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Lucky Labeling of Some Graph Families

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Abstract

Let $f: V(G) \rightarrow N$ be a labeling of the vertices of a graph G by positive integers. Let $S(v)$ denote the sum of labels of the neighbors of the vertex v in G. If isolated vertex v of G, we put $S(v) = 0$. A Labeling f is lucky if $S(u) \neq S(v)$ for every pair of adjacent vertices u and v. The lucky number of a graph G, denoted by $\eta(G)$ is the least positive integers n such that G has a lucky labelling with integer $\{1, 2, \ldots, n\}$ as the set of labels. We determined the lucky and proper labeling of spliting graph of C_{2n}. Let $f: V(G) \rightarrow N$ be a labeling of the vertices of a graph G by positive integers. Define $S(v) = \sum_{u \in N(v)} f(u)$, as the sum of neighborhood of vertex v, where $N(v)$ denotes the open neighbourhood of $v \in V$. A labeling f is lucky if $S(u) \neq S(v)$ for every pair of adjacent vertices u and v. The lucky number of a graph G, denoted by $\eta(G)$, is the least positive integer k such that G has a lucky labeling with $\{1, 2, \ldots, k\}$ as the set of labels. A Lucky labeling is proper lucky labeling if the labeling is proper as well as lucky, that is, if u and v are adjacent in G, then $f(u) \neq f(v)$ and $S(u) \neq S(v)$. The Proper Lucky number of G is denoted by $\eta_p(G)$, is the least positive integer k such that G has a proper lucky labeling with $\{1, 2, \cdots, k\}$ as the set of labels.

Keywords: Labeling, Lucky Labeling, Splitting graph.

AMS Subject Classification: 05C78

1 Introduction

In the mathematical discipline of graph theory, a graph labeling is the assignment of labels, traditionally represented by integers, to edges or vertices of a graph. Formally, given a graph $G = (V, E)$, a vertex labeling is a function of V to a set of labels; a graph with such a function defined is called vertex-labeled graph. Likewise, an edge labeling is a function of E to a set of Labels. In this case, the graph is called edge-labeled graph.

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In essence, graph labeling provides a flexible and powerful framework for representing, analyzing and solving problems in various fields by assigning labels to the elements of a graph according to specific rules and contraints.

Let $f: V\ (G) \rightarrow N$ be a labeling of the vertices of a graph G by positive integers. Let $S(v)$ denote the sum of labels of the neighbors of the vertex v in G. If isolated vertex v of G, we put $S(v) = 0$. A Labeling f is lucky if $S(u)\ \neq S(v)$ for every pair of adjacent vertices u and v. The lucky number of a graph G denoted by $\eta(G)$, is the least positive integers n such that G has a lucky labeling with integer $\{1, 2, \ldots, n\}$ as the set of labels.

A lucky labeling is proper lucky labeling if the labeling f is proper as well as lucky, that is if u and v are adjacent in G then $f(u)\ \neq f(v)$ and $S(u) \neq S(v)$. The proper lucky number of G is denoted by $\eta_p(G)$, is the least positive integer k such that G has a proper lucky labeling with $\{1, 2, \ldots, n\}$ as the set of labels.

2 Lucky and Proper lucky labelings of spiltting graphs of Cycles

Theorem 2.1. Splitting graph of C_{2n}, n ≥ 2 is lucky with $\eta(C_{2n})\ =\ 2$.

Proof: Let $C_{2n}: v_1 v_2 \ldots v_n v_{n+1} \ldots v_{2n}$ be the cycle of length 2n. Let $S'(C_{2n})$ be the splitting graph of C_{2n}. Let v_i' be the added vertex corresponding to v_i in C_{2n} for $i = 1, 2, \ldots 2n$. Let V and E be the vertex and edge sets of $S'(C_{2n})$. Therefore $V = \{v_1 v_2 \ldots v_n v_{n+1}, v_1', v_2', \ldots, v_{2n}'\}$ and $E = \{v_1 v_2, v_2 v_3, \ldots, v_{2n-1} v_{2n}, v_{2n} v_1\}\ \cup$

$\{v_1' v_2, v_2' v_3, \ldots, v_{2n-1}' v_{2n}, v_{2n}' v_1\}\ \cup\ \{v_1' v_{2n}, v_2' v_1, \ldots, v_{2n}' v_{2n-1}\}$.

Define $f: V \rightarrow \{1, 2\}$ such that $f(v_{2i-1}) = 1, f(v_{2i}) = 2$ for $1 \leq i \leq n$, $f(v_i') = 2$ for

$1 \leq i \leq 2n$. Now for $i \neq n$,

$$S(v_{2i})\ =\ f(v_{2i-1})\ +\ f(v_{2i+1})\ +\ f(v_{2i-1}')\ +\ f(v_{2i+1}')$$

$$=\ 1\ +\ 1\ +\ 2\ +\ 2\ =\ 6$$

$$S(v_{2n})\ =\ f(v_{2n-1})\ +\ f(v_1)\ +\ f(v_{2n-1}')\ +\ f(v_i')$$

$$=\ 1\ +\ 1\ +\ 2\ +\ 2\ =\ 6$$

$$\therefore\ S(v_{2i})\ =\ 6,\ \ \forall i\ =\ 1, 2, \ldots, n$$

For $i \neq 1$,

$$S(v_{2i-1})\ =\ f(v_{2i-2})\ +\ f(v_{2i})\ +\ f(v_{2n-2}')\ +\ f(v_{2i}')$$

10.48047/jocaaa.2024.33.07.103

$= 2 + 2 + 2 + 2 = 8$

For $i = 1$,

$S(v_1) = f(v_{2n}) + f(v_2) + f(v'_{2n}) + f(v'_2)$

$= 2 + 2 + 2 + 2 = 8$

$\therefore\ s(v_{2i-1}) = 8, \forall i = 1, 2, \ldots, n - 1$

For $i = n$,

$S(v_{2i}) = f(v_1) + f(v_{2i-1})$

$= 1 + 1 = 2$

Also $S(v'_{2i}) = f(v_{2i-1}) + f(v_{2i+1})$

$= 1 + 1 = 2$

$S(v'_{2i-1}) = f(v_{2i-2}) + f(v_{2i})$

$= 2 + 2 = 4$

$S(v'_1) = f(v_2) + f(v_{2n})$

$= 2 + 2 = 4$

Thus, v_1 is adjacent to $v_{i-1}, v_{i+1}, v'_{i-1}$ and v'_{i+1} we have

$$S(v_i) = \begin{cases} 8 & if\ i\ is\ odd \\ 6 & if\ i\ is\ even \end{cases}$$

$$S(v'_i) = \begin{cases} 4 & if\ i\ is\ odd \\ 2 & if\ i\ is\ even \end{cases}$$

$\therefore$ $S'(C_{2n})$ is a lucky graph with $\eta(C_{2n}) = 2.$

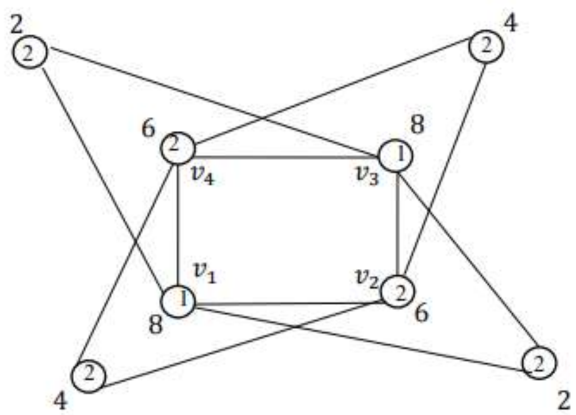

Figure 1. Lucky Labeling Splitting graph of C_4

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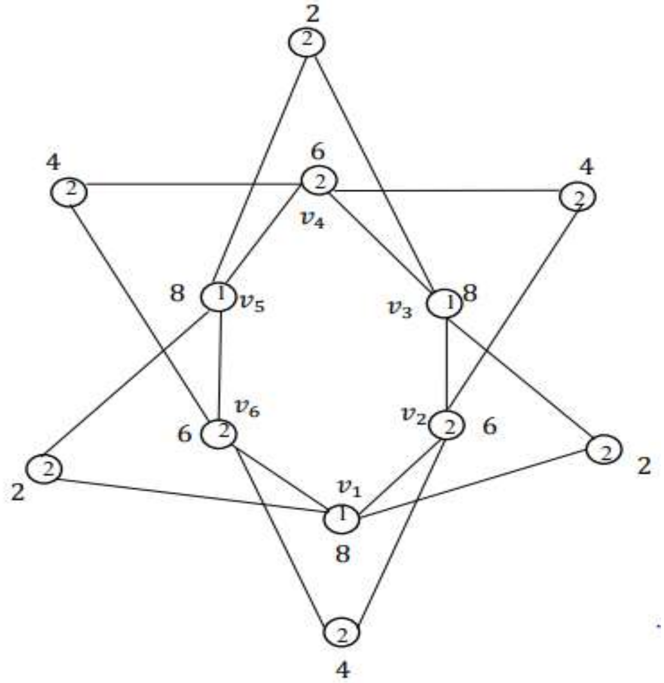

Figure 2. Lucky Labeling Splitting graph of C_6

Theorem 2.2. Splitting graph of $C_n, n \geq 3$ and n is odd, is lucky with $\eta(C_n) = 3$.

Proof: Let $C_n: v_1 v_2 \ldots v_n$ be the cycle of length n. Let $S'(C_n)$ be the splitting graph of C_n. Let v_n' be the added vertex corresponding to vi in C_n for $i = 1, 2, \ldots n$.

Let V and E be the vertex and edge sets of $S'(C_n)$.

Therefore $V = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\} \cup \{v_1'v_2, v_2'v_3, \ldots, v_{n-1}'v_n, v_n'v_1\} \cup \{v_1'v_n, v_2'v_1, \ldots, v_n'v_{n-1}\}$.

Case (i) $n \equiv 3$ (mod 6)

Define $f: V\ (S'(C_n)) \rightarrow \{1, 2, 3\}$ such that

$$f(v_i) = \begin{cases} 1 \ \ if\ i \ \equiv 1\ (mod\ 3) \\ 2 \ \ if\ i \ \equiv 2\ (mod\ 3), 1 \leq i \leq n \\ 3 \ \ if\ i \ \equiv 0\ (mod\ 3) \end{cases}$$

and $f(v_i') = 2$, for $1 \leq i \leq n$

Now $S(v_i) = f(v_{i-1}) + f(v_{i+1}) + f(v_{i-1}') + f(v_{i+1}')$

$$= \begin{cases} 3 + 2 + 2 + 2 = 9 \ \ if\ i \equiv 1\ (mod\ 3) \\ 1 + 3 + 2 + 2 = 8 \ \ if\ i \equiv 2\ (mod\ 3) \\ 2 + 1 + 2 + 2 = 7 \ \ if\ i \equiv 0\ (mod\ 3) \end{cases}$$

and $S(v_i') = f(v_{i-1}) + f(v_{i+1})$

$$= \begin{cases} 3 + 2 = 5 \ \ if\ i \equiv 1\ (mod\ 3) \\ 1 + 3 = 4 \ \ if\ i \equiv 2\ (mod\ 3) \\ 2 + 1 = 3 \ \ if\ i \equiv 0\ \ (mod\ 3) \end{cases}$$

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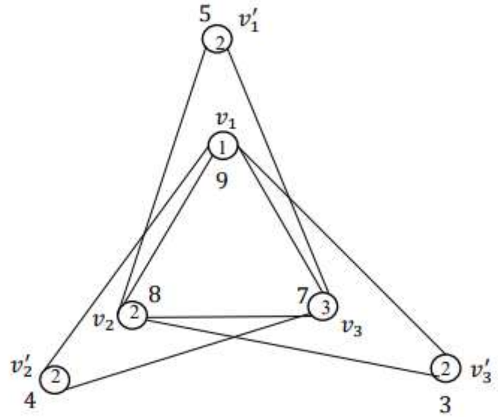

Figure 3. Lucky Labeling of $S'(C_3)$ with $\eta\big(S'(C_3)\big) = 3$

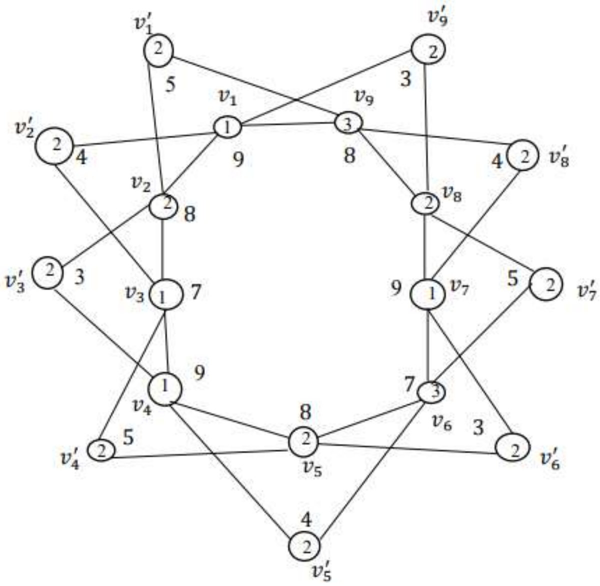

Figure 4. Lucky Labeling $S'(C_4)$ with $\eta(S'(C_4)) = 3$

Case (ii) $n \equiv 5 \pmod 6$

Define $f: V\ (S'(C_n)) \rightarrow \{1, 2, 3\}$ such that

$$f(v_i) = \begin{cases} 1 \ \ if\ i \ \equiv 1\ (mod\ 3) \\ 2 \ \ if\ i \ \equiv 2\ (mod\ 3), \\ 3 \ \ if\ i \ \equiv 0\ (mod\ 3) \end{cases} \quad 1 \le i \le n$$

and $$f(v_i') = \begin{cases} 1 \ \ if\ i \ \equiv 0\ (mod\ 3) \\ 2 \ \ if\ i \ \equiv 1\ (mod\ 3), \\ 3 \ \ if\ i \ \equiv 2\ (mod\ 3) \end{cases} \quad 1 \le i \le n$$

For $i = 1$,

$$S(v_i) = f(v_{i+1}) + f(v_n) + f(v_{i+1}') + f(v_n')$$
$$= 2 + 2 + 3 + 3 = 10$$

10.48047/jocaaa.2024.33.07.103

For $i = n$,

$S(v_i) = f(v_1) + f(v_{n-1}) + f(v_1') + f(v_{n-1}')$

$= 1 + 1 + 2 + 2 = 6$

For $i = 2, 3, \ldots, n-1$

$S(v_i) = f(v_{i-1}) + f(v_{i+1}) + f(v_{i-1}') + f(v_{i+1}')$

$$= \begin{cases} 1 + 2 + 2 + 1 = 7 & if\ i \equiv 2\ (mod\ 3) \\ 2 + 1 + 3 + 2 = 8 & if\ i \equiv 0\ (mod\ 3) \\ 3 + 2 + 1 + 3 = 9 & if\ i \equiv 1\ (mod\ 3) \end{cases}$$

For $i = 1$,

$S'(v_i') = f(v_{i+1}) + f(v_n)$

$= 2 + 2 = 4$

For $i = n$,

$S'(v_i) = f(v_1) + f(v_{n-1})$

$= 1 + 1 = 2$

For $i = 2, 3, \ldots, n-1$

$S'(v_i) = f(v_{i-1}) + f(v_{i+1})$

$$= \begin{cases} 1 + 3 = 4 & if\ i \equiv 2\ (mod\ 3) \\ 2 + 1 = 3 & if\ i \equiv 0\ (mod\ 3) \\ 3 + 2 = 5 & if\ i \equiv 1\ \ (mod\ 3) \end{cases}$$

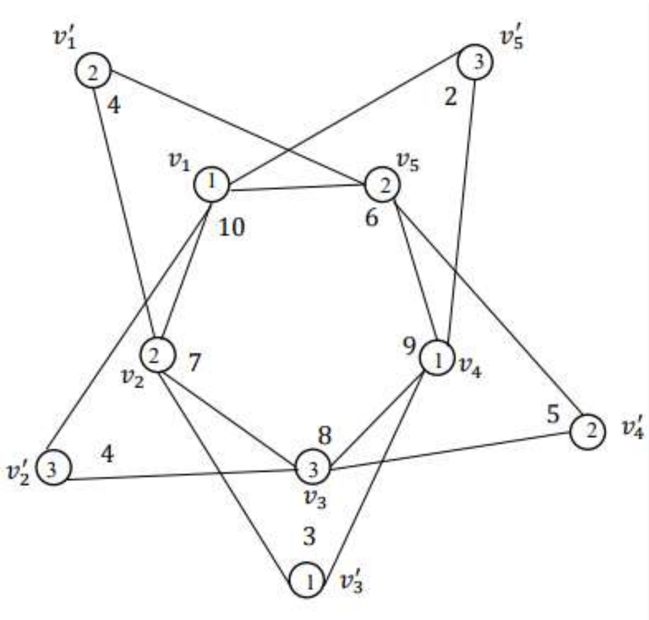

Figure 5. Lucky Labeling $S'(C_5)$ with $\eta(S'(C_5) = 3$

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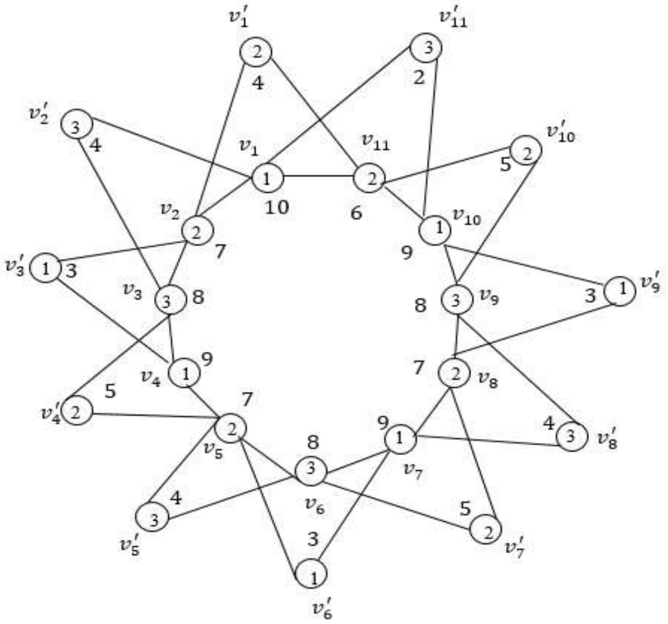

Figure 6. Lucky Labeling $S'(C_{11})$ with $\eta(S'(C_{11}) = 3$

Case (iii) $n \equiv 1$ (mod 6)

Define $f: V\ (S'(C_n)) \to \{1, 2, 3\}$ such that

$$f(v_i)\ = \begin{cases} 1\ \ if\ i\ \equiv 1\ (mod\ 3) \\ 2\ \ if\ i\ \equiv 2\ (mod\ 3), \qquad 1 \le i \le n \\ 3\ \ if\ i\ \equiv 0\ (mod\ 3) \end{cases}$$

and $f(v_i')\ = \begin{cases} 1\ \ if\ i\ \equiv 1\ (mod\ 3) \\ 2\ \ if\ i\ \equiv 2\ (mod\ 3), \quad 1 \le i \le n \\ 3\ \ if\ i\ \equiv 0\ (mod\ 3) \end{cases}$

For $i = 1,$

$$S(v_i)\ =\ f(v_{i+1})\ +\ f(v_n)\ +\ f(v_{i+1}')\ +\ f(v_n')$$

$$= 2 + 1 + 2 + 1 = 6$$

For $i = n,$

$$S(v_i)\ =\ f(v_1)\ +\ f(v_{n-1})\ +\ f(v_1')\ +\ f(v_{n=1}')$$

$$= 1 + 3 + 1 + 3 = 8$$

For $i = 2, 3, \dots, n\ -\ 1$

$$S(v_i)\ =\ f(v_{i-1})\ +\ f(v_{i+1})\ +\ f(v_{i-1}')\ +\ f(v_{i+1}')$$

$$= \begin{cases} 1 + 3 + 1 + 3 = 8\ \ \ if\ i \equiv 2\ (mod\ 3) \\ 2 + 1 + 2 + 1 = 6\ \ \ if\ i \equiv 0\ (mod\ 3) \\ 3 + 2 + 3 + 2 = 10\ \ \ if\ i \equiv 1\ (mod\ 3) \end{cases}$$

For $i = 1,$

$$S'(v_i)\ =\ f(v_{i+1})\ +\ f(v_n)$$

10.48047/jocaaa.2024.33.07.103

$= 2 + 1 = 3$

For $i = n$,

$S'(v_i) = f(v_1) + f(v_{n-1})$

$= 1 + 3 = 4$

For $i = 2, 3, \ldots, n - 1$

$S'(v_i) = f(v_{i-1}) + f(v_{i+1})$

$$= \begin{cases} 1 + 3 = 4 & if\ i \equiv 2\ (mod\ 3) \\ 2 + 1 = 3 & if\ i \equiv 0\ (mod\ 3) \\ 3 + 2 = 5 & if\ i \equiv 1\ (mod\ 3) \end{cases}$$

Hence $\eta(S'(C_n)) = 3$, if $n \geq 3$ and n is odd.

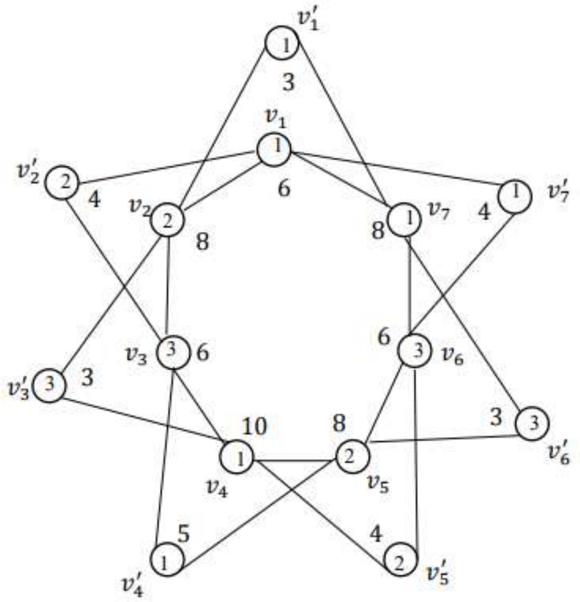

Figure 7. Lucky Labeling $S'(C_7)$ with $\eta(S'(C_7) = 3$

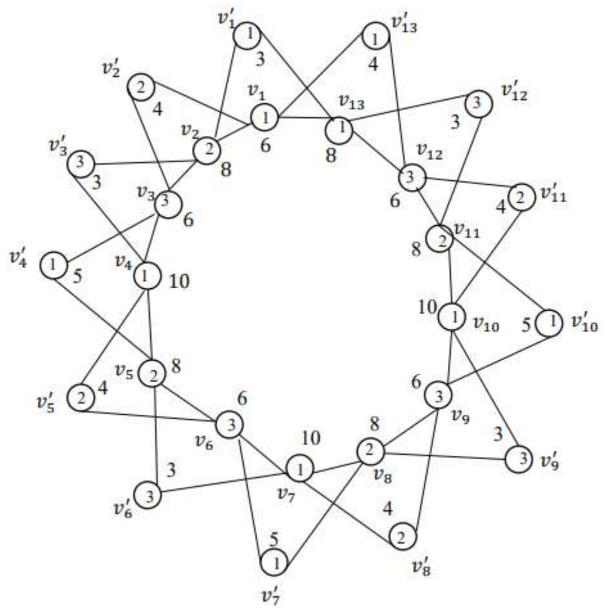

Figure 8. Lucky Labeling $S'(C_{13})$ with $\eta(S'(C_{13}) = 3$

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3 Lucky and Proper lucky labelings of bistar and its square graph

Definition 3.1. The bistar graph $B_{m,n}$ is the graph obtained from K_2 by joining n pendent edges to one end and m pendent edges to the other end of K_2. That is a bistar $B_{m,n}$ is the graph obtained by joining the centers of two stars $K_{1,m}$ and $K_{1,n}$ with an edge. It can be obtained that $B_{m,m} = B_{n,m}$.

Theorem 3.2. Bistar $B_{m,n}$ for $m, n \geq 1$ is a lucky graph with $\eta(B_{m,n}) = \begin{cases} 2 \ \ if\ m = n \\ 1\ if\ m \neq n \end{cases}$.

Proof: Let $K_2 = uv$ and construct the bistar by joining m pendent vertices $w_1, w_2, \dots, w_m$ to u and n pendent vertices $w_{m+1}, w_{m+2}, \dots, w_{m+n}$ to v. Then,

$V(B_{m,n}) = \{u, v, w_i: 1 \leq i \leq m+n\}$ and $E(B_{m,n}) = \{uv, uw_i, vw_j: 1 \leq i \ \leq m \ and$

$m+1 \ \leq i \ \leq \ m+n\}$.

We prove the theorem in two cases.

Case (i): $m = n$

Define $f: V(B_{m,m}) \to \{1, 2\}$ by

$f(u) = 1$

$f(v) = 1$

$$f(w_i) \ = \begin{cases} 1 & for \ \ 1 \leq i \ \leq m, \\ 2 & for \ \ m+1 \leq i \ \leq 2m \end{cases}$$

Let us find the neighborhood sum of each vertex.

$S(u) = m \ + \ 1$

$S(v) = 2m + 1$

$S(w_i) = 1$ for $1 \leq i \leq 2m$

Obviously $m + 1 \neq 2m + 1$ for all m.

Let S_l be the set of vertices for which the sum of the labels of the neighborhood vertices are l. Then,

$S_1 = \{w_i: 1 \leq i \leq 2m\}$

$S_{m+1} = \{u\}$

$S_{2m+1} = \{v\}$

It can be easily seen that the vertices in each of the sets S_1, S_{m+1} and S_{2m+1} are non adjacent. Hence f is a lucky labeling and $\eta(B_{m,m}) \ = \ 2$.

Case (ii): $m \neq n$

Define $f \ : V(B_{m,n}) \ \to \ 1$ by

10.48047/jocaaa.2024.33.07.103

$f(u) = 1$

$f(v) = 1$

$f(w_i) = 1$ for $1 \leq i \leq m + n$

Then,

$S(u) = m + 1$

$S(v) = n + 1$

$S(w_i) = 1$ for $1 \leq i \leq m + n$.

Obviously $1, m + 1$ and $n + 1$ are different integers.

$S_1 = \{w_i: 1 \leq i \leq 2m\}$

$S_{m+1} = \{u\}$

$S_{n+1} = \{v\}$

It can be easily seen that the vertices in each of the sets S_1, S_{m+1} and S_{n+1} are non adjacent. Hence f is a lucky labeling and $\eta(B_{m,m}) = 1$.

Example 3.3. Figure 9 illustrates a lucky labeling of the bistar $B_{3,3}$ with $\eta(B_{3,3}) = 2$ and Figure 10 illustrates a lucky labeling of the bistar $B_{4,3}$ with $\eta(B_{4,3}) = 1$.

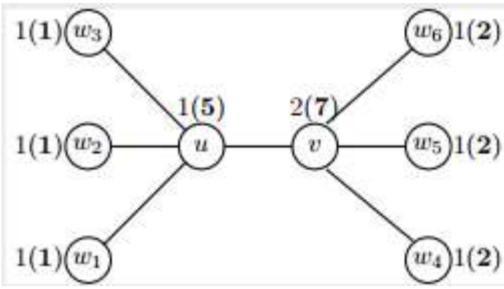

Figure 9.
Lucky Labeling $B_{3,3}$ with $\eta(B_{3,3}) = 2$

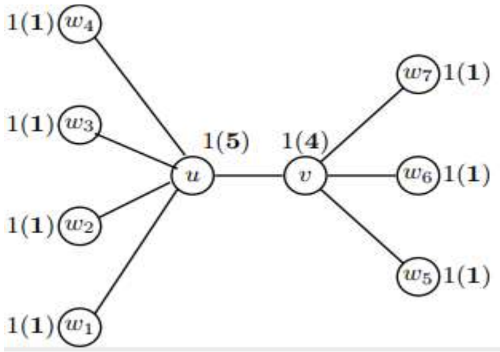

Figure 10.
Lucky Labeling $B_{4,3}$ with $\eta(B_{4,3}) = 1$

Theorem 3.4. Bistar $B_{m,n}$ for m, n $\geq$ 1 is a proper lucky graph with

$$\eta_p(B_{m,n}) = \begin{cases} 3 & if\ m = n = 1 \\ 2 & otherwise \end{cases}$$

Proof: We denote the vertices and edges of $B_{m,n}^2$ in the following way $V(B_{m,n}^2) = \{u, v, w_i: 1 \leq i \leq m + n\}$ and $E(B_{m,n}^2) = \{uv, uw_i, vw_i: 1 \leq i \leq m + n\}$.

We prove the theorem in three cases.

Case (i): $m = n \neq 1$

Define $f : V(B_{m,m}) \to \{1, 2\}$ by

10.48047/jocaaa.2024.33.07.103

$f(u) = 2$

$f(v) = 1$

$$f(w_i) = \begin{cases} 1 & for \ \ 1 \leq i \leq m, \\ 2 & for \ \ m+1 \leq i \leq 2m \end{cases}$$

Then,

$S(u) = m + 1$

$S(v) = 2m + 2$

$$S(w_i) = \begin{cases} 2 & for \ \ 1 \leq i \leq m, \\ 1 & for \ \ m+1 \leq i \leq 2m \end{cases}$$

Note that adjacent vertices have different labels. Let S_l be the set of vertices for which the sum of the labels of the neighborhood vertices are l. Then,

$S_1 = \{w_i: i + 1 \leq i \leq m + n\}$

$S_2 = \{w_i: 1 \leq i \leq m\}$

$S_{m+1} = \{u\}$

$S_{2m+2} = \{v\}$

It can be easily seen that the vertices in each of the sets S_1, S_{1m+} and $S_{2m+1}$are non adjacent. Hence f is a lucky labeling and $\eta_p(B_{m,n}) = 2$.

Case (ii): $m = n = 1$

Then the bistar $B_{1,1}$ is the path $P_4 = w_1 w_2 w_3$. Define $f: V(B_{1,1}) \rightarrow \{1, 2, 3\}$ by

$f(w_1) = 1$

$f(u) = 3$

$f(v) = 1$

$f(w_2) = 2$

Then,

$S(w_1) = 3$

$S(u) = 2$

$S(v) = 5$

$S(w_2) = 1$

Obivously f is a proper lucky labeling and $\eta_p(B_{1,1}) = 3$.

Case (iii): $m \neq n$

Let $m > n$. Define $f: V(B_{m,n}) \rightarrow \{1, 2\}$ by

$f(u) = 1$

10.48047/jocaaa.2024.33.07.103

$f(v) = 2$

$$f(w_i) = \begin{cases} 1 & for \ \ 1 \le i \le m, \\ 2 & for \ \ m+1 \le i \le m+n \end{cases}$$

Then,

$S(u) = 2(m + 1)$

$S(v) = n + 1$

$$S(w_i) = \begin{cases} 1 & for \ \ 1 \le i \le m, \\ 2 & for \ \ m+1 \le i \le m+n \end{cases}$$

Since $m > n, \ 2(m+1) \neq n+1$. It can be observed that adjacent vertices have distinct labels and any two adjacent vertices are having different neighborhood sum.

Hence f is a proper lucky labeling and $\eta p(B_{m,n}) = 2$.

Since $B_{m,n} = B_{n,m}$, the result follows if $m < n$ also.

Example 3.5. Figure 11 illustrates a proper lucky labeling of the bistar $B_{3,3}$ with $\eta_p(B_{3,3}) = 2$, Figure 12 illustrates a proper lucky labeling of the bistar $B_{1,1}$ with $\eta_p(B_{1,1}) = 3$ and Figure 12 illustrates a proper lucky labeling of the bistar $B_{4,3}$ with $\eta_p(B_{4,3}) = 2$.

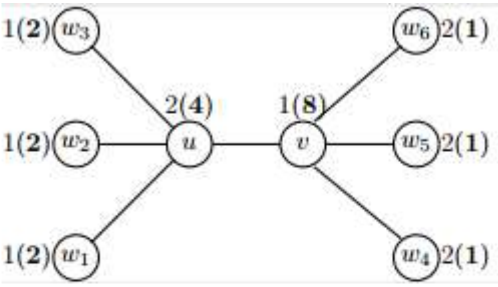

Figure 11
Proper Lucky Labeling $B_{3,3}$ with $\eta(B_{3,3}) = 2$

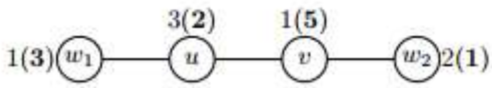

Figure 12
Proper Lucky Labeling $B_{1,1}$ with $\eta(B_{1,1}) = 3$

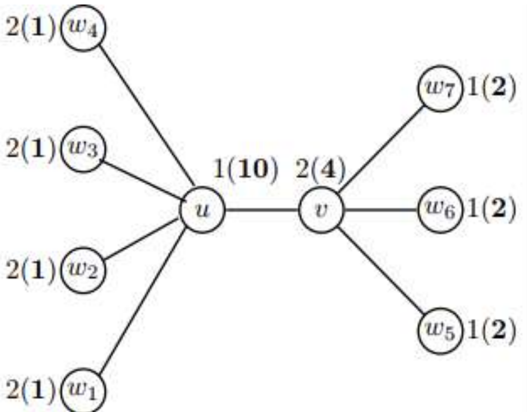

Figure 13
Proper Lucky Labeling $B_{4,3}$ with $\eta(B_{4,3}) = 1$

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Definition 3.6. For a simple connected graph G, the square of the graph G, denoted by G^2, is defined as the graph with the same vertex set as G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Theorem 3.7. The graph $B_{m,n}^2$ admits a lucky labeling for $m, n \geq 1$ and $\eta(B_{m,n}^2) = 2$.

Proof: We denote the vertices and edges of $B_{m,n}^2$ in the following way $V(B_{m,n}^2) = \{u, v, wi: 1 \leq i \leq m + n\}$ and $E(B_{m,n}^2) = \{uv, uwi, vwi: 1 \leq i \leq m + n\}$.

Case (i): $m = n$

Define $f: V(B_{m.m}) \rightarrow \{1, 2\}$ by

$f(u) = 2$

$f(v) = 1$

$$f(w_i) = \begin{cases} 1 & for \ \ 1 \leq i \leq m, \\ 2 & for \ \ m + 1 \leq i \leq 2m \end{cases}$$

Then,

$S(u) = 3m + 1$

$S(v) = 3m + 2$

$S(w_i) = 3$ for $1 \leq i \leq 2m$

$S(u), S(v)$ and S_{w_i} are all different for all values of m. Let S_1 be the set of vertices for which the sum of the labels of the neighborhood vertices are l. Then,

$S_3 = \{w_i: 1 \leq i \leq 2m\}$

$S_{3m+1} = \{u\}$

$S_{3m+2} = \{v\}$

It can be easily verified that the vertices in each of the sets S_3, S_{2m+1} and S_{2m+2} are non adjacent. Hence f is a lucky labeling and $\eta(B_{m,n}^2) = 2$.

Case (ii): $m \neq n$.

Let $m > n$. Define $f: V(B_{m,n}) \rightarrow \{1, 2\}$ by

$f(u) = 1$

$f(v) = 2$

$$f(w_i) = \begin{cases} 2 & for \ \ 1 \leq i \leq m, \\ 1 & for \ \ m + 1 \leq i \leq m + n \end{cases}$$

Now,

$S(u) = 2m + n + 2$

$S(v) = 2m + n + 1$

10.48047/jocaaa.2024.33.07.103

$S(w_i) = 3$ for $1 \leq i \leq m + n$

It can be observed that any two adjacent vertices are having different neighborhood sum. Hence f is a lucky labeling and $\eta(B^2_{m,n}) = 2$.

Since $B_{m,n} = B_{n,m}$, the result follows if $m <$ n also.

From all the above cases we have the graph $B^2_{m,n}$ admits a lucky labeling for $m, n \geq 1$ and $\eta(B^2_{m,n}) = 2$.

Example 3.8. Figure 14 illustrates a lucky labeling of the graph $B^2_{3,3}$ with $\eta(B^2_{3,3}) = 2$ and Figure 15 illustrates a proper lucky labeling of $B^2_{4,3}$ with $\eta(B^2_{4,3}) = 2$.

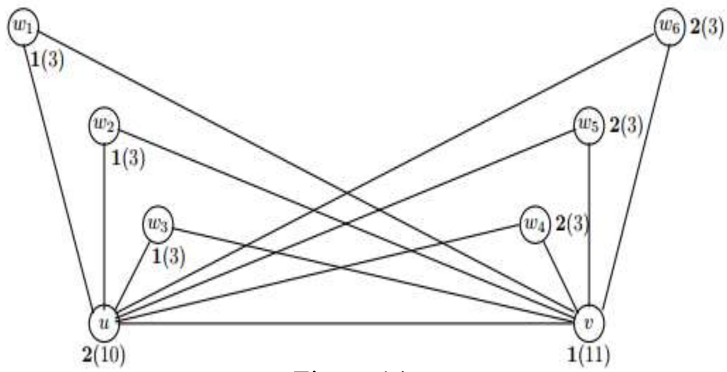

Figure 14
Proper Lucky Labeling $B^2_{3,3}$ with $\eta(B^2_{3,3}) = 2$

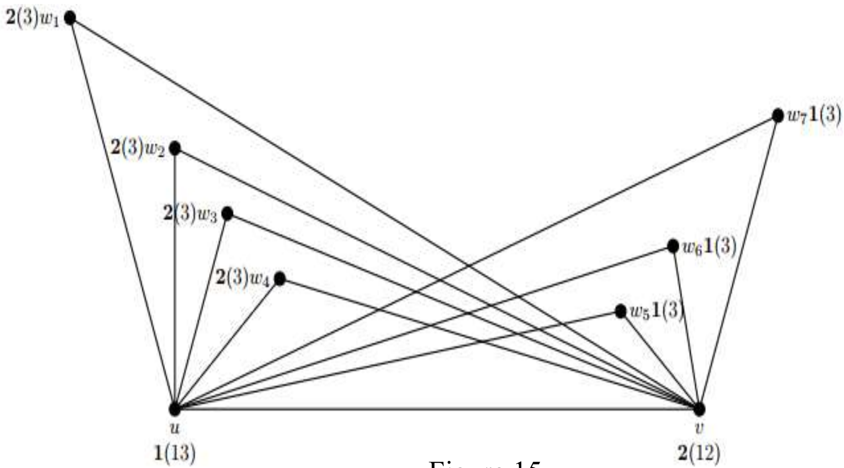

Figure 15
Proper Lucky Labeling $B^2_{4,3}$ with $\eta(B^2_{4,3}) = 2$

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