, $\mathfrak{C}d(s,t) = 5$.

From both cases, it is obvious that $\mathfrak{C}r(\overline{pP_2}) = 3$ and $\mathfrak{C}d(\overline{pP_2}) = 5$.

3.14 Result: From theorem 3.8, it is easy to attain $\mathfrak{C}(\overline{pP_2})$.

4. Coupling strength of a graph

If $e \in E(G)$ is not a bridge of G, then $\mathfrak{C}(G - e)$ is defined and $\mathfrak{C}(G) < \mathfrak{C}(G - e)$. That is removable of such e enhance the Coupling index. The least increase in $\mathfrak{C}(G)$ obtained by the removal of such e could be noted as a strength of G. Let this least increase be "Coupling strength" of G.

If G is connected but not a tree graph, the Coupling strength of G is described as

$$\mathfrak{C}\mathfrak{S}(G) = \min\{\mathfrak{C}(G - e) - \mathfrak{C}(G) : \text{ e lies in a cycle}\}$$

4.1 Examples:

$\mathfrak{C}\mathfrak{S}\left(K_p\right) = 2$, for all $p \geq 3$.

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5. Conclusion:

In this paper, we studied the coupling distance and coupling index of graphs. Various methods for computing coupling index are illustrated. Upper and lower bounds on coupling distance along with coupling index have been demonstrated. Relation between coupling radius, eccentricity and diameter also been studied. An expression for $\mathfrak{C}(G)$ is established in the base of $\mathfrak{C}diam(G)$. $\mathfrak{C}r$, $\mathfrak{C}e$ and $\mathfrak{C}d$ of classes of graphs are computed. At last, Coupling strength of graph has been introduced.

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