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CONSTRUCTION OF SYMMETRIC TOEPLITZ DETERMINANTS WHOSE ELEMENTS ARE THE COEFFICIENTS OF SOKOL-STANKIEWICZ STARLIKE FUNCTIONS ENDOWED WITH q - DERIVATIVE OPERATOR

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Abstract. In this research paper, our aim is to explore the factors influencing symmetric Toeplitz matrices, where their elements denote the coefficients from q sokol-Stankiewicz starlike functions and close to convex functions. We also established upper bounds for the matrices first four determinants. Some interesting special cases of the results presented here are also discussed.

Keywords: Starlike function, Convex function, Coefficient bounds, Univalent functions, Toeplitz matrices, q-derivative .

MSC(2010): 30C45, 33C50, 30C80.

1. Introduction, Definitions and Notation

Mathematical domains and discover numerous practical applications. A closely associated notion to Hankel determinants is the concept of Toeplitz determinants. Essentially, a Toeplitz matrix can be likened to a reversed Hankel matrix, as Hankel matrices feature constant entries along their opposite diagonal, whereas Toeplitz matrices maintain constant entries along their main diagonal. A thorough examination of the practical uses of Toeplitz matrices in both theoretical and applied mathematics can also be found in reference [7]. They possess outstanding computational characteristics and are compatible with a wide array of algorithms and determinant calculations.
Let $\mathcal{A}$ signify the class of an analytic function,which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Sahoo and Patel [18] obtained some upper bound to the second Hankel determinant for the class

$$\tilde{\mathcal{R}} = \left\{ f \in \mathcal{A} : |f^{'}(z)^2 - 1| < 1, \quad (z \in \Delta) \right\}. \tag{1.2}$$

Motivated by the above mentioned works obtained by earlier researchers, Trailokya Panigrahi and Janusz Sokől [12], introduce the new subclass $\mathcal{AR}^*_\alpha$, $0 \le \alpha \le 1$ of analytic functions as in definition 1.1. Sokǿl and Stankiewicz [19], introduced a class denoted as $\mathcal{SL}^*$, which comprises normalized analytic functions f in Δ satisfying the condition

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1.$$

This class is referred to as Sok*ó*l-Stankiewicz starlike functions.

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{AR}^*_\alpha$, $0 \leq \alpha \leq 1$, if it satisfies the condition*

$$\left| \left[\frac{zf'(z)}{(1-\alpha)f(z) + \alpha z} \right]^2 - 1 \right| < 1, \quad (z \in \Delta)\,. \tag{1.3}$$

The family $\mathcal{AR}^*_\alpha$ of new subclasses of analytic functions of the type α; $0 \leq \alpha \leq 1$ provides a transition from the class of starlike functions to the class of functions of bounded boundary rotation. To see this, we note that for the choice of $\alpha = 0$, we have $\mathcal{A}(\alpha) \equiv \mathcal{ST}(0) \equiv \mathcal{ST}$, where $\mathcal{ST}$ is the class of starlike functions $f > \mathcal{A}$, so that $\Re\left(\frac{zf'}{f}\right) > 0$ in Δ. For the choice of $\alpha = 1$, we get the family of functions $\tilde{\mathcal{R}}$ of functions $f > \mathcal{A}$, of bounded boundary rotation so that $\Re(f') > 0$ in Δ [3]. Note that for $\alpha = 0$, the class $\mathcal{AR}^*_0$, reduces to the class $\mathcal{SL}^*$, studied by Raza and Malik [17] and while $\alpha = 1$, the class $\mathcal{AR}^*_1$, reduces to $\tilde{\mathcal{R}}$ studied by Sahoo and Patel [18]. In terms of subordination, relation (1.3), can be written as

$$\mathcal{A}(\alpha) = \frac{zf'(z)}{(1-\alpha)f(z) + \alpha z} \prec p(z), \quad (z \in \Delta)\,. \tag{1.4}$$

In this research paper, our aim is to explore the factors influencing symmetric Toeplitz matrices, where their elements denote the coefficients from q sokol-Stankiewicz starlike functions and close to convex functions. Toeplitz matrices are extensively researched structured matrices utilized across diverse fields including mathematics, statistics, image processing, quantum mechanics etc., [4]. Toeplitz matrices are widely used in many fields and have become increasingly important in recent years. They appear in various areas of pure mathematics including algebra, geometry and analysis. Additionally, they are used in applied mathematics such as statistics, signal processing, and data analysis.

The symmetric Toeplitz determinant of f for $n \geq 1$ and $q \geq 1$ is defined by Thomas and Halim [1] given by

$$T_m(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_n \end{vmatrix}.$$

For example,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}, \quad T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}.$$

Quantum calculus, also known as q-calculus, diverges from classical calculus by its approach to limits. In q- calculus, derivatives are viewed as differences, while antiderivatives are seen as sums. The q-derivative of a complex valued function within the domain Δ is expressed differently. Jackson is credited with pioneering the systematic development of q-calculus [32, 33] introducing q-derivatives and q-integrals. Ismail et al. [34] were early adopters, defining the class of q-starlike functions using q-derivatives. Anastassiu and Gal [35, 36] furthered the field with q-generalizations of complex operators like q-Picard and q-Gauss-Weierstrass singular integral operators. Srivastava laid a strong foundation by applying q-calculus to geometric function theory using basic q-hypergeometric functions. Aral and Gupta [37–39] utilized q-beta functions to define the q-Baskakov-Durrmeyer operator, extending geometric results. Aldweby and Darus [40, 41] introduced q-operators via convolution of normalized analytic functions, examining their geometric structure involving q hypergeometric functions. Sahoo and Sharma provided useful results regarding the q-version of close to convex functions. Noor [42] et al., shifted focus to applications, deriving integral inequalities for relative harmonic functions. Recent contributions by researchers such as Ramachandran [43], Altinkaya [44], Bulut [9] and Mahmood and Sokől [46] have furthered the development of q-calculus in geometric function theory. Additionally, work on q-polynomials and (p, q)-polynomials has significantly contributed to the field of q-calculus [47, 48].
Below we provide some fundamental definitions and key concepts of q-calculus utilized in this study. Throughout this paper, we assume that $0 < q < 1$. Here, we recall the definitions of fractional q-calculus operators for complex-valued functions f.

Definition 1.2. *[32]*

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0\,; \\ f'(0), & z=0 \end{cases} \tag{1.5}$$

where $0 < q < 1$. *This implies the following:*

$$\lim_{q \longrightarrow 1^-} \left(D_q f\right)(z) = \lim_{q \longrightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f^{'}(z)\,, \tag{1.6}$$

provided that the function f is differentiable in the domain Δ. The function $D_q f$ has Maclaurin's series representation. In addition the q-derivative at zero is $D_q f(0) = D_{q^{-1}} f(0)$ for $|q| > 1$. The q-derivative at zero is defined as $f^{'}$ if it exists. Equivalently (1.1) can be written as

$$D_q\left(f(z)\right) = 1 + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}, z \neq 0\,. \tag{1.7}$$

Definition 1.3. *Let* $q \in (0, 1)$ *and define*

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & n \in \mathbb{C}; \\ \displaystyle\sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \ldots + q^{n-2}, & n \in \mathbb{N}\,. \end{cases} \tag{1.8}$$

Note that

$$[2]_q = 1+q, \qquad [3]_q = 1+q+q^2, \qquad [4]_q = 1+q+q^2+q^3, \qquad ....$$

for $n \in \mathbb{N}$.

2. Preliminaries

Let $\mathcal{P}$ be an analytic and univalent function with positive real part in Δ, $p(0) = 0$, $p^{'}(0) = 1$ and $Re(p(z)) > 0$. Here $\mathcal{P}$ maps the unit disk Δ onto a region of starlike functions with respect to symmetric points of the real axis. The Taylor series expansion is

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \qquad |p_n| \le 2\,. \tag{2.1}$$

Lemma 2.1. *[17] Let $p \in \mathcal{P}$, be given by (2.1), then*

$$|p_n| \le 2, \ \forall\, n \in \mathbb{N} \tag{2.2}$$

and

$$\left|p_2 - \frac{1}{2}p_1^2\right| \le 2 - \frac{1}{2}\left|p_1\right|^2. \tag{2.3}$$

Lemma 2.2. *[30], [16] Let $p \in \mathcal{P}$ be given by (2.1), then for some complex valued x with $|x| \le 1$, some complex valued ϱ with $|\varrho| \le 1$ and some complex valued ψ with $|\psi| \le 1$, we have*

$$\begin{aligned}
2p_2 &= p_1^2 + x\left(4 - p_1^2\right), && (2.4)\\
4p_3 &= p_1^3 + 2\left(4 - p_1^2\right)p_1 x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)\varrho\,, && (2.5)\\
8p_4 &= p_1^4 + \left(4 - p_1^2\right)x\left[p_1^2(x^2 - 3x + 3) + 4x\right] \\
&- 4\left(4 - p_1^2\right)(1 - |x|^2)\left[p(x-1)\varrho + \bar{x}\varrho^2 - 1 - |\varrho|^2\psi\right]. && (2.6)
\end{aligned}$$

Definition 2.3. *A function $f \in \mathcal{A}$ is said to be in the $\mathcal{B}(q,\alpha)$, if*

$$\Re\left(\frac{z(D_q f)(z)}{(1-\alpha)f(z) + \alpha z}\right) > 0, \quad (z \in \Delta)\,. \tag{2.7}$$

We note that

$$\lim_{q \longrightarrow 1^-} \mathcal{B}(q,\alpha) = \left\{ f \in \mathcal{A} : \lim_{q \longrightarrow 1^-} \Re\left(\frac{z(D_q f)(z)}{(1-\alpha)f(z) + \alpha z}\right) > 0, \quad (z \in \Delta)\right\} = \mathcal{B}. \tag{2.8}$$

The objective of this paper is to obtain the coefficient bounds using symmetric Toeplitz determinants $T_2(2)$,$T_2(3)$,$T_3(2)$ and $T_3(1)$ for belonging to the subclasses $\mathcal{B}(q,\alpha)$.

3. Coefficient estimates for Toeplitz determinant $\mathcal{B}(q,\alpha)$.

In our first theorem we determine a sharp bound for the coefficient body $T_2(2)$.

Theorem 3.1. *Let f given by (1.1), be in the class $\mathcal{B}(q,\alpha)$; $(0 \le \alpha \le 1)$. Then we have the sharp bound*

$$|T_2(2)| = \left|a_3^2 - a_2^2\right| \le \frac{4}{\alpha^2 + 2\alpha q + (2\alpha+1)q^2 + 2q^3 + q^4} \max\left\{1, \left|\frac{\mathbf{R_1}(q)}{(\alpha+q)^4}\right|\right\}$$

where,

$$\begin{aligned} \mathbf{R_1}(q) &= \alpha^3(-8q^2 - 16q - 16) + \alpha^2(-4q^4 - 24q^3 - 32q^2 - 16q + 16) \\ &+ \alpha(-8q^5 - 24q^4 - 16q^3 + 16q^2 + 32q) + (-4q^6 - 8q^5 + 16q^3 + 16q^2). \end{aligned}$$

Proof. First note that by equating the corresponding coefficients in the equation

$$\frac{zD_q\left(f(z)\right)}{(1-\alpha)f(z) + \alpha z} = p(z) \tag{3.1}$$

we get,

$$a_2 = \frac{p_1}{q+\alpha} \tag{3.2}$$

$$a_3 = \frac{p_1^2(1-\alpha)}{\alpha^2 + 2\alpha q + (\alpha+1)q^2 + q^3} + \frac{p_2}{(q^2+q+\alpha)} \tag{3.3}$$

$$\begin{aligned} a_4 &= \frac{p_1^3(1-\alpha)^2}{\alpha^3 + 3\alpha^2 q + (2\alpha^2+3\alpha)q^2 + (\alpha^2+4\alpha+1)q^3 + (3\alpha+2)q^4 + 2q^5 + q^6} \\ &+ \frac{p_1p_2(1-\alpha)(2\alpha + 2q^3 + q^2)}{\alpha^3 + 3\alpha^2 q + (2\alpha^2+3\alpha)q^2 + (\alpha^2+4\alpha+1)q^3 + (3\alpha+2)q^4 + (2+\alpha)q^5 + q^6} \\ &+ \frac{p_3}{q^3+q^2+q+\alpha} \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
a_5 \; = \; & \frac{p_1^4(1-\alpha)^3}{\begin{array}{l}\alpha^4+4\alpha^3 q+(3\alpha^3+6\alpha^2)q^2+(2\alpha^3+9\alpha^2+4\alpha)q^3+(9\alpha^2+9\alpha+1)q^4+(7\alpha^2+\\ 12\alpha+3)q^5+(3\alpha^2+12\alpha+5)q^6+(\alpha^2+8\alpha+6)q^7+(4\alpha+5)q^8+(\alpha+3)q^9+q^{10}\end{array}} \\
+ \; & \frac{p_1^2p_2(1-\alpha)^2\left((4\alpha-2)+2q+q^2\right)}{\begin{array}{l}\alpha^4+4\alpha^3 q+(3\alpha^3+6\alpha^2)q^2+(2\alpha^3+9\alpha^2+4\alpha)q^3+(9\alpha^2+9\alpha+1)q^4+(7\alpha^2+\\ 12\alpha+3)q^5+(3\alpha^2+12\alpha+5)q^6+(\alpha^2+8\alpha+6)q^7+(4\alpha+5)q^8+(\alpha+3)q^9+q^{10}\end{array}} \\
+ \; & \frac{p_1p_3(1-\alpha)}{\alpha^2+2\alpha q+(2\alpha+1)^2q^2+(2\alpha+2)q^3+(\alpha+3)q^4+3q^5+2q^6+q^7} \\
+ \; & \frac{p_1^2p_2(1-\alpha)^2}{\alpha^3+3\alpha^2q+(3\alpha+2\alpha^2)q^2+(\alpha^2+4\alpha+1)q^3+(\alpha^2+3\alpha+2)q^4+(3\alpha+2)q^5+(\alpha+2)q^6+q^7} \\
+ \; & \frac{p_2^2(1-\alpha)}{\alpha^2+2\alpha q+(2\alpha+1)q^2+(\alpha+2)q^3+(\alpha+2)q^4+2q^5+q^6} \\
+ \; & \frac{p_1p_3(1-\alpha)}{\alpha^2+2\alpha q+(\alpha+1)q^2+(\alpha+1)q^3+(\alpha+1)q^4+q^5}+\frac{p_4}{(\alpha+q+q^2+q^3+q^4)}\,. \qquad (3.5)
\end{aligned}
$$

In view of (3.2) and (3.3), a simple computation leads to

$$
\begin{aligned}
a_3^2-a_2^2 \; = \; & \frac{p_2^2}{\alpha^2+2\alpha q+(2\alpha+1)q^2+2q^3+q^4} \\
+ \; & \frac{p_1^4(1-\alpha)^2}{\alpha^4+4\alpha^3q+(2\alpha^3+6\alpha^2)q^2+(6\alpha^2+4\alpha)q^3+(\alpha^2+6\alpha+1)q^4+(2\alpha+2)q^5+q^6} \\
+ \; & \frac{2p_2p_1^2(1-\alpha)}{\alpha^3+3\alpha^2q+(2\alpha^2+3\alpha)q^2+(4\alpha+1)q^3+(\alpha+2)q^4+q^5} \\
- \; & \frac{p_1^2}{\alpha^2+2\alpha q+q^2}\,. \qquad (3.6)
\end{aligned}
$$

Note that, by lemma (2.2), we write $2p_2 = p_1^2 + x\left(4-p_1^2\right)$, where without any loss of generality, we let $0 \le p_1 = p \le 2$. Substituting this in (3.6) we obtain the following quadratic equation in x.

$$
a_3^2-a_2^2 \; = \; \left[\frac{(4-p^2)^2}{4\alpha^2+8\alpha q+(8\alpha+4)q^2+8q^3+4q^4}\right]x^2
$$

$$+ \left[\frac{p^2(4-p^2)(-q-1)}{2\alpha^3+6\alpha^2 q+(4\alpha^2+6\alpha)q^2+(8\alpha+2)q^3+(2\alpha+4)q^4+2q^5}\right]x$$
$$+ \frac{p^2\left[p^2\left(\mathbf{R_2}(q)\right)-\left(\begin{smallmatrix}4\alpha^4+16\alpha^3 q+(8\alpha^3+24\alpha^2)q^2+(24\alpha^2+16\alpha)q^3\\ +(4\alpha^2+24\alpha+4)q^4(8\alpha+8)q^5+4q^6\end{smallmatrix}\right)\right]}{4\left(\begin{smallmatrix}\alpha^6+6\alpha^5 q+(2a^5+15\alpha^4)q^2+(10\alpha^4+20\alpha^3)q^3+(\alpha^4+20\alpha^3+15\alpha^2)q^4\\ +(4\alpha^3+20\alpha^2+6\alpha)q^5+(6\alpha^2+10\alpha+1)q^6+(4\alpha+2)q^7+q^8\end{smallmatrix}\right)} \tag{3.7}$$

where,

$\mathbf{R_2}(q)=\alpha^4+4\alpha^2-4\alpha^3+(8\alpha-4\alpha^2)q+(4\alpha-2\alpha^2+4)q^2+4q^3+q^4.$

Using the triangular inequality, we get

$$\begin{aligned}\left|a_3^2-a_2^2\right| \;\leq\; & \left[\frac{(4-p^2)^2}{4\alpha^2+8\alpha q+(8\alpha+4)q^2+8q^3+4q^4}\right]\\ &+ \left[\frac{p^2(4-p^2)(-q-1)}{2\alpha^3+6\alpha^2 q+(4\alpha^2+6\alpha)q^2+(8\alpha+2)q^3+(2\alpha+4)q^4+2q^5}\right]\\ &+ \frac{p^2\left[p^2\left(\mathbf{R_2}(q)\right)-\left(\begin{smallmatrix}4\alpha^4+16\alpha^3 q+(8\alpha^3+24\alpha^2)q^2+(24\alpha^2+16\alpha)q^3\\ +(4\alpha^2+24\alpha)q^4(8\alpha+8)q^5+4q^6\end{smallmatrix}\right)\right]}{4\left(\begin{smallmatrix}\alpha^6+6\alpha^5 q+(2a^5+15\alpha^4)q^2+(10\alpha^4+20\alpha^3)q^3+(\alpha^4+20\alpha^3+15\alpha^2)q^4\\ +(4\alpha^3+20\alpha^2+6\alpha)q^5+(6\alpha^2+10\alpha+1)q^6+(4\alpha+2)q^7+q^8\end{smallmatrix}\right)} \qquad (3.8)\end{aligned} \tag{3.9}$$

where,

$$\begin{aligned}\mathbf{R_2}(q) &= \alpha^4-4\alpha^3+(8\alpha-4\alpha^2)q+(4\alpha-2\alpha^2+4)q^2+4q^3+q^4\\ &= \mathcal{M}_1(p,\alpha)\,.\end{aligned} \tag{3.10}$$

Differentiating $\mathcal{M}_1(p,\alpha)$ with respect to p, we obtain

$$\frac{\partial(\mathcal{M}_1(p,\alpha))}{\partial p}=p\left[\frac{p^2\left(4+8q+4q^2\right)+(2\alpha^2-8\alpha-4\alpha q+4\alpha q^2)+\left(-8q-6q^2+4q^3+2q^4\right)}{\alpha^4+4\alpha^3 q+(2\alpha^3+6\alpha^2)q^2+(6\alpha^2+4\alpha+1)q^3+(\alpha^2+6\alpha)q^4+(2\alpha+2)q^5+q^6}\right]. \tag{3.11}$$

Setting $\dfrac{\partial(\mathcal{M}_1(p,\alpha))}{\partial p}=0$ yields either $p=0$ or

$$p^2=-\frac{(2\alpha^2-8\alpha-4\alpha q+4\alpha q^2)+\left(-8q-6q^2+4q^3+2q^4\right)}{4+8q+4q^2}. \tag{3.12}$$

But $(2\alpha^2-8\alpha-4\alpha q+4\alpha q^2)+\left(-8q-6q^2+4q^3+2q^4\right)>0$ for $0\leq\alpha\leq 1$. Therefore the maximum value of $\left|a_3^2-a_2^2\right|$ is attained at the end points $p_1=p\in[0,2]$.

If we put $p_1 = 0$, $p_2 = 2x$ in (3.6) we get

$$\left|a_3^2 - a_2^2\right| = \frac{4|x|^2}{\alpha^2 + 2\alpha q + (2\alpha+1)q^2 + 2q^3 + q^4} \leq \frac{4}{\alpha^2 + 2\alpha q + (2\alpha+1)q^2 + 2q^3 + q^4}\,. \tag{3.13}$$

For $p_1 = p_2 = 2$, we get

$$|a_2| \leq \frac{2}{(\alpha + q + q^2)} \tag{3.14}$$

$$|a_3| \leq \frac{4(1-\alpha)}{\alpha^2 + 2\alpha q + (\alpha+1)q^2 + q^3} + \frac{2}{(\alpha + q + q^2)}. \tag{3.15}$$

Further,

$$\left|a_3^2 - a_2^2\right| \leq \left| \frac{\mathbf{R_1}(q)}{\begin{pmatrix} \alpha^6+6\alpha^5 q+(2\alpha^5+15\alpha^4)q^2+(10\alpha^4+20\alpha^3)q^3+(\alpha^4+20\alpha^3+15\alpha^2)q^4 \\ +(4\alpha^3+20\alpha^2+6\alpha)q^5+(6\alpha^2+10\alpha+1)q^6+(4\alpha+2)q^7+q^8 \end{pmatrix}} \right| \tag{3.16}$$

where,

$$\begin{aligned} \mathbf{R_1}(q) &= \alpha^3(-8q^2 - 16q - 16) + \alpha^2(-4q^4 - 24q^3 - 32q^2 - 16q + 16) \\ &+ \alpha(-8q^5 - 24q^4 - 16q^3 + 16q^2 + 32q) + (-4q^6 - 8q^5 + 16q^3 + 16q^2). \end{aligned}$$

The result is sharp for the functions given by

$$\frac{z(D_q f)(z)}{(1-\alpha)f(z) + \alpha z} = \frac{1+z}{1-z}\,.$$

□

Remark 3.2. *For $\alpha = 0$ and $q \to 1$, theorem (3.1) yields the bound $\left|a_3^2 - a_2^2\right| \leq 5$ for the class of starlike functions $\mathcal{ST}$ confirming the bound obtained by Thomas and Halim [1]. For $\alpha = 1$ and $q \to 1$ it yields the bound $\left|a_3^2 - a_2^2\right| \leq \frac{5}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathcal{R}}$ confirming the bound obtained by Radhika et al. [2].*

In our next theorem, we determine an upper bound for the coefficient body $T_2(3)$.

Theorem 3.3. *Let f given by (1.1) be in the class $\mathcal{B}(q, \alpha)$; $(0 \leq \alpha \leq 1)$. Then we have the sharp bound*

$$\begin{aligned} |T_2(3)| &= \left|a_4^2 - a_3^2\right| \\ &\leq \max\left\{ \left| \frac{64\mathbf{R_3}(q)}{16(\alpha + q + q^2 + q^3)^2(\alpha + q + q^2)^2(\alpha + q)^2} + \right.\right. \\ &+ \left.\left. \frac{16\mathbf{R_4}(q)}{4(\alpha + q + q^2)^2(\alpha + q)^2} \right|, \frac{4}{(\alpha + q + q)^2} \right\} \\ \mathbf{R_3}(q) &= (\alpha^4 - 8\alpha^3 + 24\alpha^2 - 32\alpha + 16) + (-4\alpha^3 + 24\alpha^2 - 48\alpha + 32)q \\ &+ (18\alpha^2 - 48\alpha + 40)q^2 + (6\alpha^2 - 28\alpha + 32)q^3 + (\alpha^2 - 10\alpha + 17) + (-2\alpha + 6)q^5 + q^6 \\ \mathbf{R_4}(q) &= 4 + \alpha^2 - 4\alpha + (4 - 2\alpha)q + q^2. \end{aligned}$$

Proof. In lemma(2.2) replace $4-p_1^2$ by $\mathcal{X}$ and $(1-|x|^2)\varrho$ by $\mathcal{Y}$ where $0 \le p_1 \le 2$, $|\varrho|<1$ and substitute the same in (3.3) and (3.4) to get

$$
\begin{aligned}
a_4^2 - a_3^2 \;=\; & \left[\frac{\mathbf{R}_3(q)}{16(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p_1^6 + \left[\frac{\mathbf{R}_4(q)}{4(\alpha+q+q^2)^2(\alpha+q)^2}\right]p_1^4 \\
+ & \left[\frac{\mathcal{X}^2\mathcal{Y}^2}{4(\alpha+q+q^2+q^3)^2} + \frac{p_1x^2\mathcal{X}^2\mathcal{Y}}{4(\alpha+q+q^2+q^3)^2}\right] \\
+ & \left[\frac{2\alpha-\alpha^2+2q+2q^2+q^3}{2(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)(\alpha+q)}\right]p_1x\mathcal{X}^2\mathcal{Y} \\
+ & \left[\frac{(\alpha^2-4\alpha+4)+(4-2\alpha)q+(3-\alpha)q^2+q^3}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)(\alpha+q)}\right]p_1^3\mathcal{X}\mathcal{Y} + \frac{p_1^2x^4\mathcal{X}^2}{16(\alpha+q+q^2+q^3)^2} \\
+ & \left[\frac{2\alpha-\alpha^2+2q+2q^2+q^3}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)(\alpha+q)}\right]p_1^2x^3\mathcal{X}^2 \\
+ & \left[\frac{(\alpha^4-4\alpha^3+4\alpha^2)+(-4\alpha^2+8\alpha)q+(-4\alpha^2+4\alpha+4)q^2}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p_1^2x^2\mathcal{X}^2 \\
+ & \left[\frac{(-2\alpha^2+4\alpha+8)q^3+8q^4+4q^5+q^6}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p_1^2x^2\mathcal{X}^2 \\
+ & \left[\frac{x^2\mathcal{X}^2}{4(\alpha+q+q^2)^2}\right] + \left[\frac{(\alpha^2-4\alpha+4)+(-2\alpha+4)q+(-\alpha+3)q^2+q^3}{8(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)(\alpha+q)}\right]p_1^4x^2\mathcal{X} \\
+ & \left[\frac{\mathbf{R}_5(q)}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p_1^4x\mathcal{X} \\
+ & \left[\frac{2-\alpha+q}{2(\alpha+q)(\alpha+q+q^2)^2}\right]p_1^2x\mathcal{X}
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathbf{R}_3(q) &= (\alpha^4-8\alpha^3+24\alpha^2-32\alpha+16)+(-4\alpha^3+24\alpha^2-48\alpha+32)q \\
&+ (18\alpha^2-48\alpha+40)q^2+(6\alpha^2-28\alpha+32)q^3+(\alpha^2-10\alpha+17)+(-2\alpha+6)q^5+q^6 \\
\mathbf{R}_4(q) &= 4+\alpha^2-4\alpha+(4-2\alpha)q+q^2 \\
\mathbf{R}_5(q) &= (-\alpha+6\alpha^3-12\alpha^2+8\alpha)+(2\alpha^3-6\alpha^2+8)q \\
&+ (\alpha^3-3\alpha^2-6\alpha+16)q^2+(-8\alpha+18)q^3+(-4\alpha+12)q^4+(-\alpha+5)q^5+q^6.
\end{aligned}
$$

Since $0 \le p_1 = p \le 2$, we get the following equation in x.

$$
\begin{aligned}
\left|a_4^2 - a_3^2\right| \quad & \le \frac{(2-p)^2(4-p^2)^2}{16(\alpha+q+q^2+q^3)^2}|x|^4 + \frac{(2\alpha-\alpha^2+2q+2q^2+q^3)(4-p^2)(p^2-2p)}{4(\alpha+q+q+q)^2(\alpha+q+q^2)(\alpha+q)}|x|^3 \\
& + \left[\frac{((\alpha^2-4\alpha+4)+(4-2\alpha)q+(3-\alpha)q^2+q^3)\,(4-p^2)(p-2)p^3}{8(\alpha+q+q^2+q^3)^2(\alpha+q+q^3)(\alpha+q)}\right. \\
& + \left[\frac{(\alpha^4-4\alpha^3+4\alpha^2)+(-4\alpha^2+8\alpha)q+(-4\alpha^2+4\alpha+4)q^2}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p^2(4-p^2) \\
& + \left[\frac{(-2\alpha^2+4\alpha+8)q^3+8q^4+4q^5+q^6}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p^2(4-p^2) \\
& + \left[\frac{p(4-p^2)^2}{4(\alpha+q+q^2+q^3)^2}\right] - \left[\frac{\mathbf{R_6}(q)}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2}\right](4-p^2)^2\Bigg]|x|^2 \\
& \quad + \left[\frac{(q-1)(4-p^2)p^2}{2(\alpha+q)(\alpha+q+q^2)^2} + \frac{(-\alpha^2+2\alpha+2q+2q^2+q^3)(4-p^2)p}{2(\alpha+q)(\alpha+q+q^2)(\alpha+q+q^2+q^3)^2}\right. \\
& \quad + \left.\frac{\mathbf{R_5}(q)}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}p^4(4-p^2)\right]|x| \\
& \quad + \left|\mathbf{R_3}(q)p^6 - \mathbf{R_4}(q)p^4\right| + \left[\frac{(\alpha^2-4\alpha+4)+(-2\alpha+4)q+(-\alpha+3)q^2+q^3}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)(\alpha+q)}\right]p^3(4-p^2) \\
& \quad + \frac{(4-p^2)^2}{4(\alpha+q+q^2+q^3)^2} \\
& \quad = \mathcal{M}_2(p,|x|)
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathbf{R_3}(q) &= (\alpha^4-8\alpha^3+24\alpha^2-32\alpha+16)+(-4\alpha^3+24\alpha^2-48\alpha+32)q \\
&+ (18\alpha^2-48\alpha+40)q^2+(6\alpha^2-28\alpha+32)q^3+(\alpha^2-10\alpha+17)+(-2\alpha+6)q^5+q^6 \\
\mathbf{R_4}(q) &= 4+\alpha^2-4\alpha+(4-2\alpha)q+q^2 \\
\mathbf{R_5}(q) &= (-\alpha+6\alpha^3-12\alpha^2+8\alpha)+(2\alpha^3-6\alpha^2+8)q \\
&+ (\alpha^3-3\alpha^2-6\alpha+16)q^2+(-8\alpha+18)q^3+(-4\alpha+12)q^4+(-\alpha+5)q^5+q^6 \\
\mathbf{R_6}(q) &= \alpha^2+2\alpha q+(2\alpha+1)q^2+(-2\alpha+2)q^3-q^4-2q^5-q^6.
\end{aligned}
$$

It is necessary to prove that the maximum value of $\mathcal{M}_2(p,|x|)$ lies in the region $[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\mathcal{M}_2(p_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $\mathcal{M}_2(p,|x|)$ with respect to $|x|$ and equating it to 0 implies that $p = p_0 = 2$, which is a contradiction. Thus to obtain the maximum of $\mathcal{M}_2(p,|x|)$, we need to consider only the end points of $[0,2]\times[0,1]$.

For $p=0$, we get

$$\begin{aligned}\mathcal{M}_2(0,|x|) &= \frac{4|x|^4}{(\alpha+q+q+q)^2}+\frac{4}{(\alpha+q+q^2+q^3)^2}\\ &- \frac{4\left(\alpha^2+2\alpha q+(2\alpha+1)q^2+(-2\alpha+2)q^3-q^4-2q^5-q^6\right)|x|}{(\alpha+q+q^2+q^2)(\alpha+q+q^3)^2}\\ &\leq \frac{4}{(\alpha+q+q^2)^2}.\end{aligned}$$

For $p=2$, we obtain

$$\mathcal{M}_2(2,|x|)=\left|64\mathbf{R_3}(q)-16\mathbf{R_4}(q)\right|. \tag{3.17}$$

For $|x|=0$, we get

$$\begin{aligned}\mathcal{M}_2(p,0) &= \left|\mathbf{R_3}(q)p^6-\mathbf{R_4}(q)p^4\right|\\ &+ \left[\frac{\mathbf{R_8}(q)}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)(\alpha+q)}\right]p_1^3(4-p^2)+\frac{(4-p^2)^2}{4(\alpha+q+q^2+q^3)^2}\end{aligned}$$

where,

$$\mathbf{R_8}(q) = (\alpha^2-4\alpha+4)+(-2\alpha+4)q+(-\alpha+3)q^2+q^3 .$$

Here $\mathcal{M}_2(p,0)$ has the maximum values $\left|64\mathbf{R_3}(q)-16\mathbf{R_4}(q)\right|$ for $p=2$ and $\dfrac{4}{(\alpha+q+q^2)^2}$ for $p=0$.

For $|x|=1$, we get

$$\begin{aligned}\mathcal{M}_2(p,1) &= \frac{(2-p)^2(4-p^2)^2}{16(\alpha+q+q^2+q^3)^2}+\frac{(2\alpha-\alpha^2+2q+2q^2+q^3)(4-p^2)(p^2-2p)}{4(\alpha+q+q+q)^2(\alpha+q+q^2)(\alpha+q)}\\ &+\left[\frac{\left((\alpha^2-4\alpha+4)+(4-2\alpha)q+(3-\alpha)q^2+q^3\right)(4-p^2)(p-2)p^3}{8(\alpha+q+q^2+q^3)^2(\alpha+q+q^3)(\alpha+q)}\right.\\ &+\left[\frac{(\alpha^4-4\alpha^3+4\alpha^2)+(-4\alpha^2+8\alpha)q+(-4\alpha^2+4\alpha+4)q^2}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p^2(4-p^2)\\ &+\left[\frac{(-2\alpha^2+4\alpha+8)q^3+8q^4+4q^5+q^6}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p^2(4-p^2)\\ &+\left.\left[\frac{p(4-p^2)^2}{4(\alpha+q+q^2+q^3)^2}\right]-\left[\frac{\mathbf{R_6}(q)}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2}\right](4-p^2)^2\right]\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{(q-1)(4-p^2)p^2}{2(\alpha+q)(\alpha+q+q^2)^2}+\frac{(-\alpha^2+2\alpha+2q+2q^2+q^3)(4-p^2)p}{2(\alpha+q)(\alpha+q+q^2)(\alpha+q+q^2+q^3)^2}\right.\\
&+\left.\frac{\mathbf{R_5}(q)}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)^2(\alpha+q)^2}p^4(4-p^2)\right]\\
&+\left|\mathbf{R_3}(q)p^6-\mathbf{R_4}(q)p^4\right|+\left[\frac{(\alpha^2-4\alpha+4)+(-2\alpha+4)q+(-\alpha+3)q^2+q^3}{4(\alpha+q+q^2+q^3)^2(\alpha+q+q^2)(\alpha+q)}\right]p^3(4-p^2)\\
&+\frac{(4-p^2)^2}{4(\alpha+q+q^2+q^3)^2}
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathbf{R_3}(q) &= (\alpha^4-8\alpha^3+24\alpha^2-32\alpha+16)+(-4\alpha^3+24\alpha^2-48\alpha+32)q\\
&+ (18\alpha^2-48\alpha+40)q^2+(6\alpha^2-28\alpha+32)q^3+(\alpha^2-10\alpha+17)+(-2\alpha+6)q^5+q^6,\\
\mathbf{R_4}(q) &= 4+\alpha^2-4\alpha+(4-2\alpha)q+q^2\\
\mathbf{R_5}(q) &= (-\alpha+6\alpha^3-12\alpha^2+8\alpha)+(2\alpha^3-6\alpha^2+8)q\\
&+ (\alpha^3-3\alpha^2-6\alpha+16)q^2+(-8\alpha+18)q^3+(-4\alpha+12)q^4+(-\alpha+5)q^5+q^6\\
\mathbf{R_6}(q) &= \alpha^2+2\alpha q+(2\alpha+1)q^2+(-2\alpha+2)q^3-q^4-2q^5-q^6.
\end{aligned}
$$

Here $\mathcal{M}_2(p,1)$ has the maximum values $\left|64\mathbf{R_3}(q)-16\mathbf{R_4}(q)\right|$ for $p=2$ and $\dfrac{4}{(\alpha+q+q^2)^2}$ for $p=0$. □

Remark 3.4. *For $\alpha=0$ and $q\to 1$, theorem (3.3) yields the bound $\left|a_4^2-a_3^2\right|\le 7$ for the class of starlike functions $\mathcal{ST}$ confirming the bound obtained by Thomas and Halim [1]. For $\alpha=1$ and $q\to 1$ it yields the bound $\left|a_4^2-a_3^2\right|\le \frac{4}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathcal{R}}$ confirming the bound obtained by Radhika et al. [2].*

Theorem 3.5. *Let f given by (1.1), be in the class $\mathcal{B}(q,\alpha)$; $(0\le\alpha\le 1;\alpha\ne\alpha_0)$. Then we have the sharp bound*

$$
|T_3(2)|=\left|\left(\begin{vmatrix} a_2 & a_3 & a_4\\ a_3 & a_2 & a_3\\ a_4 & a_3 & a_2\end{vmatrix}\right)\right|\le\begin{cases}\max\left\{|\mathbf{R_9}(q)\mathbf{R_{10}}(q)|,\dfrac{8|\mathbf{R_9}(q)|}{(\alpha+q+q^2)^2}\right\}, if\ \alpha\ne\alpha_0,\\ \max\left\{\left|\mathbf{R_{10}}(q)\mathbf{R_{11}}(q)\right|,\dfrac{8|\mathbf{R_{11}}(q)|}{(\alpha+q+q^2)^2}\right\}, if\ \alpha=\alpha_0\end{cases}
$$

where $\alpha_0 \approx 0.5$ is the positive root of the polynomial,
$(8\alpha-8)+(-8+8\alpha)q+(6\alpha-4)q^2+(2\alpha+2)q^3+4q^4+2q^5=0$, *when* $q \longrightarrow 1$. *Also*

$$\mathbf{R_9}(q) = \frac{(8\alpha-8)+(-8+8\alpha)q+(6\alpha-4)q^2+(2\alpha+2)q^3+4q^4+2q^5}{(\alpha+q+q^2+q^3)(\alpha+q+q^2)(\alpha+q)} \tag{3.18}$$

$$\mathbf{R_{10}}(q) = \frac{\mathbf{R_{12}}(q)}{(\alpha+q+q^2+q^3)(\alpha+q+q^2)^2(\alpha+q)^2} \tag{3.19}$$

$$\mathbf{R_{11}}(q) = \frac{(4\alpha^2-8\alpha+16)+6q+(2\alpha+10)q^2+(2\alpha+8)q^3+4q^4+2q^5}{(\alpha+q+q^2+q^3)(\alpha+q+q^2)(\alpha+q)} \tag{3.20}$$

$$\begin{aligned}\mathbf{R_{12}}(q) &= (16\alpha^2-16\alpha+3)+(16\alpha^2-14)q+(4\alpha^2+40\alpha-31)q^2+(-4\alpha^2+64\alpha-40)q^3 \\ &+ (32\alpha-15)q^4+(8\alpha+10)q^5+11q^6+4q^7. \end{aligned} \tag{3.21}$$

Proof. We have ,

$$|T_3(2)| = \left|a_2^3-2a_2a_3^2+2a_3^2a_4-a_2a_4\right| \tag{3.22}$$

$$= \left|(a_2-a_4)\left(a_2^2-2a_3^2+a_2a_4\right)\right|. \tag{3.23}$$

Using the same technique as in theorem (3.1), one can obtain with simple computations that

$$|a_2-a_4| \leq \left|\mathbf{R_9}(q)\right| \ for\ \alpha \neq \alpha_0\ . \tag{3.24}$$

We need to show that

$$\left|a_2^3-2a_3^2+a_2a_4\right| \leq |\mathbf{R_{10}}(q)|\ . \tag{3.25}$$

In lemma (2.2) replace $4-p^2$ by $\mathcal{X}$ and $(1-|x|^2)\,\varrho$ by Y where $0 \leq p \leq 2$, $|\varrho|<1$ and substitute the same in (3.2),(3.3) and (3.5) to get

$$\begin{aligned}\left|a_2^2-2a_3^2+a_2a_4\right| &= \left|\left[\frac{\mathbf{R_{13}}(q)}{4(\alpha+q+q^2+q^3)(\alpha+q+q^2)^2(\alpha+q)^2}\right]\right|p_1^4 \\ &+ \left|\frac{p_1^2}{(\alpha+q)^2}-\frac{p_1^2\mathcal{X}x^2}{4(\alpha+q+q^2+q^3)(\alpha+q)}\right| \\ &+ \left|\frac{\mathcal{X}Yp_1}{2(\alpha+q+q^2+q^3)(\alpha+q)}-\frac{\mathcal{X}^2x^2}{2(\alpha+q+q^2)^2}\right| \\ &+ \left|\left[\frac{\mathbf{R_{14}}(q)}{2(\alpha+q+q^2+q^3)(\alpha+q+q^2)^2(\alpha+q)^2}\right]p_1^2\mathcal{X}x\right|\end{aligned}$$

where,

$$\begin{aligned}\mathbf{R_{13}}(q) &= (-\alpha^3+4\alpha^2-4\alpha+1)+(\alpha^2-3)q+(-2\alpha^2+7\alpha-7)q^2 \\ &+ (-2\alpha^2+10\alpha-12)q^3+(3\alpha-7)q^4-2q^5\end{aligned}$$

and

$$\mathbf{R_{14}}(q) = (\alpha^3 - 2\alpha^2) + (\alpha^2 - 4\alpha)q + (\alpha^2 - 2\alpha - 2)q^2 + (2\alpha^2 - 3\alpha - 2)q^3 - 3q^4 - q^5.$$

Applying the triangle inequality and assuming that $p_1 = p$, where $0 \leq p \leq 2$, we obtain

$$\begin{aligned}
\left|a_2^2 - 2a_3^2 + a_2a_4\right| &\leq \left[\frac{p^2(4-p^2)}{4(\alpha+q+q^2+q^3)(\alpha+q)} + \frac{p^2(4-p^2)}{2(\alpha+q+q^2+q^3)(\alpha+q)} + \frac{(4-p^2)^2}{2(\alpha+q+q^2)^2}\right]|x|^2 \\
&+ \left[\left[\frac{\mathbf{R_{14}}(q)}{2(\alpha+q+q^2+q^3)(\alpha+q+q^2)(\alpha+q)^2}\right]p(4-p^2)\right]|x| \\
&+ \left[\frac{p(4-p^2)}{2(\alpha+q+q^2+q^3)(\alpha+q)} + \frac{p^2}{(\alpha+q)^2}\right] \\
&+ \left[\frac{\mathbf{R_{13}}(q)}{4(\alpha+q+q^2+q^3)(\alpha+q+q^2)^2(\alpha+q)^2}\right]p^4 \\
&= \mathcal{M}_3(p, |x|)
\end{aligned}$$

where,

$$\begin{aligned}
\mathbf{R_{13}}(q) &= (-\alpha^3 + 4\alpha^2 - 4\alpha + 1) + (\alpha^2 - 3)q + (-2\alpha^2 + 7\alpha - 7)q^2 \\
&+ (-2\alpha^2 + 10\alpha - 12)q^3 + (3\alpha - 7)q^4 - 2q^5
\end{aligned}$$

and

$$\mathbf{R_{14}}(q) = (\alpha^3 - 2\alpha^2) + (\alpha^2 - 4\alpha)q + (\alpha^2 - 2\alpha - 2)q^2 + (2\alpha^2 - 3\alpha - 2)q^3 - 3q^4 - q^5.$$

As in theorem (3.3), in order to obtain the maximum of $\mathcal{M}_3(p, |x|)$, we need to consider only the end points of $[0, 2] \times [0, 1]$.
For $p = 0$, we obtain

$$\mathcal{M}_3(0, |x|) = \frac{8|x|^2}{(\alpha+q+q^2)^2} \leq \frac{8}{(\alpha+q+q^2)^2}. \tag{3.26}$$

For $p = 2$, we have

$$\mathcal{M}_3(2, |x|) = \frac{\mathbf{R}_{12}(q)}{(\alpha+q)(\alpha+q+q^2)^2(\alpha+q+q^2+q^3)} = \mathbf{R_{10}}(q) \tag{3.27}$$

where,

$$\begin{aligned}
\mathbf{R_{12}}(q) &= (16\alpha^2 - 16\alpha + 3) + (16\alpha^2 - 14)q + (4\alpha^2 + 40\alpha - 31)q^2 + (-4\alpha^2 + 64\alpha - 40)q^3 \\
&+ (32\alpha - 15)q^4 + (8\alpha + 10)q^5 + 11q^6 + 4q^7.
\end{aligned}$$

For $|x| = 0$, we get

$$\mathcal{M}_3(p, 0) = \left|\frac{p^2}{\alpha+q)^2} + \frac{p(4-p^2)}{2\alpha+q+q^2+q^3)\alpha+q)} + \frac{\mathbf{R_{13}}(q)}{(\alpha+q+q^2)^2(\alpha+q)^2}p^4\right| \tag{3.28}$$

where,

$$\begin{aligned}\mathbf{R_{13}}(q) &= (-\alpha^3+4\alpha^2-4\alpha+1)+(\alpha^2-3)q+(-2\alpha^2+7\alpha-7)q^2 \\ &+ (-2\alpha^2+10\alpha-12)q^3+(3\alpha-7)q^4-2q^5.\end{aligned}$$

Here $\mathcal{M}_3(p,0)$ attains the maximum values $\dfrac{8}{(\alpha+q+q^2)^2}$ at $p=0$ and $\mathbf{R_{10}}(q)$ at $p=2$.
For $|x|=1$, we obtain

$$\begin{aligned}\mathcal{M}_3(p,1) &= \left[\frac{p^2(4-p^2)}{4(\alpha+q+q^2+q^3)(\alpha+q)}+\frac{p^2(4-p^2)}{2(\alpha+q+q^2+q^3)(\alpha+q)}+\frac{(4-p^2)^2}{2(\alpha+q+q^2)^2}\right] \\ &+ \left[\left[\frac{\mathbf{R_{14}}(q)}{2(\alpha+q+q^2+q^3)(\alpha+q+q^2)(\alpha+q)^2}\right]p(4-p^2)\right] \\ &+ \left[\frac{p(4-p^2)}{2(\alpha+q+q^2+q^3)(\alpha+q)}+\frac{p^2}{(\alpha+q)^2}\right] \\ &+ \left[\frac{\mathbf{R_{13}}(q)}{4(\alpha+q+q^2+q^3)(\alpha+q+q^2)^2(\alpha+q)^2}\right]p^4\end{aligned}$$

where,

$$\begin{aligned}\mathbf{R_{13}}(q) &= (-\alpha^3+4\alpha^2-4\alpha+1)+(\alpha^2-3)q+(-2\alpha^2+7\alpha-7)q^2 \\ &+ (-2\alpha^2+10\alpha-12)q^3+(3\alpha-7)q^4-2q^5\end{aligned}$$

and

$$\mathbf{R_{14}}(q) = (\alpha^3-2\alpha^2)+(\alpha^2-4\alpha)q+(\alpha^2-2\alpha-2)q^2+(2\alpha^2-3\alpha-2)q^3-3q^4-q^5.$$

Here $\mathcal{M}_3(p,1)$ attains the maximum values $\dfrac{8}{(\alpha+q+q^2)^2}$ at $p=0$ and $\mathbf{R_{10}}(q)$ at $p=2$. Therefore

$$\left|a_2^2-2a_3^2+a_2a_4\right| \le max\left\{|\mathbf{R_{10}}(q)|, \frac{8}{(\alpha+q+q^2)^2}\right\}. \tag{3.29}$$

Thus

$$|T_3(2)| = \left|(a_2-a_4)\left(a_2^2-2a_3^2+a_2a_4\right)\right| \le max\left\{\left|\mathbf{R_9}(q)\mathbf{R_{10}}(q)\right|, \frac{8\left|\mathbf{R_9}(q)\right|}{(\alpha+q+q^2)^2}\right\}. \tag{3.30}$$

For the case $\alpha = \alpha_0$, we compute $|a_2 - a_4|$ as follows:

$$|a_2 - a_4| = \left| \frac{p_1}{(\alpha+q)} - \left[\left[\frac{p_1 p_2 (1-\alpha)(4\alpha - 2 + 2q + q^2)}{(\alpha+q+q^2+q^3)(\alpha+q+q^2)(\alpha+q)} \right] \right. \right. \tag{3.31}$$

$$\left. \left. + \frac{p_1^3 (1-\alpha)^2}{(\alpha+q+q^2)(\alpha+q)(\alpha+q+q^2+q^3)} + \frac{p_3}{(\alpha+q+q^2+q^3)} \right] \right|. \tag{3.32}$$

Since $|p_n| \leq 2$, an application of triangle inequality shows that

$$|a_2 - a_4| \leq |\mathbf{R_{11}}(q)| = \frac{(4\alpha^2 - 8\alpha + 16) + 6q + (2\alpha + 10)q^2 + (2\alpha + 8)q^3 + 4q^4 + 2q^5}{(\alpha+q+q^2+q^3)(\alpha+q+q^2)(\alpha+q)}. \tag{3.33}$$

Therefore,

$$|T_3(2)| = \left| (a_2 - a_4)\left(a_2^2 - 2a_3^2 + a_2 a_4\right) \right| \leq max \left\{ |\mathbf{R_{10}}(q)\mathbf{R_{11}}(q)|, \frac{8|\mathbf{R_{11}}(q)|}{(\alpha+q+q^2)^2} \right\}. \tag{3.34}$$

This completes the proof of Theorem (3.5). $\square$

Remark 3.6. *For $\alpha = 0$ and $q \to 1$, theorem (3.5) yields the bound $|T_3(2)| \leq 12$ for the class of starlike functions $\mathcal{ST}$ confirming the bound obtained by Thomas and Halim [1]. For $\alpha = 1$ and $q \to 1$ it yields the bound $|T_3(2)| \leq \frac{8}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathfrak{R}}$ confirming the bound obtained by Radhika et al. [2].*

Theorem 3.7. *Let f given by (1.1), be in the class $\mathcal{B}(q, \alpha)$; $0 \leq \alpha \leq 1$. Then we have the sharp bound*

$$|T_3(1)| = \left| \left(\begin{vmatrix} 1 & a_1 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix} \right) \right| \leq max \left\{ 1 + \frac{1}{4(\alpha+q+q^2)^2}, \left| \frac{\mathbf{R_{15}}(q)}{(\alpha+q)^3(\alpha+q+q^2)^2} \right| \right\}$$

where,

$$\begin{aligned} \mathbf{R_{15}}(q) &= (\alpha^5 - 16\alpha^3 + 15\alpha) + (5\alpha^4 - 6\alpha^3 - 20\alpha^2 - \alpha + 16)q + (2\alpha^4 - 4\alpha^3 - 16\alpha^2 - 36\alpha + 32)q^2 \\ &+ (16\alpha^3 + 10\alpha^2 - 31\alpha + 4)q^3 + (11\alpha^3 + 12\alpha^2 - 2\alpha - 16)q^4 \\ &+ (6\alpha^3 + 3\alpha^2 + 8\alpha - 7)q^5 + (3\alpha + 2)q^6 + q^7. \end{aligned}$$

Proof. Expanding the determinant by using equations (3.2) and (3.3), we get

$$\begin{aligned} T_3(1) &= 1 + 2a_2^2(a_3 - 1) - a_3^2 \\ &= 1 + \frac{2p_1^2}{(\alpha+q)^2}\left(\frac{p_1^2(1-\alpha)}{(\alpha+q+q^2)(\alpha+q)} + \frac{p_2}{(\alpha+q+q^2)} - 1 \right) \\ &- \left[\frac{p_1^4(1-\alpha)}{(\alpha+q+q^2)^2(\alpha+q)^2} + \frac{p_2^2}{(\alpha+q+q^2)^2} + \frac{2(1-\alpha)p_1^2 p_2}{(\alpha+q+q^2)^2(\alpha+q)} \right] \end{aligned}$$

In lemma(2.2) replace $4-p_1^2$ by $\mathcal{X}$ and substitute in $T_3(1)$ to get

$$\begin{aligned} T_3(1) &= 1+\left[\frac{\mathbf{R_{16}}(q)}{4(\alpha+q+q^2)^2(\alpha+q)^3}\right]p_1^4+\left[\frac{\alpha^2-1+2q^2}{2(\alpha+q+q^2)^2(\alpha+q)^2}\right]p_1^2x\mathcal{X} \\ &- \frac{2p_1^2}{(\alpha+q)^2}-\frac{x^2\mathcal{X}^2}{4(\alpha+q+q^2)^2} \end{aligned}$$

where,

$$\mathbf{R_{16}}(q) = -\alpha^3+4\alpha+(\alpha^2+4)q-3\alpha q^2+3q^3.$$

Since $0 \leq p_1 = p \leq 2$, on applying triangle inequality we obtain the following quadratic equation in x.

$$\begin{aligned} T_3(1) &\leq \left[\frac{(4-p^2)^2}{4(\alpha+q+q^2)^2}\right]|x|^2+\left[\frac{p^2(4-p^2)(\alpha^2+\alpha^2-1+2q^2)}{2(\alpha+q+q^2)^2(\alpha+q)^2}\right]|x| \\ &+ \left[1+\frac{8(\alpha+q+q^2)^2(\alpha+q)+\mathbf{R_{16}}(q)p^2}{4(\alpha+q+q^2)^2(\alpha+q)^3}p^2\right] \end{aligned}$$

where,

$$\mathbf{R_{16}}(q) = -\alpha^3+4\alpha+(\alpha^2+4)q-3\alpha q^2+3q^3.$$

$$\begin{aligned} T_3(1) &\leq \left[\frac{(4-p^2)}{4(\alpha+q+q^2)^2}\right]+\left[\frac{p^2(4-p^2)(\alpha^2+2[3]_q-2[2]_q-1))}{2(\alpha+q+q^2)^2(\alpha+q)^2}\right] \\ &+ \left[1+\frac{8\left(\alpha+q+q^2\right)^2(\alpha+q)+\mathbf{R_{16}}(q)p^2}{\left(4\alpha+q+q^2\right)^2(\alpha+q)^3}p^2\right] \\ &= \mathcal{M}_4(p,\alpha) \end{aligned}$$

where,

$$\mathbf{R_{16}}(q) = -\alpha^3+4\alpha+(\alpha^2+4)q-3\alpha q^2+3q^3.$$

Differentiating $\mathcal{M}_4(p,\alpha)$ with respect to p, we obtain

$$\frac{\partial(\mathcal{M}_4(p,\alpha))}{\partial p} = p\left[\frac{p^2\mathbf{R_{17}}(q)+\mathbf{R_{18}}(q)}{(\alpha+q+q^2)^2(\alpha+q)^3}\right] \tag{3.35}$$

where,

$$\mathbf{R_{17}}(q) = -2\alpha^3+6\alpha+(-\alpha^2+6)q+(3\alpha^2-4\alpha+8)q^2$$

and

$$\mathbf{R_{18}}(q) = 4\alpha^3+12\alpha+(4\alpha^2+4\alpha-4)q+(8\alpha^2+8\alpha)q^2+8q^3+8q^4+4q^5.$$

Setting $\dfrac{\partial(\mathcal{M}_4(p,\alpha))}{\partial p} = 0$ yields either $p = 0$ or

$$p^2 = \frac{-\mathbf{R}_{18}(q)}{\mathbf{R}_{17}(q)} \tag{3.36}$$

where,

$$\mathbf{R}_{17}(q) \;=\; -2\alpha^3 + 6\alpha + (-\alpha^2 + 6)q + (3\alpha^2 - 4\alpha + 8)q^2$$

and

$$\mathbf{R}_{18}(q) \;=\; 4\alpha^3 + 12\alpha + (4\alpha^2 + 4\alpha - 4)q + (8\alpha^2 + 8\alpha)q^2 + 8q^3 + 8q^4 + 4q^5 \,.$$

But $-\mathbf{R}_{18}(q) < 0$ for $0 \le \alpha \le 1$ and therefore the maximum value of $T_3(1)$ is attained at the end points $p_1 = p \in [0, 2]$. For $p_1 = 0$ and $p_2 = 2x$ we have

$$\left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| = 1 - \frac{4|x|^2}{(\alpha + q + q^2)^2} \le 1 + \frac{4}{(\alpha + q + q^2)^2} \,. \tag{3.37}$$

In the view of (3.2), (3.3) and $p_1 = p_2 = 2$, we get

$$2a_2^2 a_3 \;=\; \frac{32(1-\alpha)}{(\alpha + q + q^2)(\alpha + q)^3} + \frac{16}{(\alpha + q + q^2)(\alpha + q)^2} \tag{3.38}$$

$$-2a_2^2 \;=\; \frac{-8}{(\alpha + q)^2} \tag{3.39}$$

$$-a_3^2 \;=\; -\frac{16(1-\alpha)^2}{(\alpha + q + q^2)(\alpha + q)^2} - \frac{4}{(\alpha + q + q^2)^2} - \frac{16(1-\alpha)}{(\alpha + q + q^2)^2(\alpha + q)} \,. \tag{3.40}$$

Substitute the values of (3.38), (3.39) and (3.40) in (3.22) to get

$$\left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \le \left|\frac{\mathbf{R}_{15}(q)}{(\alpha + q)^3(\alpha + q + q^2)^2}\right| \tag{3.41}$$

where,

$$\begin{aligned}
\mathbf{R}_{15}(q) \;=\;& (\alpha^5 - 16\alpha^3 + 15\alpha) + (5\alpha^4 - 6\alpha^3 - 20\alpha^2 - \alpha + 16)q + (2\alpha^4 - 4\alpha^3 - 16\alpha^2 - 36\alpha + 32)q^2 \\
+\;& (16\alpha^3 + 10\alpha^2 - 31\alpha + 4)q^3 + (11\alpha^3 + 12\alpha^2 - 2\alpha - 16)q^4 + (6\alpha^3 \\
+\;& 3\alpha^2 + 8\alpha - 7)q^5 + (3\alpha + 2)q^6 + q^7.
\end{aligned}$$

This completes the proof of the theorem (3.7). $\square$

Remark 3.8. *For $\alpha = 0$ and $q \to 1$, theorem (3.7) yields the bound $\left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \le 8$ for the class of starlike functions $\mathcal{ST}$ confirming the bound obtained by Thomas and Halim [1]. For $\alpha = 1$ and $q \to 1$ it yields the bound $\left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \le \frac{13}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathfrak{R}}$ confirming the bound obtained by Radhika et al. [2].*

Conclusion

We investigate Toeplitz matrices defined by coefficients from q-Sokol-Stankiewicz Starlike functions in the present work. Our analysis yields unique and differentiable results by establishing maximum bounds for the first four determinants of these matrices. Interestingly, our results coincide with previous studies by Radhika et al. [2] on functions with limited boundary rotation and by Thomas and Halim [1] on starlike and near-convex functions. This advances the body of knowledge in the area and emphasizes how unique our findings are in relation to current research.

Data Availability

No data were used in this paper.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors confirm no competing interests.

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