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Certified Dominating Sets and Certified Domination Polynomials of Pan graph

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Abstract: Let $G = (V, E)$ be a simple graph of order n. A dominating set S is a certified dominating set of G if S has either zero or at least two neighbours in $V - S$. Let $D_{cer}(\mathcal{P}_n, i)$ be the family of all certified dominating sets of a pan graph $\mathcal{P}_n$ with cardinality i and let $d_{cer}(\mathcal{P}_n, i) = |D_{cer}(\mathcal{P}_n, i)|$. In this article, we obtain the recursive formula for $d_{cer}(\mathcal{P}_n, i)$. Using this recursive formula, we obtain the polynomial $D_{cer}(\mathcal{P}_n, x) = \sum_{i=\left\lceil\frac{n}{3}\right\rceil}^{n+1} d_{cer}(\mathcal{P}_n, i)x^i$, which we call the certified domination polynomial of pan graph $\mathcal{P}_n$.

Key words: Certified dominating set, certified domination number, certified domination polynomial.

Mathematics Subject Classification: 05C69, 05C31, 05

1. INTRODUCTION

The concept of dominating sets and the domination number in Graph Theory were first introduced by Oystein Ore and Claude Berge in 1962. The concept of domination polynomial in Graph Theory was introduced by Saied Alikhani and Yee-hock Peng in 2009. The concept of certified domination in graphs was introduced by Dettlaf et al.,2020. They also further studied the concept in their subsequent work including its applications in real life situations. This motivaited us to investigate the certified domination polynomial of certain graphs.

A dominating set is a subset of vertices in a graph where every vertex in the graph is either in the set or adjacent to a vertex in the set. The domination number (denoted as $\gamma(G)$) is the minimum size of a dominating set [5]. A dominating set S is a certified dominating set of G if S has either zero or at least two neighbours in $V - S$. The certified domination number $\gamma_{cer}(G)$

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of G is the minimum size of a certified dominating set. For a detailed treatment of certified domination of a graph, the reader is referred to [3]. Let $D_{cer}(\mathcal{P}_n, i)$ be a family of all certified dominating sets of $\mathcal{P}_n$ with cardinality i and let $d_{cer}(\mathcal{P}_n, i) = |D_{cer}(\mathcal{P}_n, i)|$. We call the polynomial $D_{cer}(\mathcal{P}_n, x) = \sum_{i=\gamma_{cer}(\mathcal{P}_n)}^{|V(\mathcal{P}_n)|} d_{cer}(\mathcal{P}_n, i)\, x^i$ as the certified domination polynomial of pan graph. Pan graph is a graph obtained by joining a cycle C_n to a singleton graph K_1 with a bridge. It is denoted by $\mathcal{P}_n$. Figure 1 shows the example of pan graph $\mathcal{P}_n$[4].

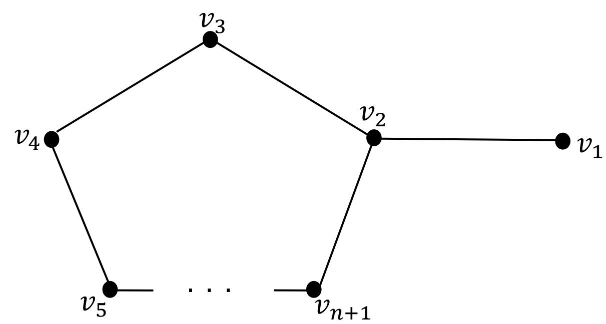

Figure 1: Pan graph $\mathcal{P}_n$

2. CERTIFIED DOMINATING SETS OF PAN GRAPH

In this section, we shall investigate certified dominating sets of Pan graphs. We need the following lemmas to prove our main results in this section.

Lemma 2.1: [4]

Let $\mathcal{P}_n$ be a pan graph with $n+1$ vertices. Then $\gamma_{cer}(\mathcal{P}_n) = \left\lceil \frac{n}{3} \right\rceil$ for all $n \geq 3$

Lemma 2.2: For all $n \geq 5$,

(i) $D_{cer}(\mathcal{P}_n, i) = \emptyset$ if and only if $i < \left\lceil \frac{n}{3} \right\rceil$ or $\left\lfloor \frac{n}{2} \right\rfloor + 1 < i \leq n$ or $i > n+1$

(ii) $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$ if and only if $\left\lceil \frac{n}{3} \right\rceil \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ or $i = n+1$

Lemma 2.3: For all $i < n-1$ and $n \geq 6$,

(i) If $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$ then $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$

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(ii) If $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$ then $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$

Proof:

(i) Since $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$, by Lemma 2.2(ii) , $\left\lceil \frac{n-3}{3} \right\rceil \leq i-1 \leq \left\lfloor \frac{n-3}{2} \right\rfloor + 1$ or $i-1 = n-2$ (Neglect this case since $i = n-1$). Which gives $\left\lceil \frac{n}{3} \right\rceil \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. Therefore $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$ (by Lemma 2.2(ii))

(ii) Since $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$, by Lemma 2.2(ii), $\left\lceil \frac{n-2}{3} \right\rceil \leq i-1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor + 1$ or $i-1 = n-2$ (Neglect this case since $i = n$). Which gives $\left\lceil \frac{n+1}{3} \right\rceil \leq i < \left\lfloor \frac{n}{2} \right\rfloor + 1$. Therefore $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$ (by Lemma 2.2(ii))

Lemma 2.4: Suppose $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$

(i) For all $i < n-1$ and $n \geq 6$, $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$ if and only if $n = 2m$ and $i = m+1$, $(m \geq 3)$.

(ii) For all $i < n-1$ and $n \geq 6$, $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) = \emptyset$ if and only if $n = 3p$ and $i = p$, $(i \geq 2)$.

(iii) For all $i < n-1$ and $n \geq 8$, $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$ if and only if $\left\lceil \frac{n+1}{3} \right\rceil \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

(iv) $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) = \emptyset$ if and only if $i = n+1$.

Proof:

(i) ($\Rightarrow$) Since $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset$, by Lemma 2.2(i), $i-1 < \left\lceil \frac{n-3}{3} \right\rceil$ or $\left\lfloor \frac{n-3}{2} \right\rfloor + 1 < i-1 \leq n-3$ or $i-1 > n-2$ (Neglect Since $i > n-1$). If $i-1 < \left\lceil \frac{n-3}{3} \right\rceil$, then $i < \left\lceil \frac{n}{3} \right\rceil$. Hence $D_{cer}(\mathcal{P}_n, i) = \emptyset$, a paradox. Since $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$, by Lemma 2.2(ii), $\left\lceil \frac{n-2}{3} \right\rceil \leq i-1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor + 1$ or $i-1 = n-1$ (Neglect this case since $i = n$). Also since $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$, by

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Lemma 2.2(ii), $\left\lceil \frac{n}{3} \right\rceil \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$ or $i = n+1$ (Neglect this case). Hence we have the following three relations.

$$\left\lfloor \frac{n-3}{2} \right\rfloor + 1 < i - 1 \le n - 3 , \left\lceil \frac{n-2}{3} \right\rceil \le i - 1 \le \left\lfloor \frac{n-2}{2} \right\rfloor + 1 \text{ and } \left\lceil \frac{n}{3} \right\rceil \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$\Rightarrow \left\lfloor \frac{n-1}{2} \right\rfloor + 1 < i \le n - 2 , \left\lceil \frac{n+1}{3} \right\rceil \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } \left\lceil \frac{n}{3} \right\rceil \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$$

Comparing these relations, we get $\left\lfloor \frac{n-1}{2} \right\rfloor + 1 < i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$. This gives $n = 2m (m \in \mathbb{N})$ and $i = m + 1$. Since $n \ge 6$, $m \ge 3$

($\Leftarrow$)Suppose $n = 2m$ and $i = m + 1$, by Lemma 2.2, $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset, D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$ and $D_{cer}(\mathcal{P}_n, i) \neq \emptyset$

Similarly we can prove (ii), (iii) and (iv)

The following theorem constructs the families of certified dominating sets of $\mathcal{P}_n$.

Theorem 2.5: For all $n \ge 5$,

(i) If $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$, then $D_{cer}(\mathcal{P}_n, i) = \{X \cup \{v_n\} / X \in D_{cer}(\mathcal{P}_{n-2}, i-1)\} = \{v_1, v_2, v_4, \ldots, v_n\}$, for all $i < n - 1$.

(ii) If $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) = \emptyset$, then $D_{cer}(\mathcal{P}_n, i) = \{X \cup \{v_{n-1}\} / X \in D_{cer}(\mathcal{P}_{n-3}, i-1)\} = \{v_2, v_5, \ldots v_{n-4}, v_{n-1}\}$, for all $i < n - 1$.

(iii) If $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$, then $D_{cer}(\mathcal{P}_n, i) = \{X_1 \cup \{v_{n-1}\} / X_1 \in D_{cer}(\mathcal{P}_{n-3}, i-1)\} \cup \{X_2 \cup \{v_n\} / X_2 \in D_{cer}(\mathcal{P}_{n-2}, i-1)\}$, for all $i < n - 1$.

(iv) If $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) = \emptyset$, then $D_{cer}(\mathcal{P}_n, i) = \{v_1, v_2, \ldots, v_{n+1}\}$

Proof: Let $\mathcal{P}_n$ be the pan graph with $n + 1$ vertices. Let $V(\mathcal{P}_n) = \{v_1, v_2, \ldots, v_{n+1}\}$, where v_1 is the pendant vertex.

(i) Since $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$, by Lemma 2.4 (i) ,$n = 2m$ and $i = m + 1$, $(m \ge 3)$. Therefore $D_{cer}(\mathcal{P}_n, i) = \{v_1, v_2, v_4, \ldots, v_n\}$. Also we denote $\{X \cup \{v_n\} / X \in D_{cer}(P_{n-2}, i-1)\}$ simply by Y_1. Suppose $X \in D_{cer}(\mathcal{P}_{n-2}, i-1)$. At least one of the elements

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are labeled with v_{n-2}. If v_{n-2} is in X, then $X \cup \{v_n\} \in D_{cer}(\mathcal{P}_n, i)$. Hence $Y_1 \subseteq D_{cer}(\mathcal{P}_n, i)$. Conversely suppose $Y \in D_{cer}(\mathcal{P}_n, i)$. Here all elements are ends with v_n. If v_n is in Y, then $Y = X \cup \{v_n\}$, where X ends with v_{n-2} and $X \in D_{cer}(\mathcal{P}_{n-2}, i-1)$. Hence $Y \subseteq Y_1$

(ii) Since $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) = \emptyset$, by Lemma 2.4 (ii) ,$n = 3p$ and $i = p$, $(i \geq 2)$. Therefore $D_{cer}(\mathcal{P}_n, i) = \{v_2, v_5, \dots v_{n-4}, v_{n-1}\}$. Also we denote $\{X \cup \{v_{n-1}\}/X \in D_{cer}(\mathcal{P}_{n-3}, i-1)\}$ simply by Y_2. Suppose $X \in D_{cer}(\mathcal{P}_{n-3}, i-1)$. All the elements of X are ends with v_{n-4}. If v_{n-4} is in X, then $X \cup \{v_{n-1}\} \in D_{cer}(\mathcal{P}_n, i)$. Hence $Y_2 \subseteq D_{cer}(\mathcal{P}_n, i)$. Conversely suppose $Y \in D_{cer}(\mathcal{P}_n, i)$. Here all elements are ends with v_{n-1}. If v_{n-1} is in Y, then $Y = X \cup \{v_{n-1}\}$, where X ends with v_{n-4} and $X \in D_{cer}(\mathcal{P}_{n-3}, i-1)$. Hence $Y \subseteq Y_2$

(iii) We have $D_{cer}(\mathcal{P}_{n-3}, i-1) \neq \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) \neq \emptyset$. We denote $\{X_1 \cup \{v_{n-1}\}/X_1 \in D_{cer}(\mathcal{P}_{n-3}, i-1)\}$ simply by Y_1. Let $X_1 \in D_{cer}(\mathcal{P}_{n-3}, i-1)$. At least one of the elements are labeled with v_{n-4} or v_{n-3}. If v_{n-4} or v_{n-3} is in X_1, then $X_1 \cup \{v_{n-1}\} \in D_{cer}(\mathcal{P}_n, i)$. Next we denote $\{X_2 \cup \{v_n\}/X_2 \in D_{cer}(\mathcal{P}_{n-2}, i-1)\}$ simply by Y_2. Suppose $X_2 \in D_{cer}(\mathcal{P}_{n-2}, i-1)$. All the elements of X are ends with v_{n-3} or v_{n-2} If v_{n-3} or v_{n-2} is in X_2, then $X_2 \cup \{v_n\} \in D_{cer}(\mathcal{P}_n, i)$. Hence $Y_1 \cup Y_2 \subseteq D_{cer}(\mathcal{P}_n, i)$. Conversely suppose $Y \in D_{cer}(\mathcal{P}_n, i)$. Here all elements are ends with v_{n-1} or v_n If v_{n-1} is in Y, then $Y = X_1 \cup \{v_{n-1}\}$, for some $X_1 \in D_{cer}(\mathcal{P}_{n-3}, i-1)$. If v_n is in Y, then $Y = X_2 \cup \{v_n\}$, for some $X_2 \in D_{cer}(\mathcal{P}_{n-2}, i-1)$. Hence $Y \subseteq Y_1 \cup Y_2$

(iv) Since $D_{cer}(\mathcal{P}_{n-3}, i-1) = \emptyset$ and $D_{cer}(\mathcal{P}_{n-2}, i-1) = \emptyset$ then by Lemma 2.4(iv), $i = n+1$. Therefore $D_{cer}(\mathcal{P}_n, i) = \{v_1, v_2, \dots, v_{n+1}\}$

Lemma 2.6: For all $n \geq 6$,

$$d_{cer}(\mathcal{P}_n, i) = \begin{cases} 0 & if\ i = n-1, n \\ 1 & if\ i = n+1 \\ d_{cer}(\mathcal{P}_{n-3}, i-1) + d_{cer}(\mathcal{P}_{n-2}, i-1) & otherwise \end{cases}$$

CERTIFIED DOMINATION POLYNOMIAL OF A PAN GRAPH

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In this section we introduce and investigate the certified domination polynomial of pan graph.

Theorem 3.1: For all $n \geq 6$, $D_{cer}(\mathcal{P}_n, x) = x\{D_{cer}(\mathcal{P}_{n-3}, x) + D_{cer}(\mathcal{P}_{n-2}, x)\} + x^{n+1} - x^n - x^{n-1}$ with initial values $D_{cer}(\mathcal{P}_3, x) = x + x^2 + x^4$, $D_{cer}(\mathcal{P}_4, x) = x^2 + x^3 + x^5$ and $D_{cer}(\mathcal{P}_5, x) = 2x^2 + 2x^3 + x^6$

Proof: From Lemma 2.6, we have for $1 \leq i \leq n-2$,

$$d_{cer}(\mathcal{P}_n, i) = d_{cer}(\mathcal{P}_{n-3}, i-1) + d_{cer}(\mathcal{P}_{n-2}, i-1)$$

$$\Rightarrow \sum d_{cer}(\mathcal{P}_n, i)x^i = \sum d_{cer}(\mathcal{P}_{n-3}, i-1)x^i + \sum d_{cer}(\mathcal{P}_{n-2}, i-1)x^i$$

$$\Rightarrow D_{cer}(\mathcal{P}_n, x) = x\{D_{cer}(\mathcal{P}_{n-3}, x) + D_{cer}(\mathcal{P}_{n-2}, x)\} \quad (1)$$

Since $d_{cer}(P_n, i) = 0$ for $i = n-1$ and n (by Lemma 2.6), we remove x^{n-1} and x^n from (1). Also since $d_{cer}(P_n, i) = 1$ for $i = n+1$ (by Lemma 2.6), we add the term x^{n+1} to (1). Therefore $D_{cer}(P_n, x) = x\{D_{cer}(P_{n-3}, x) + D_{cer}(P_{n-2}, x)\} + x^{n+1} - x^n - x^{n-1}$ with initial values $D_{cer}(\mathcal{P}_3, x) = x + x^2 + x^4$, $D_{cer}(\mathcal{P}_4, x) = x^2 + x^3 + x^5$ and $D_{cer}(\mathcal{P}_5, x) = 2x^2 + 2x^3 + x^6$

Theorem 3.2: The following properties hold for the co-efficients of $D_{cer}(\mathcal{P}_n, x)$.

(i) $d_{cer}(\mathcal{P}_{2n}, n+1) = 1$, for all $n \in \mathbb{N}$

(ii) $d_{cer}(\mathcal{P}_{2n}, n) = \frac{n^2-3n+4}{2}$, for all $n \geq 2$

(iii) $d_{cer}(\mathcal{P}_{2n+1}, n+1) = n$, for all $n \in \mathbb{N}$

(iv) $d_{cer}(\mathcal{P}_{3n}, n) = 1$, for all $n \in \mathbb{N}$

(v) $d_{cer}(\mathcal{P}_{3n+1}, n+1) = \frac{n(n+1)}{2}$, for all $n \in \mathbb{N}$

(vi) $d_{cer}(\mathcal{P}_{3n+2}, n+1) = n+1$, for all $n \in \mathbb{N}$

Proof: Proof follows from the Table

i \ $\mathcal{P}_n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

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$\boldsymbol{\mathcal{P}_3}$	1	1	0	1																
$\boldsymbol{\mathcal{P}_4}$	0	1	1	0	1															
$\boldsymbol{\mathcal{P}_5}$	0	2	2	0	0	1														
$\boldsymbol{\mathcal{P}_6}$	0	1	2	1	0	0	1													
$\boldsymbol{\mathcal{P}_7}$	0	0	3	3	0	0	0	1												
$\boldsymbol{\mathcal{P}_8}$	0	0	3	4	1	0	0	0	1											
$\boldsymbol{\mathcal{P}_9}$	0	0	1	5	4	0	0	0	0	1										
$\boldsymbol{\mathcal{P}_{10}}$	0	0	0	6	7	1	0	0	0	0	1									
$\boldsymbol{\mathcal{P}_{11}}$	0	0	0	4	9	5	0	0	0	0	0	1								
$\boldsymbol{\mathcal{P}_{12}}$	0	0	0	1	11	11	1	0	0	0	0	0	1							
$\boldsymbol{\mathcal{P}_{13}}$	0	0	0	0	10	16	6	0	0	0	0	0	0	1						
$\boldsymbol{\mathcal{P}_{14}}$	0	0	0	0	5	20	16	1	0	0	0	0	0	0	1					
$\boldsymbol{P_{15}}$	0	0	0	0	1	21	27	7	0	0	0	0	0	0	0	1				

Table 1: $d_{cer}(\mathcal{P}_n, i)$, the number of co-efficients of $\mathcal{P}_n$ with cardinality i.

CONCLUSION

This paper discusses and analyses the certified dominating sets and certified domination polynomial of pan graph. Using the recursive formula we constructed the certified domination polynomial of pan graph $\mathcal{P}_n$ as $D_{cer}(\mathcal{P}_n, x) = \sum_{i=\left\lceil\frac{n}{3}\right\rceil}^{|V(\mathcal{P}_n)|} d_{cer}(\mathcal{P}_n, i)\, x^i$.

REFERENCES

1. K. Lal Gipson and M.J. Angelin Jenisha, "Certified Dominating Sets and Certified Domination Polynomial of Complete Bipartite Graph $K_{m,n}$", Advances and Appliations in Discrete Mathematics, Volume 42, Number 3, 2025 pp.273-283.

2. K. Lal Gipson and Subha T, "Secure Domination Polynomials of Paths", International Journal of Scientific & Technology Research, Volume 9, Issue 02, 2020,pp. 6516 – 6520.

3. Magda Dettlaf, Maagdalena Lemanska, Jerzy Topp, Radoslaw Ziemann and Pawel Zylinski, 2020, "Certified domination", AKCE International Journal of graphs and comibinatorics, Volume 17, Issue 1, 2020, pp.86-97.

10.48047/jocaaa.2024.33.07.92

4. P. Paul Hawkins, A.M. Anto and T. Shyla Issac Mary, “Perfect Domination Sets and Perfect Domination Polynomial of a Pan Graph”, Journal of Critical Reviews, Volume 7, Issue 04, 2020, pp. 909-913.

5. Saeid Alikhani and Yee-hock Peng, “Dominating Sets and Domination Polynomials of Paths”, International Journal of Mathematics and Mathematical Sciences, Vol.2009, Article ID 542040, (2009)

6. A. Vijayan and K. Lal Gipson K, “Dominating Sets and Domination Polynomials in Square of Paths”, Open Journal in Discrete Mathematics, Volume 3, No.1, 2013, pp.60 – 69.

7. Vinisha C and K. Lal Gipson , “Twain Secure Perfect Dominating Sets and Twain Secure Perfect Domination Polynomials of Stars” , Journal of Computational Analysis and Applications, Volume 33, No. 6, 2024, pp.416-419.