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BOUNDS OF SOME RADIO CONTRA HARMONIC MEAN GRAPHS

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Abstract

Let G be a simple graph with p vertices and q edges. For a connected graph G, the radio contra harmonic mean labeling is defined as an injective function $f: V(G) \to Z^+$ where $d(u,v) + \left\lceil \frac{(f(u))^2 + (f(v))^2}{f(u)+f(v)} \right\rceil \geq 1 + diam(G)$ or $d(u,v) + \left\lfloor \frac{(f(u))^2 + (f(v))^2}{f(u)+f(v)} \right\rfloor \geq 1 + diam(G)\ \forall\ u,v \in V(G)$. The radio contra harmonic mean number of f, denoted by $rchmn(f)$ is the maximum integer assigned to any vertex $v \in V(G)$ under the mapping f. Further, the radio contra harmonic mean number of G, denoted by $rchmn(G)$ is the smallest value of $rchmn(f)$ taken over all radio contra harmonic mean labeling of G. It is obvious from the definition that $rchmn(G) = |V(G)|$. If $rchmn(G) = V(G)$, then G is called a radio contra harmonic mean graceful graph. In this paper, we analyze bounds of radio contra harmonic mean graphs and radio contra harmonic mean number of path graph is also calculated.

Keywords: distance, order, diameter, path.

1. Introduction

Graph labeling is a recently developing area in graph theory, for detailed survey we refer Gallian[3]. For all other standard terminology we follow Harray[4]. Hale's radio frequency classification problems were implemented in 1980. Motivated by channel assignment of FM radio stations, Chartrand defined the concept of radio labeling of graphs in 2001[2]. Radio labeling is used for X-ray, crystallography, coding theory, routing, computing etc. In this sequence, Ponraj[5] introduced the notation radio mean labeling of graphs and calculated the radio mean number of some graphs. Motivated by those things we introduced radio contra harmonic mean labeling and radio contra harmonic mean number of graphs[1].

Definition 1.1 [1] The radio contra harmonic mean labeling of a connected graph G is defined as an injective function $f: V(G) \to \mathbb{Z}^+$ where

$$d(u,v) + \left\lceil \frac{(f(u))^2 + (f(v))^2}{f(u)+f(v)} \right\rceil \geq 1 + diam(G)\ \forall\, u,v \in V(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

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or $d(u,v) + \left\lfloor \frac{(f(u))^2+(f(v))^2}{f(u)+f(v)} \right\rfloor \geq 1 + diam(G)\ \forall\, u,v \in V(G)$.

The radio contra harmonic mean number of f denoted by $rchmn(f)$ is the maximum integer assigned to any $v \in V(G)$ under this mapping f. Further the radio contra harmonic mean number of G denoted by $rchmn(G)$ is the smallest value of $rchmn(f)$ taken over all radio contra harmonic mean labeling of G. It is obvious from the definition that $rchmn(G) = |V(G)|$. If $rchmn(G) = V(G)$, then G is called a radio contra harmonic mean graceful graph.

2. Main Results:

Theorem 2.1 For any graph G, with diameter less than 5 admits radio contra harmonic mean labeling and the radio contra harmonic mean number of that graph G is its order.

Proof: Let G be a graph with diameter less than 5. Define $f: V(G) \to \mathbb{Z}^+$ as follows:

$f(v_i) = i\ \ \forall\, i \in V(G)$

Case (i): $diam(G) = 1$ or 2

Then $d(v_i, v_j) + \left\lceil \frac{(f(v_i))^2+(f(v_j))^2}{f(v_i)+f(v_j)} \right\rceil \geq 1 + 2 = 3$ for $i \neq j$.

Case (ii): $diam(G) = 3$

For $f(v_i) = i\ \ \forall\, i \in V(G)$ such that $d(v_1, v_2) = 2$.

Then $d(v_i, v_j) + \left\lceil \frac{(f(v_i))^2+(f(v_j))^2}{f(v_i)+f(v_j)} \right\rceil \geq 4$ for $i \neq j$.

Case (iii): $diam(G) = 4$

Consider the three vertices v_1, v_2, v_3 of G such that

(i) $d(v_1, v_3) = d(v_2, v_3) = 2$

(ii) $d(v_1, v_2) = 3$

Then $d(v_i, v_j) + \left\lceil \frac{(f(v_i))^2+(f(v_j))^2}{f(v_i)+f(v_j)} \right\rceil \geq 4$ for $i \neq j$.

Hence the inequality (1) holds.

Thus for every graph G with diameter less than 5 admit radio contra harmonic mean labeling. In this way of labeling we get, $rchmn(G) =$ order of G for $diam(G) \leq 4$. Hence for any graph G, with diameter less than 5 admit radio contra harmonic mean labeling and the radio contra harmonic mean number of that graph G is its order.

Corollary 2.2 Converse of the above theorem need not be true.

Theorem 2.3 For any connected graph G, $rchmn(G + v) = |V(G + v)| \leq 1 + rchmn(G)$.

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Proof: Let G be a connected graph.

Claim 1: $rchmn(G+v) = |V(G+v)|$

Let v be the new vertex. Obviously $|V(G)|+1 = |V(G+v)|$. Without loss of generality let $diam(G) = n$. Then $G+v$ is a graph with diameter 2. By theorem 2.1, $rchmn(G+v)$ is its order. ie., $rchmn(G+v) = |V(G+v)|$.

Claim 2: $|V(G+v)| \leq 1 + rchmn(G)$.

If $diam(G) \leq 4$, then $|V(G+v)| = 1 + |V(G)| = rchmn(G+v)$. Therefore $rchmn(G+v) = 1 + rchmn(G)$.

If $diam(G) > 4$ then $rchmn(G) \leq |V(G)| = |V(G+v)| - 1$.

$$rchmn(G) \leq |V(G+v)| - 1$$

$$rchmn(G) + 1 \leq |V(G+v)| = rchmn(G+v)$$

ie., $rchmn(G+v) \leq 1 + rchmn(G)$.

Hence $rchmn(G+v) = |V(G+v)| \leq 1 + rchmn(G)$.

Theorem 2.4 If $G = G_1 \cup G_2 \cup \ldots . \cup G_n$ then

(i) $rchmn(G)$ does not exist.

(ii) $rchmn(G+v) = |V(G)| + 1$

Proof: Let $G = G_1 \cup G_2 \cup \ldots . \cup G_n$ and $|V(G)| = |V(G_1) + |V(G_2)| + \ldots . + |V(G_n)|$.

(i) Since G is disconnected, G will not admit radio contra harmonic mean labeling. Therefore G is not a radio contra harmonic mean graph. Hence $rchmn(G)$ does not exist.

(ii) $G+v$ is a connected graph with diameter 2. By theorem 2.1, $rchmn(G+v)$ is its order. $rchmn(G+v) = |V(G+v)| = |V(G_1) + |V(G_2)| + \ldots . + |V(G_n)| + 1$.

Hence $rchmn(G+v) = |V(G)| + 1$.

Lemma 2.4 For any path $P_n, n \in [3 + S_k + k, 3 + S_{k+1} + k]$ where $S_k = 1 + 2 + 3 + \cdots + k$ and $k = 1, 2, 3, \ldots$, there exists a radio contra harmonic mean labeling of $P_n, n \geq 5$ with radio contra harmonic mean number $rchmn(P_n) = 2n - k - 4$.

Proof: Let $v_1, v_2, v_3, \ldots, v_n$ be n vertices of path P_n with its diameter is $n - 1$.

A labeling $f: V(P_n) \to Z^+$ is a radio contra harmonic mean labeling of path P_n if f is an injective function and satisfies the following condition:

$$d(v_i, v_j) + \left\lceil \frac{(f(v_i))^2 + (f(v_j))^2}{f(v_i) + f(v_j)} \right\rceil \geq n \ \forall\, v_i, v_j \in V(G), i \neq j \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Suppose $n \in [3 + S_k + k, 3 + S_{k+1} + k]$ where $S_k = 1 + 2 + 3 + \cdots + k$.

Let us define a function $f: V(P_n) \to Z^+$ using indices $d_0, d_1, d_2, \ldots d_{k+1}$.

$d_0 = n - h,\ h = n - (3 + S_k + k)$

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$d_1 = 1$

Case (i): k is odd

(i) for $i = 2, 3, \dots, k,$

$d_i = d_{i-2} + k - i + 4,\ i$ is odd

$d_i = d_{i-2} - k + i - 2,\ i$ is even.

(ii) for $i = k + 1,$

$d_i = d_{i-2} - 2.$

The function f is given by

(i) for $i = 0, i$ is even and $i \le k,\ f\left(v_{d_i}\right) = n - k - 2 + i$

(ii) for $i = 1, i$ is odd and $i \le k,\ f\left(v_{d_i}\right) = n - k - 4 + i$

(iii) for $i = k + 1, k$ is odd, $f\left(v_{d_i}\right) = f\left(v_{d_{i-2}}\right) + 1 = f\left(v_{d_{i-1}}\right) + 2$

(iv) (a) for $k = 1,\ i \in \{(d_2, d_0) \cup (d_1, d_2)\}$

(b) for $k \ge 3, i \in \{\{(d_{k+1}, d_{k-1}) \cup (d_k, d_{k+1})\} \cup \{(d_{k-1}, d_{k-3}) \cup (d_{k-2}, d_k) \cup (d_{k-3}, d_{k-5}) \cup (d_{k-4}, d_{k-2}) \cup \dots . \cup (d_2, d_0) \cup (d_1, d_3)\}\}$

ie) $for\ i \in \{c_r : 1 \le r \le s, s = n - h - k - 2\},$

$f(v_i) = f\left(v_{c_r}\right) = f\left(v_{d_{k+1}}\right) + r, 1 \le r \le s$

(v) if $h \ge 1, i = d_0 + l, 1 \le l \le h, f(v_i) = f\left(v_{d_0+l}\right) = f\left(v_{d_{k+1}}\right) + s + l$

Case (ii): k is even

Let $i = 2, 3, \dots, k + 1.$

$d_i = d_{i-2} - k + i - 2, i$ is even

$d_i = d_{i-2} + k - i + 4, i$ is odd

The function f is given by

(i) for $i = 0, i$ is even and $i \le k + 1, f\left(v_{d_i}\right) = n - k - 2 + i$

(ii) for $i = 1, i$ is odd and $i \le k + 1, f\left(v_{d_i}\right) = n - k - 4 + i$

(iii) $i \in \left\{\{(d_{k+1}, d_k) \cup (d_k, d_{k-2}) \cup (d_{k-1}, d_{k+1})\} \cup \{(d_{k-2}, d_{k-4}) \cup (d_{k-3}, d_{k-1}) \cup (d_{k-4}, d_{k-6}) \cup (d_{k-5}, d_{k-3}) \cup \dots . \cup (d_2, d_0) \cup (d_1, d_3)\}\right\}$

ie.,$i \in \{c_r : 1 \le r \le s, s = n - h - k - 2\},$

$f(v_i) = f\left(v_{c_r}\right) = f\left(v_{d_k}\right) + r, 1 \le r \le s$

(iv) if $h \ge 1, i = d_0 + l, 1 \le l \le h, f(v_i) = f\left(v_{d_0+l}\right) = f\left(v_{d_k}\right) + s + l$

It is clear that f is an injective function.

Now to show that f satisfy equation (2).

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For that we consider the following cases:

Subcase (i): Consider the pair (v_i, v_j) of distinct vertices where

$v_i, v_j \in \{v_{d_0}, v_{d_1}, v_{d_2}, \dots, v_{d_{k+1}}\}, i \neq j$, we get

$$\left\lceil \frac{(f(v_i))^2 + (f(v_j))^2}{f(v_i) + f(v_j)} \right\rceil \geq n - d(v_i, v_j)$$

Subcase (ii): Consider the pair (v_i, v_j) of distinct vertices where $i, j \in [d_0 + 1, n], i \neq j$ we get

$$\left\lceil \frac{(f(v_i))^2 + (f(v_j))^2}{f(v_i) + f(v_j)} \right\rceil \geq n - d(v_i, v_j)$$

Subcase (iii): Consider the pair (v_i, v_j) of distinct vertices where

$i \in \{v_{d_0}, v_{d_1}, v_{d_2}, \dots, v_{d_{k+1}}\},\ j \in [d_0 + 1, n]$, we get

$$\left\lceil \frac{(f(v_i))^2 + (f(v_j))^2}{f(v_i) + f(v_j)} \right\rceil \geq n - d(v_i, v_j)$$

Subcase (iv): Consider the pair (v_i, v_j) of distinct vertices where

$v_i, v_j \in V(P_n) \setminus (\{v_{d_0}, v_{d_1}, v_{d_2}, \dots, v_{d_k}\} \cup \{v_{d_0+1}, v_{d_0+2}, \dots, v_n\})$, we get

$$\left\lceil \frac{(f(v_i))^2 + (f(v_j))^2}{f(v_i) + f(v_j)} \right\rceil \geq n - d(v_i, v_j)$$

Subcase (v): Consider the pair (v_i, v_j) of distinct vertices where

$v_i \in \{v_{d_0}, v_{d_1}, v_{d_2}, \dots, v_{d_{k+1}}\}, v_j \in V(P_n) \setminus (\{v_{d_0}, v_{d_1}, v_{d_2}, \dots, v_{d_k}\} \cup \{v_{d_0+1}, v_{d_0+2}, \dots, v_n\})$

$$\left\lceil \frac{(f(v_i))^2 + (f(v_j))^2}{f(v_i) + f(v_j)} \right\rceil \geq n - d(v_i, v_j)$$

Subcase (vi): Consider the pair (v_i, v_j) of distinct vertices where

$v_i \in \{v_{d_0+1}, v_{d_0+2}, \dots, v_n\}, v_j \in V(P_n) \setminus (\{v_{d_0}, v_{d_1}, v_{d_2}, \dots, v_{d_k}\} \cup \{v_{d_0+1}, v_{d_0+2}, \dots, v_n\})$

$$\left\lceil \frac{(f(v_i))^2 + (f(v_j))^2}{f(v_i) + f(v_j)} \right\rceil \geq n - d(v_i, v_j)$$

It is clear that the function f defined above satisfies the condition (2) and hold radio contra harmonic mean labeling of path $P_n, n \geq 5$. Under this mapping the maximum number assigned to any vertex of P_n is 2n − k − 4. Hence the radio contra harmonic mean number of f, $rchmn(f) = 2n - k - 4$.

Therefore it is clear that any radio contra harmonic mean labeling of P_n defined under the mapping f whose range contains only integers greater than that of f. Now

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we shall investigate the radio contra harmonic mean labeling of P_n about whose range consist of integers less than $n - k - 3$. It is observed that under the labeling f, for odd k,

$$\sum_{i=odd\leq k-2} d\left(v_{d_i}, v_{d_{i+2}}\right) + d\left(v_{d_k}, v_{d_{k+1}}\right) + \sum_{i=odd\leq k} d\left(v_{d_{k+2-i}}, v_{d_{k-i}}\right) \leq diam(P_n)$$

and for even k,

$$\sum_{i=odd\leq k-1} d\left(v_{d_i}, v_{d_{i+2}}\right) + d\left(v_{d_{k+1}}, v_{d_k}\right) + \sum_{i=even\leq k} d\left(v_{d_{k+2-i}}, v_{d_{k-i}}\right) \leq diam(P_n)$$

ie) for both cases we get,

$$(k+1) + k + (k-1) + (k-2) + \ldots\ldots + 4 + 3 + 2 + 2 = 3 + S_k + k - 1$$
$$\leq n - 1$$

Consider any radio contra harmonic mean labeling g of P_n, $g : V(P_n) \to \{n - k - 4, n - k - 3, \ldots\ldots\}$. Then it follows from radio contra harmonic mean condition that any two vertices receives labels $n - k - 4$ and $n - k - 2$ must be atleast $k + 2$ distance apart, any two vertices receives labels $n - k - 3$ and $n - k - 1$ must be atleast $k + 1$ distance apart, any two vertices receives labels $n - k - 2$ and $n - k$ must be atleast k distance apart, any two vertices receives labels $n - k - 1$ and $n - k + 1$ must be atleast $k - 1$ distance apart, ..., any two vertices receives labels $(n - k - 3) + k$ and $(n - k - 3) + (k + 1)$ must be at least 2 distance apart. If $\{(n - k - 4), (n - k - 3), (n - k - 2), \ldots, (n - k - 3) + k, (n - k - 3) + (k + 1)\}$ are in the range of g, then we must have

$$(k+2) + (k+1) + k + (k-1) + \ldots\ldots + 4 + 3 + 2 + 2 \leq n - 1.$$

This implies $(k + 2) + 3 + S_k + k - 1 \leq n - 1$, a contradiction since $3 + S_k + k \leq n \leq 3 + S_{k+1} + k$. This means not all of the integers {(n – k – 4), (n – k – 3), (n – k – 2), ..., $(n - k - 3) + k, (n - k - 3) + (k + 1)\}$ are in the range of g which implies the maximum integer assigned to any vertex of P_n under the mapping g, $rchmn(g)$ greater than $2n - k - 6$. In other words, $rchmn(g) \geq 2n - k - 4$. Similarly, we can show that any radio contra harmonic mean labeling of P_n whose range set includes integer less than $n - k - 3$ has span greater than or equal to $2n - k - 4$.

Hence, for any path $P_n, n \in [\,3 + S_k + k \leq n \leq 3 + S_{k+1} + k]$ the $rchmn(P_n) = 2n - k - 4$.

Example 2.5 Radio contra harmonic mean number of path graph P_{10}, $rchmn(P_{10}) = 14$ is depicted in figure 1.

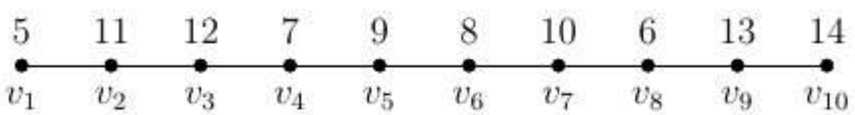

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Figure 1: Radio contra harmonic mean labeling of P_{10}

Theorem 2.6 For path graph P_n, $n \geq 2$, $rchmn(P_n) = \begin{cases} n & if\ n \leq 4 \\ 2n-k-4 & if\ n > 4 \end{cases}$

Proof:

Case (i): $n \leq 4$

By theorem 2.1, $rchmn(P_n) = n$ for $n \leq 4$.

Case (ii): $n > 4$

Proof follows from lemma 2.4.

Hence $rchmn(P_n) = \begin{cases} n & if\ n \leq 4 \\ 2n-k-4 & if\ n > 4 \end{cases}$

3. Open Problem

1. Find $rchmn(C_n)$.
2. Generalize the properties of tree family with respect to radio contra harmonic mean labeling.

4. Conclusion

In this paper, we have discussed bound of radio contra harmonic mean graph and least upper bound of path graph under radio contra harmonic mean labeling. It is planned to explore different graph properties in future work regarding to this concept.

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