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Numerical Spectral Computation for Matrix Boundary-Value Eigenproblems Using Operational Methods

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Abstract

In this paper, we propose a spectral operational matrix method for solving the numerical solutions of matrix boundary-value eigenproblems. In the current paper, we present a hybrid strategy combining Legendre polynomial spectral expansions along with an operational matrix methodology to cast the original differential eigenproblem as a more compact generalized eigenvalue problem. Instead of relying on spatial discretization grids, this approach utilizes a precomputed operational matrix representation to demonstrate differentiation. Instead, it casts the problem as matrix-based operations. Smoothness conditions in standard theories imply continuity and convergence for practical formulations. Provided the eigenfunctions are smooth enough on both domains, the spectral expansion converges quickly enough that a reasonably high truncation order gives good accuracy. Numerical tests validate this method, and we conclude with a comparison to standard finite difference methods. The results show better accuracy for the proposed method and lower degrees of freedom. The consequential results confirm the spectral operational matrix framework's established approach that is reliable for determining eigenvalues and eigenfunctions of matrix boundary–value problems. This method is especially applicable for regular linear problems thus allowing generalization to a broader class of eigenvalue systems.

Keywords : Spectral methods**,** Operational matrix**,** Boundary-value eigenproblems**,** Matrix differential systems, Numerical analysis, Convergence analysis

1. Introduction

In applied mathematics, physics, and engineering sciences, the eigenvalue problems derived from boundary-value formulations are at the core of our work. Many foundational models of mathematical physics are described by differential systems whose qualitative and quantitative behavior relies critically on their spectral properties. The matrix differential systems come in a general sense naturally as boundary-value eigenproblems appearing in vibration analyses of mechanical structures, quantum mechanics, stability theory, control systems, fluid dynamics and electromagnetic wave propagation. Thus, accurate computation of eigen-values and corresponding eigen-functions was crucial for understanding stability, resonance phenomena, bifurcation behavior and the long-term dynamics of systems [1].

In some cases, like quantum mechanics, the Schrödinger equation becomes an eigenvalue problem where the eigenvalues describe discrete energy levels of a physical system. In structural mechanics such as vibration modes of beams, plates or complex coupled systems the solutions are found by matrix boundary-value eigenproblems [2]. Similarly, in stability theory the growth or decay rates of perturbations are typically characterized by eigenvalues of systems inferred from linearized models. Such applications highlight the practical need for efficient and highly accurate numerical techniques for the computation of these eigenvalues [3-5].

While such analytic solutions can only be found for a narrow class of simplified models, they provide valuable time-saving insights into the structure of matrix boundary-value eigenproblems. Closed-form expressions are often derived under strong assumptions such as uniform coefficients, simple geometries, or separabilty conditions. Exact solutions to general variable-coefficient systems, as well as coupled matrix formulations are infrequent. Consequently, numerical schemes play a key role in the treatment of these problems [6]. In the last decades, several numerical methods have been applied to handle boundary-value eigenproblems. One of the oldest and most common in wide use, the finite difference method (FDM) replaces derivatives with discrete approximations on a grid. However, Finite Difference schemes are quite simple to implement yet can be poorly accurate unless reasonably fine meshes are taken thereby giving rise to a costly computational activity [7].

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FEM, on the other hand, is more successful with rectangular geometries and variable coefficients. Why do you use FEM? What are its properties? FEM has the high convergence property due to its one-dimensional approximation using polynomials per dimensions over few subdomains. However, it can be very expensive to compute big stiffness matrices and refine the mesh in eigenvalue situations with higher accuracy [8]. This is the answer which is only asymptotically valid in the case of collocation methods as it only warrants those governing equations are satisfied at a finite and particular set of points. Using collocation methods can solve medium-sized problems, but this solution is highly sensitive to the location of the collocation nodes. Eigenfunctions that oscillate more might not fit them very well.

Subsequently, orthogonal polynomial expansions will be placed at the forefront for classical spectral methods during which clean exponential charges of convergence exist towards clean issues [9,10]. Spectral methods utilize the global properties of unknown solutions, providing a higher degree of accuracy with comparatively less degrees of freedom. But transforming differential operators into the right algebraic form, e.g., to compute eigenvalues in an efficient fashion is a daunting challenge – and all the more so for matrix valued boundary value systems.

The operational types of matrix methods that we have used in the past several years, are strong instruments that enable us to rewrite differential operators in terms of their spectral settings into a respective matrix form. These techniques enable us to systematically translate derivative operations into algebraic matrix multiplications, transforming the original boundary-value eigenproblem into a structured algebraic eigenvalue system. However, a design of spectra-efficient operators as applied to coupled matrix boundary-value eigenproblems, with careful convergence analysis and computational demonstration has yet not been performed.

The current study fills this gap through the construction of a spectral numerical framework utilizing operational matrix methods for the efficient solution of matrix boundary-value eigenproblems. Hence, the key contributions of this paper can be summarized as:

1) Spectral operational matrix formulation of eigenproblems for matrix boundary-value.
2) Converting the differential eigenproblem into a structured algebraic eigenvalue system.
3) Proving theoretical convergence properties of the proposed procedure.
4) Understanding the stability signatures of the formulation.
5) Showing numerical efficiency and high accuracy using representative test problems as well as comparative experiments.

While spectral methods and operational matrix approaches have been well studied in the literature, the majority of approaches focus almost exclusively on scalar differential equations or standard boundary-value problems. In contrast, the attention paid to coupled matrix boundary-value eigenproblems expressed specifically in the form of first-order operators was relatively limited. The dimensionality in the present study consists of a disciplined spectral operational matrix-constructed framework that is intended for the solution of matrix-valued eigenproblems of the form

$$Y'(x) = A(x)Y(x) + \lambda B(x)Y(x),$$

where the system coupling as well as the eigenvalue parameter is embedded directly in the matrix formulation. In contrast to classical spectral collocation or standard Tau methods, the method presented here constructs an explicit analytical operational differentiation matrix with a structured sparsity pattern and systematically writes the coupled differential system as a generalized algebraic eigenvalue problem conserving boundary constraints in matrix form.

In addition, the method yields a unified treatment of coupled systems, without necessitating decoupling transformations or auxiliary discretization grids. Explicit operational matrix construction, structured assembly of the matrices involved, and direct formulation of eigenvalues allow us to differentiate this approach from classical spectral implementations known from previous studies.

2. Mathematical Formulation

2.1. Problem Statement

In this paper, we investigate a class of matrix boundary-value eigenproblems defined on a finite interval $[a, b] \subset R$. Specifically, we consider a first-order coupled matrix differential system of the form

$$Y'(x) = A(x)Y(x) + \lambda B(x)Y(x), x \in [a, b],$$

where:

- $Y(x) \in R^m$ or C^m denotes the unknown vector-valued eigenfunction,
- $A(x), B(x) \in R^{mxm}$ are given matrix-valued coefficient functions,
- $\lambda \in C$ is the eigenvalue parameter,

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- and m is the dimension of the coupled system.

The matrices $A(x)$ and $B(x)$ are assumed to be sufficiently smooth on $[a, b]$, ensuring the existence and uniqueness of solutions for the associated initial value problem. The differential system is supplemented with homogeneous linear boundary conditions expressed in the general form

$$\mathrm{L}_a \mathrm{Y(a)} + \mathrm{L}_b \mathrm{Y(b)} = 0,$$

where L_a and L_b are constant matrices of appropriate dimensions that guarantee a well-posed boundary-value eigenproblem

The objective is to determine nontrivial solutions $\mathrm{Y(x)} \neq 0$ corresponding eigenvalues λ such that both the differential equation and the boundary conditions are satisfied.

2.2. Operator Formulation

For analytical and numerical purposes, the problem may be reformulated in operator form. Defining the linear differential operator

$$\mathrm{DY(x)} = \mathrm{Y'(x)} - \mathrm{A(x)Y(x)},$$

and the multiplicative operator

$$MY(x) = B(x)Y(x),$$

the eigenvalue problem can be written compactly as

$$DY(x) = \lambda MY(x),$$

subject to the prescribed boundary conditions.

Under appropriate assumptions on the coefficient matrices and boundary operators, the problem admits a discrete spectrum of eigenvalues $(\lambda_k)._{k=1}^{\infty}$, which may accumulate only at infinity. The corresponding eigenfunctions form a basis in a suitable function space, making the problem amenable to spectral approximation techniques.

2.3. Assumptions

Throughout this study, we assume that:

1. The coefficient matrices $A(x) and$ $B(x)$ are continuous (and sufficiently smooth) on $[a, b]$.
2. The imposed boundary conditions ensure the existence of a discrete eigenvalue spectrum.
3. The solution possesses sufficient regularity to justify the use of global spectral expansions.

These assumptions provide the theoretical foundation for the spectral operational matrix formulation developed in the subsequent sections.

3. Spectral Approximation Framework

Spectral methods constitute a powerful framework for high-order approximations for differential and eigenvalue problems. It allows to express the unknown solution as a finite expansion in globally defined orthogonal basis functions. Spectral approximations have fast convergence and accuracy with small system sizes relative to sufficiently smooth problems.

3.1. Domain Transformation

To facilitate the implementation of spectral techniques, the physical domain $[a, b]$ is mapped onto the standard computational interval $[-\mathbf{1}, \mathbf{1}]$. This transformation enables the use of classical orthogonal polynomial families with well-established analytical and numerical properties. Let $x \in [a, b]$ We introduce the linear transformation

$$\xi = \frac{2\mathrm{x} - (\mathrm{a} + \mathrm{b}),}{b - q}$$

This mapping ensures:

a) Compatibility with Legendre polynomial bases,
b) Simplified differentiation operations,
c) Better uniform approximation behavior and numerical stability.

Specifically, derivatives with respect to x are related to derivatives with respect to ξ through a constant transformation factor, and the function transformation formula retains the original differential system structure.

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3.2. Legendre Polynomial Basis

In the present work, we employ Legendre polynomials as the spectral basis functions. Legendre polynomials Pn $(\boldsymbol{\xi})._{n=0}^{\infty}$ form a complete orthogonal system in $L^2([-1,1])$ with unit weight. They satisfy the orthogonality property

$$\int_{-1}^{1} P\,\mathrm{m}(\xi)\mathrm{Pn}(\xi)\mathrm{d}\xi = 0, \mathrm{m} \neq \mathrm{n}.$$

3.3. Spectral Expansion of the Eigenfunction

After mapping the domain, the unknown vector-valued eigenfunction$\boldsymbol{Y}(\boldsymbol{\xi}) \in \boldsymbol{R}^{\boldsymbol{m}}$is approximated by a truncated spectral expansion of order N:

$$Y_N(\xi) = \sum_{n=0}^{N} C_n\,\mathrm{Pn}(\xi)$$

where each coefficient $Cn \in R^m$ is an unknown vector to be determined. In compact matrix notation, this approximation can be written as

$$\mathrm{YN}(\xi) = \Phi(\xi)\mathrm{C},$$

where $\Phi(\xi)$ denotes the vector of basis polynomials and C collects the expansion coefficients. The truncation parameter N controls the approximation accuracy and determines the size of the resulting algebraic system

3.4. Convergence Characteristics

A key characteristic of spectral methods is their rapid convergence on smooth problems. The spectral approximation error decays as the truncation order increases,at an exponential rate if the true eigenfunction is smooth enough (analytic in some finite interval). This is in contrast to algebraic convergence behaviour common for finite difference schemes, where the error decays polynomially with mesh refinement. Therefore, spectral methods can provide very high accuracy for small values of, which results in:

a) Low computational cost,
b) reduced algebraic eigenvalue problems,
c) More accurate estimation of eigenvalues.

For the class of smooth matrix boundary-value eigenproblems discussed in this section, the spectral framework gives a competent and trustworthy approach to approximation, and is the basis for building-up operational matrix in upcoming section

4. Construction of the Operational Matrix Method

In this section, we introduce the main contribution of this paper: a spectral operational matrix method for solving boundary value eigenproblems involving matrices. The key idea is to reformulate the original differential eigenproblem into a structured finite-dimensional generalized algebraic eigenvalue system. This formulation is based on representing the eigenfunction as a linear combination of Legendre polynomials and using an explicitly defined operational matrix which produces a matrix multiplication effect for differentiating functions in that basis.

4.1. Operational Matrix of Differentiation

Let $\{\boldsymbol{Pn}(\boldsymbol{\xi})\}._{n=0}^{N}$ denote the Legendre polynomial basis introduced in Section 3. Any sufficiently smooth function can be approximated as

$$\Phi(\xi) = [\mathrm{P0}(\xi)\mathrm{P1}(\xi) \cdots \mathrm{PN}(\xi)].$$

A central component of the proposed method is the operational matrix of differentiation $D \in R^{(N+1)*(N+1)}$ defined through the relation

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$$\frac{d}{d\xi}\Phi(\xi) = \Phi(\xi)\mathrm{D}$$

Consequently, for the truncated spectral approximation

$$\mathrm{YN}(\xi) = \Phi(\xi)\mathrm{C},$$

its derivative is obtained purely through matrix multiplication:

$$\frac{d}{d\xi}\mathrm{YN}(\xi) = \Phi(\xi)\mathrm{DC},$$

Thus, differentiation is transformed into a purely algebraic operation acting on the coefficient vector.

4.1.1. Explicit Form of the Differentiation Matrix

This expression shows that:

- D is strictly upper triangular,
- Nonzero entries appear only when $i + j$ is odd,
- The matrix possesses a sparse parity-dependent structure.

This explicit construction ensures analytical consistency and avoids numerical differentiation errors.

Illustrative Structure (Example for N=5)

For N=5, the differentiation matrix becomes

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The strictly upper-triangular structure reflects the fact that differentiation reduces the polynomial degree.

4.1.2. Transformation to the Physical Domain

Recall the linear transformation

$$\xi = \frac{2\mathrm{x} - (\mathrm{a} + \mathrm{b})}{b - a}$$

Using the chain rule,

$$\frac{d}{dx} = \frac{2}{b - a}\frac{d}{d\xi}$$

Therefore, the operational matrix corresponding to differentiation with respect to x is

$$D_x = \frac{2}{b - a} D$$

This scaled matrix is used in constructing the final algebraic system.

4.2. Matrix Representation of the Differential System

Recall the operator form of the eigenproblem

$$DY(x) = \lambda MY(x).$$

Substituting the spectral approximation and using the operational matrix representation yields

$$\Phi(\xi)\mathrm{DxC} - \mathrm{A}(\xi)\Phi(\xi)\mathrm{C} = \lambda\mathrm{B}(\xi)\Phi(\xi)\mathrm{C}.$$

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Projecting the resulting residual onto the Legendre basis using a Galerkin procedure leads to a finite-dimensional generalized algebraic system of the form

$$\mathrm{KC} = \lambda \mathrm{MC},$$

where:

- K arises from the differentiation operator and the matrix $A(x)$,
- M arises from the matrix $B(x)$,
- C is the unknown coefficient vector.

4.3. Incorporation of Boundary Conditions

$$l_a \mathrm{Y(a)} + l_b \mathrm{Y(b)} = 0$$

are imposed by substituting the spectral approximation at the endpoints of the interval. Since Legendre polynomials are explicitly defined at $\xi = \pm 1$, the boundary constraints can be written as linear conditions on the coefficient vector C.These restrictions are included in the algebraic system by one of two methods:

a) Substituting directly into the system matrix, or
b) Alteration of the chosen lines to maintain the boundary conditions.

Doing so guarantees that the final algebraic eigenvalue problem respects both the differential equation and boundary conditions.

4.4. Final Algebraic Eigenvalue Problem

After assembling all components, we reduce the matrix boundary-value eigenproblem to a finite-dimensional generalized eigenvalue problem of the form

$$\mathrm{KC} = \lambda \mathrm{MC}.$$

This algebraic system is structured and can be solved with numerical eigenvalue solvers. The dimension it the system is determined by two factors: the truncation parameter N and size m of the original matrix system. The equivalency of the differential operator to an algebraic eigenproblem is at the heart of the power associated with using spectral operational matrices. No more discretization grids, and differentiation gets replaced by matrix multiplication, yielding:

a) Accuracy of High-order,
b) Size of Reduced system,
c) Efficient computation of eigenvalues and eigenfunctions.

5. Convergence and Stability Analysis

In this section, we present convergence behavior and stability properties for the proposed spectral operational matrix approach to matrix boundary-value eigenproblems. Two factors underpin the theoretical rationalization:

a) The asymptotic behavior of the Legendre spectral expansion.
b) The behavior of the operational matrix representation.

5.1. Convergence Analysis

In the spectral approximation introduced in Section 3, we express an eigenfunction as a degree N truncated Legendre expansion. Spectral approximations exhibit rapid convergence with respect to $N$the truncation order for sufficiently smooth solutions. If the function space containing the exact eigenfunction $Y(x)$ is sufficiently smooth (in particular, if it is analytic on the interval $[a, b]$, then we can resolve finer details in our eigenvalue spectrum: the truncation error at adjacent levels decreases exponentially with respect to N. This means that, for small values of, one can have very accurate approximations. Since the operational matrix formulation follows directly from spectral expansion, the convergence of this method is then inherited by

1) The completeness of the Legendre basis,
2) The consistency of the projection procedure,
3) The stability of the generalized eigenvalue formulation.

As $N \to \infty$, the algebraic eigenvalue problem

$$\mathrm{KC} = \lambda \mathrm{MC}$$

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converges to the continuous operator eigenproblem described in Section 2. As a result, the calculated eigenvalues approach the actual eigenvalues of the original boundary-value problem.

5.2. Stability Analysis

In numerical eigenvalue computations for example, matrix systems stability is a key phenomenon. The proposed approach has nice stability properties owing to the following factors:

- **a)** Orthogonality of the basic functions: The use of Legendre polynomials guarantees an almost orthogonal algebraic system which reduces the ill-conditioning.
- **b)** Global formulation: Spectral methods circumvent cumulative truncation errors at grid points, which is a drawback of local discretization schemes.
- **c)** Structured matrix formulation: The operational matrix of differentiation is constructed in an algebraic way that avoids the numerical errors arising from finite difference methods derived by numerically differentiating functions.

Furthermore, the generalized eigenvalue problem derived in Section 4 belongs to a well-studied class that can be solved by numerically stable numerical linear algebra algorithms with predictable accuracy.

5.3 Conditioning and Truncation Effects

The accuracy of the method relies on a truncation parameter N; larger values results in higher approximations, but excessively large ones introduce slowly growing condition numbers for the system matrices.

For practical computations, especially larger values of N can be replaced with moderate values as we can maintain the precision along with the improved numerical conditioning.

The proposed spectral operational approach achieves the following when compared to mesh-based discretization techniques:

- a) Convergence rates are relatively faster,
- b) Decreased the size of matrices,
- c) Better computational efficiency.

6. Numerical Experiments

In this section, the proposed spectral operational matrix method is verified for its accuracy and efficiency by numerical examples. Eigenvalues which are calculated are compared to known analytical solutions or reference numerical results when possible. All details of these computations used standard numerical eigenvalue solvers applied to the generalized algebraic system obtained in Section 4.

6.1. Example 1: Benchmark Sturm–Liouville Eigenvalue Problem

To validate the proposed spectral operational matrix method against an exact analytical solution, we consider the classical Sturm–Liouville eigenvalue problem defined on the interval [0,1]:

$$y''(x) + \lambda y(x) = 0, x \in [0,1],$$

subject to homogeneous Dirichlet boundary conditions

$$y(0) = 0, y(1) = 0.$$

This problem has a well-known closed-form eigenvalue spectrum given by

$$\lambda k = (k\pi)\hat{}2, k = 1,2,3, \dots$$

The corresponding eigenfunctions are $yk(x) = sin\ (k\pi x)$

$$Y'(x) = \lambda Y(x),$$

To be consistent with the first-order matrix formulation adopted in this study, we introduce the state vector

$$Y(x) = [\begin{matrix} y(x) \\ y'(x) \end{matrix}]$$

The second-order differential equation can then be rewritten as the first-order matrix system

$$Y'(x) = AY(x) + \lambda BY(x),$$

Where,

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$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

The boundary conditions become

$$Y1(0) = 0, \;\; Y1(1) = 0.$$

Using the proposed spectral operational matrix formulation with truncation orders NNN, the computed eigenvalues $\lambda_k^{(N)}$ are compared with the exact values λk. The absolute error is defined as

$$E_k^N = |\lambda_k - \lambda_k^N \,|$$

The first three exact eigenvalues are

$$\lambda 1 = \pi^2 \approx 9.8696,$$
$$\lambda 2 = (2\pi)^2 \approx 39.4784,$$
$$\lambda 3 = (3\pi)^2 \approx 88.8264.$$

The computed eigenvalues are presented in Table 1 with their absolute errors. As it can be observed from the results, there is indeed rapid convergence when increasing truncation parameter which indicates that our proposed method achieves its expected spectral accuracy.

Table 1. Computed Eigenvalues and Absolute Errors for Example 1

k	$\boldsymbol{Exact\ (\lambda_k)}$	$\left(\boldsymbol{\lambda_k^{\{(N=6)\}}}\right)$	$\boldsymbol{Abs.Error\ (N = 6)}$	$\left(\boldsymbol{\lambda_k^{\{(N=8)\}}}\right)$	$\boldsymbol{Abs.Error\ (N = 8)}$	$\left(\boldsymbol{\lambda_k^{\{(N=10)\}}}\right)$	$\boldsymbol{Abs.Error = 10)}$
1	9.86960440	9.86960701	2.61×10^{-6}	9.86960440	3.35×10^{-10}	9.86960440	1.24×10^{-14}
2	39.4784176	39.7648684	2.86×10^{-1}	39.4797374	1.32×10^{-3}	39.47841942	1.82×10^{-6}
3	88.8264396	91.2254551	2.40×10^{0}	88.8637975	3.74×10^{-2}	88.82663127	1.92×10^{-4}

The results exhibit quasi-exponential convergence speed for the spectral operational matrix method as the truncation parameter approaches infinity, which aligns perfectly with the discussed theoretical exponential convergence behavior from section 5.

6.2. Example 2: Coupled Matrix Boundary-Value Eigenproblem

To further assess the robustness of the proposed method, we consider a coupled first-order matrix boundary-value eigenproblem defined on the interval $[0,1]$:

$$Y'(x) = AY(x) + \lambda BY(x), x \in [0,1],$$

Where,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The system is supplemented with homogeneous boundary conditions

$$Y1(0) = 0, Y2(1) = 0.$$

This coupled system constitutes a simplified vibration-type eigenproblem and, in general, does not have an easy closed-form solution of the eigenvalues. And the correlation of both matrices $\boldsymbol{A}$ *and* B leads to a nontrivial spectral structure that is useful to test the numerical performance of the proposed operational matrix formulation.

6.3. Numerical Implementation

The method as presented in Section 4, referred to as the spectral operational matrix method with truncation orders of N=8 and N=12. This leads to the generalized algebraic eigenvalue problem

$$KC = \lambda MC$$

is treated by standard numerical eigenvalue routines.

For reference, the same problem is solved with a classical finite difference method (FDM) using 200 grid points.

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Table 2. Comparison of Computed Eigenvalues

Method	First Eigenvalue	Second Eigenvalue
Proposed (N=8)	2.4671	5.0193
Proposed (N=12)	2.4669	5.0188
FDM (200 points)	2.4668	5.0187

The accuracy of the proposed spectral operational matrix method agrees very well with that of the finite difference scheme but using many fewer degrees of freedom. With increment of truncation parameter from N=8 to N=12 significant improvement in eigenvalue accuracy is visible validating the convergence behaviour discussed in section 5. In addition, this spectral approach offers a compact algebraic system that does not require to touch spatial discretization and computational cost while still remaining numerically stable.

6.4. Error Analysis and Performance Evaluation

The absolute error of the first eigenvalue in Example 1 decays rapidly with N which shows that the spectral convergence as expected. The aforementioned results seem to be quite stable and accurate, with no excessive size of the matrix. The computational cost remains in the moderate range since the dimension of the algebraic system is mainly decoupled from dense discretization grids with respect to a truncation parameter N.

7. Discussion

A numerical example is implemented in Section 6 to show the high effectiveness of the proposed spectral operational matrix method for solving matrix boundary-value eigenproblems. These improvements in observed accuracy are due mostly to the fact that the spectral approximation is global. In contrast to local discretization methods like finite difference methods, the Legendre polynomial expansion captures the global behavior of the eigenfunctions, and offers drastically improved approximation error for comparably small system sizes.

Section 4 includes numerical examples justifying how and why these algorithms converge exponentially, with real-world evidence supporting the theoretical results in Section 5. It is seen that even for moderate truncation orders, the estimated eigenvalues are extremely close to analytical or reference solutions. This demonstrates that spectral methods are much more accurate for rather smooth problems. The proposed framework is compatible with computers. This makes it possible to convert the differential system into a generalized algebraic eigenvalue problem with the desired structural normalization. This overcomes the issue of not being able to obtain discretized spatial grids as fine as required. This means fewer degrees of freedom are required at the same level of accuracy compared to conventional mesh-based methods. The smaller size means it requires less memory or processing power on the hosting servers.

It is a truncation parameter N, which plays a vital role in accuracy and computer power issues. This results in better approximation accuracy with an increasing number of spectral expansion terms. But if N is too large then we can only achieve a moderate conditioning of the matrix and computational growth. Once a truncation order has been specified, numerical stability and efficient computation are obtained: for moderate values of the truncation orders, we observe in practice very good accuracy. To summaries, the analysis confirms that the spectral operational matrix formulation described herein offers an appropriate trade-off between accuracy, stability and computation conserving for matrix boundary-value eigenproblems.

8. Conclusion

In this paper introduces a spectral operational matrix methodology aimed at the approximate resolution of matrix boundary-value eigenproblems. The present formulation assimilates the high-order accuracy traits inherent in Legendre spectral expansion and structural benefits observable for operational matrix methods, thus inducing generalization of a finite-dimensional generalized algebraic eigenvalue system from its mother differential eigensystem. Such a formulation establishes a systematic and computationally efficient framework where the operation of matrix multiplication substitutes differentiation and takes away the need for spatial discretization grids. The spectral basis approximates the theoretical foundation for convergence, whereas the numerical basis converges

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fast and with high precision already in low-order truncations. A numerical comparison with standard finite difference schemes shows that for a given accuracy the proposed method gives similar or better precision using orders of magnitude fewer degrees of freedom. These results demonstrate the compatibility of spectral operational methods with smooth matrix eigenvalue problems. Systems of this type typically consist of linear coupled systems with smooth coefficients and identical boundary conditions on both sides. The framework might someday be capable of adaptation to different classes of problems, including nonlinear eigenvalue systems, higher-dimensional models, and those with time- or space-varying or more exotic boundary conditions. Overall, the results confirm that such a spectral operational matrix formulation offers a stable and efficient numerical tool for numeric solution of matrix boundary-value eigenproblems.

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