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Coupling Index of Some Algebraic Graphs

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Abstract:

Consider the (p,q) simple connected graph $G = (V, E)$. The length of the shortest path between any two vertices is added to the total number of vertices on the path to get the coupling distance among any two vertices. The objective of this study is to examine the coupling index of identity graphs and zero-divisor graphs $\Gamma(R)$of commutative rings using group theory, graph theory, and applications. In this study, the identity graphs $I(Z_p)$ connected to the group $(Z_p, +)$ and a few classes of zero-divisor graphs $\Gamma(R)$ of the commutative ring R are examined.

Keywords: Zero-Divisor Graphs, Identity Graphs, Coupling Index

1. Introduction:

In this study, connected, finite graphs are taken into consideration. The graph theory distance idea has been illustrated to be helpful in designing software. Some indispensable graph theory parameters, including eccentricity, radius, diameter, even metric dimension, have been established using this distance conception. The length of the shortest x − y path in a graph is the geodesic distance, or d(x, y), between any two vertices. In a graph, x and y are neighboring if and only if d(x, y) equals unity, whereas x and y are not neighbors if and only if d(x, y) ≥ 2. Detour distance outcomes varied and were compared to geodesic distance [4-5]. The length of the shortest path between any two vertices

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

is added to the total number of vertices on the path to get the coupling distance among any two vertices.

Beck [2] first raised the possibility of a graph relating to the zero divisors of a commutative ring in 1998. The zero divisor graph (R) is a simple graph with vertex set $Z(R)^*$, the set of all non-zero divisors in R, and any two different vertices x and y are adjacent if xy=0. It was modulated by Anderson D.F. and Livingson P.S. [1] after that.

Vasantha Kandasamy W.B. first proposed the concept of groups as graphs [6]. Identity graphs are graphs $\mathcal{I}(Z_p)$ associated to group Z_p and $\mathcal{I}(Z_p)$ is defined as (V, E) where $\mathrm{V} = \mathrm{V}(Z_p)$, and two elements $x, y \in Z_p$ are adjacent iff $x.y = e$ or $x = e$ or $y = e$. For graph-theoretic parlance one may refer Harary [3]. In this study, we examine the quotient energy of a few zero-divisor commutative ring graphs $\Gamma(R)$ and identity graphs of groups.

2. Coupling Index of Zero-Divisor graph

2.1 Definition: Let be a (p, q) graph. The coupling matrix is

$$\mathfrak{C}(G) = \begin{cases} \mathfrak{C}d(s,t) & if\ s \neq t \\ 0 & otherwise \end{cases}$$

And the Coupling index is

$$\mathfrak{C}(G) = \frac{1}{2} \sum_{\{s,t\} \subseteq \in V(G)} \mathfrak{C}d(s,t)$$

2.2Theorem: $\mathfrak{C}\left(\Gamma(Z_{2p})\right) = \frac{p}{2}[5\mathrm{p} - 9] + 2$ for $p > 2$ where p is a prime.

Proof: For any prime $p > 2, V\left(\Gamma(Z_{2p})\right) = \{v_1, v_2, \dots, v_p\} = \{2,4, \dots, 2(p-1), p\}$ and $E\left(\Gamma(Z_{2p})\right) = \{v_i v_p / 1 \leq i \leq p-1\}$. As there is p-1 pendant vertices, these p-1 vertices are of distance 1 from central vertex with degree p-1.

Therefore, $\mathfrak{C}d(v_1, v_i) = 3, 1 \leq i \leq p-1$.

Also among those p-1 pendant vertices, one can reach to another pendant vertex by distance 2. So, there is a path of distance 2 among all p-1 pendant vertices.

Therefore, $\mathfrak{C}d(v_i, v_j) = 3, 1 \leq i, j \leq p-1, i \neq j$. Then the corresponding Coupling matrix $\mathfrak{C}$ of $\Gamma(Z_{2p})$ is of the form $\mathfrak{C} = \begin{pmatrix} 0 & 3J_{1\times p-1} \\ 3J_{p-1\times 1} & 5(J_{p-1} - I_{p-1}) \end{pmatrix}$, where J and I denotes the matrix of ones and identity matrix, respectively.

Therefore the Coupling Index of $\Gamma(Z_{2p})$ is,

$$\mathfrak{C}\left(\Gamma(Z_{2p})\right) = \frac{1}{2}\left[6(\mathrm{p}-1) + 10\binom{\mathrm{p}-1}{2}\right]$$

$$= \frac{1}{2}\left[6(\mathrm{p}-1) + 10\left(\frac{\mathrm{p}^2 - 3\mathrm{p} + 2}{2}\right)\right]$$

$$= \frac{1}{2}[6p - 6 + 5(\mathrm{p}^2 - 3\mathrm{p} + 2)]$$

$$= \frac{1}{2}[5\mathrm{p}^2 - 9\mathrm{p} + 4]$$

Hence,

$$\mathfrak{C}\left(\Gamma(Z_{2p})\right) = \frac{p}{2}[5\mathrm{p} - 9] + 2$$

2.3 Theorem: $\mathfrak{C}\left(\Gamma(Z_{3p})\right) = \frac{p}{2}(5p-3) + 4$ for $p > 2$ where p is a prime.

Proof: For any prime $p > 3, V\left(\Gamma(Z_{3p})\right) = \{u_1, u_2, v_1, v_2, \dots, v_{p-1}\}$ where $u_1 = p, u_2 = 2p$ and $v_i = 3i \; for \; 1 \leq i \leq p-1$. And $E\left(\Gamma(Z_{3p})\right) = \{u_1v_i, u_2v_i, 1 \leq i \leq p-1\}$. Then $\left|V\left(\Gamma(Z_{3p})\right)\right| = p+1$and $\left|E\left(\Gamma(Z_{3p})\right)\right| = 2(p-1)$.

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

Here $d(u_1, u_2) = d(v_i, v_j) = 2$.

$$\therefore \mathfrak{C}d(u_1, u_2) = \mathfrak{C}d(v_i, v_j) = 5$$

Also, $d(u_i, v_j) = 1 \ \forall \leq i \leq 2; 1 \leq j \leq p-1$.

$$\therefore \mathfrak{C}d(u_i, v_j) = 3$$

Then the corresponding coupling matrix is of the form $\mathfrak{C} = \begin{pmatrix} 5(J_2 - I_2) & 3J_{2\times p-1} \\ 3J_{p-1\times 2} & 5(J_{p-1} - I_{p-1}) \end{pmatrix}$, where

J and I denotes the, matrix of ones and identity matrix, respectively.

Therefore the coupling index of $\Gamma(Z_{3p})$ is,

$$\mathfrak{C}\left(\Gamma(Z_{3p})\right) = \frac{1}{2}\left[3(4p-4) + 10\left(\binom{p-1}{2} + 1\right)\right]$$

Now, $\binom{p-1}{2} + 1 = \frac{p^2-3p+4}{2}$

Therefore,

$$\mathfrak{C}\left(\Gamma(Z_{3p})\right) = \frac{1}{2}\left[12p - 12 + 10\left(\frac{p^2-3p+4}{2}\right)\right]$$

$$= \frac{1}{2}[12p - 12 + 5p^2 - 15p + 20]$$

$$= \frac{1}{2}[5p^2 - 3p + 8]$$

Hence,

$$\mathfrak{C}\left(\Gamma(Z_{3p})\right) = \frac{p}{2}(5p-3) + 4$$

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

2.4 Theorem: $\mathfrak{C}\left(\Gamma(Z_{5p})\right) = \frac{p}{2}(5p+9)+23$ for $p > 2$ where p is a prime.

Proof: For any prime $p > 3, V\left(\Gamma(Z_{5p})\right) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, u_3, u_4\} = \{p, 2p, 3p, 4p\}$ and $V_2 = \{v_1, v_2, \dots, v_{p-1}\} = \{5,10, \dots, 5(p-1)\}$ and $E\left(\Gamma(Z_{5p})\right) = \{u_1v_i, u_2v_i, u_3v_i, u_4v_i, 1 \le i \le p-1\}$. Then $\left|V\left(\Gamma(Z_{5p})\right)\right| = p+3$and $\left|E\left(\Gamma(Z_{5p})\right)\right| = 4(p-1)$.

Here, $d(u_i, u_j) = d(v_s, v_t) = 2; 1 \le i,j \le 4 \; and \; 1 \le s,t \le p-1$, such that $i \ne j \; and \; s \ne t$.

Also, $d(u_i, v_s) = 1$. Hence, $\mathfrak{C}d(u_i, u_j) = \mathfrak{C}d(v_s, v_t) = 5$ & $\mathfrak{C}d(u_i, v_s) = 3$.

The corresponding coupling matrix is of the form $\mathfrak{C}(\Gamma(Z_{5p})) = \begin{pmatrix} 5(J_4 - I_4) & 3J_{4\times p-1} \\ 3J_{p-1\times 4} & 5(J_{p-1} - I_{p-1}) \end{pmatrix}$,

where J and I denotes the matrix of ones and identity matrix, respectively.

so that,

$$\mathfrak{C}\left(\Gamma(Z_{5p})\right) = \frac{1}{2}\left[(8)(3)(p-1) + (2)\left(6 + \binom{p-1}{2}\right)(5)\right]$$

$$= \frac{1}{2}\left[(8)(3)(p-1) + (2)\left(\frac{p^2-3p+14}{2}\right)(5)\right]$$

$$= \frac{1}{2}[24p - 24 + 5p^2 - 15p + 70]$$

$$= \frac{1}{2}[5p^2 + 9p + 46]$$

Hence $\mathfrak{C}\left(\Gamma(Z_{5p})\right) = \frac{p}{2}(5p+9)+23$.

2.5 Theorem: $\mathfrak{C}\left(\Gamma(Z_{pq})\right) = \frac{1}{2}[5(p^2+q^2) - 21(p+q) + 6pq + 26]$ for $p, q > 2$ where p, q are distinctprimes p $< q$.

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

Proof: For any distinct primes $p, q > 2$, with $\mathrm{p} < q, V\left(\Gamma(Z_{pq})\right) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \dots, u_{q-1}\} = \{p, 2p, 3p, \dots p(q-1)\}$; $V_2 = \{v_1, v_2, \dots, v_{p-1}\} = \{q, 2q, \dots, q(p-1)\}$; and $E\left(\Gamma(Z_{pq})\right) = \{u_i v_j; u_i \in V_1, v_j \in V_2; 1 \le i \le q-1; 1 \le i \le p-1\}$.Then

$\left|V\left(\Gamma(Z_{pq})\right)\right| = pq - p - q + 1$and $\left|E\left(\Gamma(Z_{pq})\right)\right| = 2(q-1)$.

The corresponding coupling matrix is of the form

$\mathfrak{C} = \begin{pmatrix} 5(J_{p-1} - I_{p-1}) & 3J_{p-1\times q-1} \\ 3J_{q-1\times p-1} & 5J_{q-1} \end{pmatrix}$, where J and I denotes the matrix of ones and identity matrix, respectively.

Hence the coupling index is,

$$\mathfrak{C}\left(\Gamma(Z_{pq})\right) = \frac{1}{2}\left[(2)(3)(p-1)(q-1) + (2)\left(\binom{\mathrm{q}-1}{2} + \binom{\mathrm{p}-1}{2}\right)(5)\right]$$

$$= \frac{1}{2}\left[6(pq - p - q + 1) + 10\left(\frac{p^2 + q^2 - 3(p+q) + 4}{2}\right)\right]$$

$$= \frac{1}{2}[5p^2 + 5q^2 - 21p - 21q + 6pq + 26]$$

$$\mathfrak{C}\left(\Gamma(Z_{pq})\right) = \frac{1}{2}[5(p^2 + q^2) - 21(p+q) + 6pq + 26]$$

2.6 Theorem:$\mathfrak{C}\left(\Gamma(Z_{2p}) + \Gamma(Z_4)\right) = \frac{1}{2}[5p^2 - 3p + 4]$ for $p > 2$ where p is a prime.

Proof: For any prime $p > 2, V\left(\Gamma(Z_{2p}) + \Gamma(Z_4)\right) = \{u_1, u_2, \dots, u_{p-1}, u_p, x\}$

$= \{2,4, \dots, 2(p-1), p, x\}$, where $\mathrm{x} = 2 \in Z_4$and $E\left(\Gamma(Z_{2p}) + \Gamma(Z_4)\right) = \{u_i u_p, u_i x, u_p x / 1 \le \mathrm{i} \le \mathrm{p} - 1\}$. Then $\left|V\left(\Gamma(Z_{2p}) + \Gamma(Z_4)\right)\right| = p + 1$ and $\left|E\left(\Gamma(Z_{2p}) + \Gamma(Z_4)\right)\right| = 2p - 1$. The

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

corresponding coupling matrix $\mathfrak{C}$ is of the form $\mathfrak{C} = \begin{pmatrix} 3(J_2 - I_2) & 3J_{2\times p-1} \\ 3J_{p-1\times 2} & 5(J_{p-1} - I_{p-1}) \end{pmatrix}$, where I, J denotes the identity matrix, matrix of ones, respectively.

Therefore the coupling index of $\Gamma(Z_{2p}) + \Gamma(Z_4)$ is,

$$\mathfrak{C}\left(\Gamma(Z_{2p}) + \Gamma(Z_4)\right) = \frac{1}{2}\left[(2)(3)(p-1) + (2)\left(\binom{p-1}{2}\right)(5)\right]$$

$$= \frac{1}{2}\left[12p - 6 + 10\left(\frac{p^2 - 3p + 2}{2}\right)\right]$$

$$= \frac{1}{2}[12p - 6 + 5p^2 - 15p + 10]$$

Hence,

$$\mathfrak{C}\left(\Gamma(Z_{2p}) + \Gamma(Z_4)\right) = \frac{1}{2}[5p^2 - 3p + 4]$$

2.7 Theorem: $\mathfrak{C}\left(\Gamma(Z_{2p}) + \Gamma(Z_6)\right) = \frac{5p^2+9p+26}{2}$ for $p > 2$ where p is a prime.

Proof: For any prime $p > 2$,

$V\left(\Gamma(Z_{2p}) + \Gamma(Z_6)\right) = \{u_1, u_2, \dots, u_{p-1}, u_p, x, y, z\} = \{2,4,\dots,2(p-1), p, x, y, z\}$, where $x = 2, y = 3$ *and* $z = 4 \in Z_6$ and $E\left(\Gamma(Z_{2p}) + \Gamma(Z_6)\right) = \{u_iu_p, u_ix, u_iy, u_iz, u_px, u_py, u_pz, xy, yz / 1 \leq i \leq p-1\}$. Then $\left|V\left(\Gamma(Z_{2p}) + \Gamma(Z_6)\right)\right| = p + 3$ and $\left|E\left(\Gamma(Z_{2p}) + \Gamma(Z_6)\right)\right| = 4p + 1$. The corresponding coupling matrix $\mathfrak{C}$ is of the form $\mathfrak{C} = \begin{pmatrix} 3(J_2 - I_2) & 3J_{2\times p-1} & 3J_2 \\ 3J_{2\times p-1} & 5(J_{p-1} - I_{p-1}) & 3J_{2\times p-1} \\ 3J_2 & 3J_{2\times p-1} & 5(J_2 - I_2) \end{pmatrix}$,

Therefore the coupling index of $\Gamma(Z_{2p}) + \Gamma(Z_6)$ is,

Journal of Computational Analysis and Applications
VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

$$\mathfrak{C}\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=\frac{1}{2}\left[2(4p+1)(3)+10\left(\binom{p-1}{2}+1\right)\right]$$

$$\mathfrak{C}\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=\frac{1}{2}\left[\frac{24p+6+5p^2-15p+20}{2}\right]$$

$$\mathfrak{C}\left(\Gamma(Z_{2p})+\Gamma(Z_6)\right)=\frac{5p^2+9p+26}{2}$$

2.8 Theorem: $\mathfrak{C}\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\frac{5p^2+3p+10}{2}$ for $p>2$ where p is a prime.

Proof: For any prime $p>2$,

$V\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\{u_1,u_2,\dots,u_{p-1},u_p,x,y\}=\{2,4,\dots,2(p-1),p,x,y\}$, where $x=3$ and $y=6\in Z_9$ and $E\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\{u_iu_p,u_ix,u_iy,u_px,u_py,xy/\ 1\le i\le p-1\}$. Then $\left|V\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)\right|=p+2$and $\left|E\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)\right|=3p$. The corresponding coupling matrix $\mathfrak{C}$ is of the form $\mathfrak{C}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, where $A=3J_3$, $B=3J_{3\times p-1}$, $C=3J_{p-1\times 3}$ and $D=5J_{p-1}$.

Therefore the coupling index of $\Gamma(Z_{2p})+\Gamma(Z_9)$ is,

$$\mathfrak{C}\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\frac{1}{2}\left[6p(3)+10\binom{p-1}{2}\right]$$

$$\mathfrak{C}\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\frac{1}{2}\left[\frac{18p+5p^2-15p+10}{2}\right]$$

$$\mathfrak{C}\left(\Gamma(Z_{2p})+\Gamma(Z_9)\right)=\frac{5p^2+3p+10}{2}$$

2.9 Theorem: $\mathfrak{C}\left(\Gamma(Z_{p^2})\right)=\frac{3p^2-9p+6}{2}$ for $p>2$ where p is a prime.

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

Proof: Since $\Gamma(Z_{p^2}) \cong K_{p-1}$, $f$or any prime $p > 2, V\left(\Gamma(Z_{p^2})\right) = \{u_1, u_2, \dots, u_{p-1}\}$. Here each u_i and u_j is of distance 1 from every vertex. ie, $d(u_i, u_j) = 1\ \forall i \neq j$.

The corresponding coupling matrix $\mathfrak{C}$ is of the form $\mathfrak{C} = 3(J_{p-1} - I_{p-1})$.

Therefore the coupling index of $\Gamma(Z_{p^2})$ is,

$$\mathfrak{C}\left(\Gamma(Z_{p^2})\right) = \frac{1}{2}\left[(2)\binom{\mathrm{p}-1}{2}(3)\right]$$

$$\mathfrak{C}\left(\Gamma(Z_{p^2})\right) = \frac{3p^2 - 9p + 6}{2}$$

2.10 Theorem: $\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_4)\right) = \frac{p}{2}[5\mathrm{p} - 9] + 2$ for $p > 2$ where p is a prime.

Proof: Since the resulting graph of $\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_4) \cong \Gamma(Z_{2p})$,

$$\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_4)\right) = \mathfrak{C}\left(\Gamma(Z_{2p})\right) = \frac{p}{2}[5\mathrm{p} - 9] + 2$$

2.11 Theorem: $\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right) = \frac{1}{2}[5p^2 + 3p + 14]$ for $p > 2$ where p is a prime.

Proof: For any prime $> 2, V\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right) = \{u_1, u_2, \dots, u_{p-1}, x, y, z\} = \{2,4, \dots, p(p-1), x, y, z\}$, where $\mathrm{x} = 2, \mathrm{y} = 3$ and $z = 4 \in Z_6$. And $E\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right) = \{u_i x, u_i y, u_i z, \mathrm{xy}, \mathrm{yz} / 1 \leq \mathrm{i} \leq \mathrm{p} - 1\}$. Then $\left|V\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right)\right| = p + 2$and $\left|E\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right)\right| = 3p - 1$. And the corresponding coupling matrix $\mathfrak{C}$ is of the form $\mathfrak{C} = \begin{pmatrix} 0 & 3J_{1\times 2} & 3J_{1\times p-1} \\ 3J_{2\times 1} & 5(J_2 - I_2) & 3J_{2\times p-1} \\ 3J_{p-1\times 1} & 3J_{p-1\times 2} & 5(J_{p-1} - I_{p-1}) \end{pmatrix}$,

Therefore the coupling index of $\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)$ is,

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

$$\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right) = \frac{1}{2}\left[6(3p-1) + 10\left(\binom{p-1}{2} + 1\right)\right]$$

$$\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right) = \frac{1}{2}\left[18p - 6 + 10\left[\frac{p^2-3p+2}{2} + 1\right]\right]$$

$$\Rightarrow \mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right) == \frac{1}{2}[18p - 6 + 5p^2 - 15p + 20]$$

$$\Rightarrow \mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_6)\right) = \frac{1}{2}[5p^2 + 3p + 14]$$

2.12 Theorem:$\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right) = \frac{1}{2}[5p^2 + 3p + 10]$ for $p > 2$ where p is a prime.

Proof: For any prime $p > 2$,

$V\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right) = \{u_1, u_2, \dots, u_{p-1}, x, y\} = \{2,4, \dots, p(p-1), x, y\}$, where $\mathrm{x} = 3$ and $z = 6 \in Z_9$. And $E\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right) = \{u_i x, u_i y, \mathrm{xy} \,/\, 1 \le \mathrm{i} \le \mathrm{p}-1\}$. Then $\left|V\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right)\right| = p + 1$ and $\left|E\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right)\right| = 2p - 1$. Then the corresponding coupling matrix $\mathfrak{C}$ is of the form

$$\mathfrak{C} = \begin{pmatrix} 3(J_3 - I_3) & 3J_{3\times p-1} \\ 3J_{p-1\times 3} & 5(J_{p-1} - I_{p-1}) \end{pmatrix}.$$

Therefore the coupling index of $\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)$ is,

$$\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right) = \frac{1}{2}\left[18p + 10\left(\frac{p^2-3p+2}{2}\right)\right]$$

$$\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right) = \frac{1}{2}[18p + 5p^2 - 15p + 10]$$

$$\mathfrak{C}\left(\Gamma(\overline{Z_{p^2}}) + \Gamma(Z_9)\right) = \frac{1}{2}[5p^2 + 3p + 10]$$

3. Coupling Index of Identity graph $\mathcal{J}(Z_p)$ of the group $(Z_p, +)$

VOL. 31, NO. 4,2023
10.48047/jocaaa.2023.31.04.70

3.1 Theorem: For even $p > 2$, $\mathfrak{C}(\mathcal{I}(Z_p, +))) = \frac{1}{2}[5p^2 - 11p + 6]$.

Proof: For the Identity graph $\mathcal{I}(Z_p)$ of the group $(Z_p, +)$,

$V(\mathcal{I}(Z_p, +)) = \{0,1, \dots, p-1\} = \{u_1, u_2, \dots, u_p\}$. Then the coupling matrix of $\mathcal{I}(Z_p, +)$ is of the form $\mathfrak{C} = \begin{pmatrix} 0 & 3J_{1\times p-1} \\ 3J_{p-1\times 1} & T_{p-1} \end{pmatrix}$, where

$$T = \begin{pmatrix} 3(J_2 - I_2) & 5J_2 & 5J_2 & \dots & 5J_2 \\ 5J_2 & 3(J_2 - I_2) & 5J_2 & \dots & 5J_2 \\ 5J_2 & 5J_2 & 3(J_2 - I_2) & \dots & 5J_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 5J_2 & 5J_2 & 5J_2 & \dots & 3(J_2 - I_2) \end{pmatrix}$$

Now,

$$\mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}\left[6\binom{3p-3}{2} + 10\left\{\binom{p-1}{2} - \binom{p-1}{2}\right\}\right]$$

$$\mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}\left[3(3p-3) + 10\left\{\frac{p^2 - 4p + 3}{2}\right\}\right]$$

$$\mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}[9p - 9 + 5p^2 - 20p + 15]$$

$$\Rightarrow \mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}[5p^2 - 11p + 6]$$

3.2 Theorem: For odd $p > 2$, $\mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}[5p^2 - 14p + 20]$.

Proof: For the Identity graph $\mathcal{I}(Z_p, +)$ of the group $(Z_p, +)$, $V(\mathcal{I}(Z_p, +)) = \{0,1, \dots, p-1\} = \{u_1, u_2, \dots, u_p\}$. The coupling matrix of $\mathcal{I}(Z_p, +)$ is of the form $\mathfrak{C} = \begin{pmatrix} 0 & 3J_{1\times p-1} \\ 3J_{p-1\times 1} & D_{p-1} \end{pmatrix}$, such that

10.48047/jocaaa.2023.31.04.70

$$D = \begin{pmatrix} 3(J_2 - I_2) & 5J_2 & 5J_2 & \dots & 5J_{2\times 1} \\ 5J_2 & 3(J_2 - I_2) & 5J_2 & \dots & 5J_{2\times 1} \\ 5J_2 & 5J_2 & 3(J_2 - I_2) & \dots & 5J_{2\times 1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 5J_{2\times 1} & 5J_{2\times 1} & 5J_{2\times 1} & \dots & 0 \end{pmatrix}$$

Now,

$$\mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}\left[6p + 10\left\{\binom{p-1}{2} - \binom{p-2}{2}\right\}\right]$$

$$\mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}\left[6p + 10\left\{\frac{p^2 - 4p + 4}{2}\right\}\right]$$

$$\mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}[6p + 5p^2 - 20p + 20]$$

$$\Rightarrow \mathfrak{C}(\mathcal{I}(Z_p, +)) = \frac{1}{2}[5p^2 - 14p + 20]$$

Conclusion:

Coupling Index of certain zero divisor graphs of ring and identity graphs of groups are evaluated in this present article. This method can be extended to any kind of graphs as well.

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