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Degree Based Topological Indices of Some GCD-Sharing Graph

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Abstract

Topological indices constitute as numerical descriptors derived from graph-theoretical representations of molecular structures, capturing features like branching patterns, structural symmetry, and interconnection networks. This key holds an importance in quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) studies, enabling the prediction of various molecular properties, including physicochemical, biological, and pharmacological ones. This paper undertakes a systematic study of the GCD-sharing graph Z_n, when $n = \rho^{\kappa}, \rho\delta, \rho^{\kappa}\delta$ (ρ, δ is prime and $\kappa \geq 2$) focusing on certain degree-based topological indices and examining the correlation among the indices to uncover underlying structural relationships and mathematical patterns.

Keywords: Graph, Topological Indices, GCD-sharing Graphs
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Introduction

Graph theory is a mathematical framework for modeling and analyzing interconnected systems across numerous scientific disciplines, from chemical informatics and molecular biology to computer science and social network analysis. In this work, we consider simple, finite, and undirected graphs $G(V, E)$ with n vertices and m edges. Our terminology and notation are based on Harary [5].

Topological indices are numerical graph invariants that encode structural information and provide powerful tools for analyzing molecular properties, network robustness, and biological interactions. The field of topological indices traces its origins to Harold Wiener (1947), the work establishing correlations between hydrocarbon structures and physical properties.

This foundation is the basis for creating numerous topological indices, each capturing different aspects of graph structure through distinct mathematical formulations. These tools have become indispensable in quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) studies, demonstrating predictive capabilities across chemical and biological contexts.

The first Zagreb Index and Second Zagreb Index were introduced by Gutman and Trinajstic in 1972, defined as $M_1(G) = \sum_{uv\in E(G)}\big(d(u) + d(v)\big)$ and $M_2(G) = \sum_{uv\in E(G)}\big(d(u)d(v)\big)$ [4].

The Randic Index, proposed by Milan Randic in 1975 to quantify the branching of chemical molecules, is defined as $R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$ [7].

Zhou and Trinajstic developed the Sum Connectivity Index to provide a variant emphasis on molecular branching. It is given by $SCI(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)}}$ [11].

$\sqrt{d(u)+d(v)}$

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Proposed by Fath-Tabar et al. (2011) for the analysis of Hamiltonian properties, the Harmonic Index is a graph-theoretic measure defined as $H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}$ [6].

The Geometric-Arithmetic Index, proposed by Vukičević and Furtula in 2009, is defined as $GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$ [9].

The Inverse Sum Indeg Index was advanced as a simple index by Gupta et al. It is mathematically defined as $ISI(G) = \sum_{uv\in E(G)} \frac{d(u)d(v)}{d(u)+d(v)}$ [10].

Estrada et al. (1998) developed the Atom-Bond Connectivity Index, which is defined as $ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}$ [2].

A novel graph invariant known as the Sombor index was proposed by Gutman, based on the degrees of a vertices, and is defined by $SO(G) = \sum_{uv\in E(G)} \sqrt{d(u)^2 + d(v)^2}$ [3]

Furtula and Gutman in 2017 proposed that the Forgotten Index is a higher-degree variant $F(G) = \sum_{uv\in E(G)} d(u)^2 + d(v)^2$[1]

Here, $d(u)$ and $d(v)$ are the degrees of the vertices u and v respectively.

Algebraic graph theory has explored structures arising from mathematical constructs, among which GCD-sharing graphs represent a particularly interesting class. These graphs embody deep connections between number theory and graph theory. Their regular yet complex structure, determined by the prime factorization of n, makes them ideal candidates for systematic investigation using topological indices.

The GCD-sharing graph defined on the set of integers modulo n, denoted Z_n, has vertex set $\{0,1,2,\dots,n-1\}$ satisfies the condition gcd $(\alpha,\beta) > 1$ such that an edge connects two distinct vertices α and β precisely when the $\gcd(\gcd(\alpha, n), \gcd(\beta, n)) > 1$.

The Pearson correlation coefficient, which quantifies the linear relationship between two variables, is defined as $r = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2+(y_i-\bar{y})^2}}$, where x_i represents the non-isolated vertices $(n-\varphi(n))$, ($\varphi(n)$ is Euler's totient function) and y_i represents the topological index. This Correlation analysis reveals which descriptors capture unique versus more structural information. It confirms index sensitivity to graph architecture changes and helps to optimize descriptor selection. These patterns provide insights into fundamental graph properties and guide practical applications in QSPR studies. This approach ensures comprehensive characterization of GCD-sharing graphs while minimizing computational redundancy. Here, the correlation is calculated from the values of non-isolated vertices and topological indices.

As a result, we define a strong correlation as a value greater than 0.8, and one greater than 0.9 indicates a very strong association.

In this study, we calculate these degree-based topological indices for GCD-sharing graphs corresponding to specific values of n, such as prime powers, product of primes, product of prime powers, and we conduct a systematic correlation analysis to identify relationships and patterns among these indices, revealing how different descriptors capture complementary aspects of graph architecture.

Definition: 1.1 A GCD-sharing graph Z_n is defined on the vertex set $\{0,1,2,\dots,n-1\}$, in which gcd $(\alpha,\beta) > 1$, the vertex exists. Two distinct vertices a and b are connected by an edge if and only if $\gcd(\gcd(\alpha, n), \gcd(\beta, n)) > 1$.

1. Main Results

Theorem: 2.1

Suppose Z_{ρ^κ} is a gcd sharing graph. Then

$M_1(Z_{\rho^\kappa}) = \rho^{\kappa-1}(\rho^{\kappa-1}-1)^2$, $M_2(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)^3}{2}$, $R(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}}{2}$, $SCI(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2\sqrt{2(\rho^{\kappa-1}-1)}}$,

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$H(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}}{2}$, $GA(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}$, $ISI(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)^2}{4}$, $ABC(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}\sqrt{2(\rho^{\kappa-1}-2)}}{2(\rho^{\kappa-1}-1)}$,

$SO(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)^2}{\sqrt{2}}$, $F(Z_{\rho^\kappa}) = \rho^{\kappa-1}(\rho^{\kappa-1}-1)^3$

Proof

Let Z_{ρ^κ} be a gcd sharing graph with ρ^κ vertices, here ρ is prime and $\kappa \geq 2$

For $\kappa = 1$, there is one vertex and no edge

Here, Z_{ρ^κ} graph has $\rho^{\kappa-1}$ vertices and $\binom{\rho^{\kappa-1}}{2}$ edges

Since the graph Z_{ρ^κ} is complete, the degree of each vertex is $\rho^{\kappa-1} - 1$

Now, First Zagreb Index, $M_1(Z_{\rho^\kappa}) = \binom{\rho^{\kappa-1}}{2}\left((\rho^{\kappa-1} - 1) + (\rho^{\kappa-1} - 1)\right)$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1} - 1)}{2}(2\rho^{\kappa-1} - 2)$$

$$= \rho^{\kappa-1}(\rho^{\kappa-1} - 1)^2$$

Second Zagreb Index, $M_2(Z_{\rho^\kappa}) = \binom{\rho^{\kappa-1}}{2}\left((\rho^{\kappa-1} - 1)(\rho^{\kappa-1} - 1)\right)$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1} - 1)^3}{2}$$

Randic Index, $R(Z_{\rho^\kappa}) = \binom{\rho^{\kappa-1}}{2}\frac{1}{\sqrt{(\rho^{\kappa-1})(\rho^{\kappa-1})}}$

$$= \frac{\rho^{\kappa-1}}{2}$$

Sum Connectivity Index, $SCI(Z_{\rho^\kappa}) = \binom{\rho^{\kappa-1}}{2}\frac{1}{\sqrt{(\rho^{\kappa-1})+(\rho^{\kappa-1})}}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1} - 1)}{2\sqrt{2(\rho^{\kappa-1} - 1)}}$$

Harmonic Index, $H(Z_{\rho^\kappa}) = \binom{\rho^{\kappa-1}}{2}\frac{2}{(\rho^{\kappa-1})+(\rho^{\kappa-1})}$

$$= \frac{p^{k-1}}{2}$$

Geometric Arithmetic Index, $GA(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}\frac{2\sqrt{(\rho^{\kappa-1})(\rho^{\kappa-1})}}{(\rho^{\kappa-1})+(\rho^{\kappa-1})}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1} - 1)}{2}$$

Inverse Sum Indeg Index, $ISI(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}\frac{(\rho^{\kappa-1})(\rho^{\kappa-1})}{(\rho^{\kappa-1})+(\rho^{\kappa-1})}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1} - 1)^2}{4}$$

Atom Bond Connectivity Index, $ABC(Z_{\rho^\kappa}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}\sqrt{\frac{(\rho^{\kappa-1}-1)+(\rho^{\kappa-1}-1)-2}{(\rho^{\kappa-1})(\rho^{\kappa-1})}}$

$$= \frac{\rho^{\kappa-1}\sqrt{2(\rho^{\kappa-1} - 2)}}{2(\rho^{\kappa-1} - 1)}$$

Sombor Index, $SO(Z_{\rho^\kappa}) = \binom{\rho^{\kappa-1}}{2}\sqrt{(\rho^{\kappa-1})^2 + (\rho^{\kappa-1})^2}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1} - 1)^2}{\sqrt{2}}$$

Forgotten Index, $F(Z_{\rho^\kappa}) = \binom{\rho^{\kappa-1}}{2}((\rho^{\kappa-1})^2 + (\rho^{\kappa-1})^2)$

$$= \rho^{\kappa-1}(\rho^{\kappa-1} - 1)^3$$

Theorem: 2.2

Let $Z_{\rho\delta}$ be a gcd sharing graph then

$M_1(Z_{\rho\delta}) = 2(\rho - 1)^2 + 2(\delta - 1)^2 + 2(\rho - 1)(\delta - 1) + (\rho - 1)^2(\rho - 2) + (\delta - 1)^2(\delta - 2)$

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$$M_2(Z_{\rho\delta}) = (\rho-1)^3 + (\delta-1)^3 + (\rho-1)^2(\delta-1) + (\rho-1)(\delta-1)^2 + \frac{(\rho-1)^3(\rho-2)}{2} + \frac{(\delta-1)^3(\delta-2)}{2}$$

$$R(Z_{\rho\delta}) = \frac{\rho-1}{\sqrt{\rho(\rho-3)+\delta(\rho-1)+2}} + \frac{\delta-1}{\sqrt{\delta(\delta-3)+\rho(\delta-1)+2}} + \frac{(\rho-2)}{2} + \frac{(\delta-2)}{2}$$

$$SCI(Z_{\rho\delta}) = \frac{\rho-1}{\sqrt{2\rho+\delta-3}} + \frac{\delta-1}{\sqrt{\rho+2\delta-3}} + \frac{(\rho-1)(\rho-2)}{2\sqrt{2(\rho-1)}} + \frac{(\delta-1)(\delta-2)}{\sqrt{2(\delta-1)}}$$

$$H(Z_{\rho\delta}) = \frac{2(\rho-1)}{2\rho+\delta-3} + \frac{2(\delta-1)}{\rho+2\delta-3} + \frac{(\rho-2)}{2} + \frac{(\delta-2)}{2}$$

$$GA(Z_{\rho\delta}) = \frac{2(\rho-1)\sqrt{\rho(\rho-3)+\delta(\rho-1)+2}}{2\rho+\delta-3} + \frac{2(\delta-1)\sqrt{\delta(\delta-3)+\rho(\delta-1)+2}}{\rho+2\delta-3} + \frac{(\rho-1)(\rho-2)}{2} + \frac{(\delta-1)(\delta-2)}{2}$$

$$ISI(Z_{\rho\delta}) = \frac{(\rho-1)(\rho(\rho-3)+\delta(\rho-1)+2)}{2\rho+\delta-3} + \frac{(q-1)(\delta(\delta-3)+\rho(\delta-1)+2)}{\rho+2\delta-3} + \frac{(\delta-2)(\rho-1)^2}{4} + \frac{(\delta-2)(\delta-1)^2}{4}$$

$$ABC(Z_{\rho\delta}) = (\rho-1)\sqrt{\frac{2\rho+\delta-5}{\rho(\rho-3)+\delta(\rho-1)+2}} + (q-1)\sqrt{\frac{\rho+2\delta-5}{\delta(\delta-3)+\rho(\delta-1)+2}} + \frac{(\rho-2)\sqrt{2(\rho-2)}}{2} + \frac{(\delta-2)\sqrt{2(\delta-2)}}{2}$$

$$SO(Z_{\rho\delta}) = (\rho-1)\sqrt{2\rho^2+\delta^2-6\rho-4\delta+2\rho\delta+5} + (\delta-1)\sqrt{\rho^2+2\delta^2-6\delta-4\rho+2\rho\delta+5} + \frac{(\rho-1)^2(\rho-2)}{\sqrt{2}} + \frac{(\delta-1)^2(\delta-2)}{\sqrt{2}}$$

$$F(Z_{\rho\delta}) = (\rho-1)(2\delta^2+\delta^2-6\rho-4\delta+2\rho\delta+5) + (\delta-1)(\delta^2+2\rho^2-6\delta-4\rho+2\rho\delta+5) + (\rho-1)^3(\rho-2) + (\delta-1)^3(\delta-2)$$

Proof

Let $Z_{\rho\delta}$ be a gcd sharing graph with $(\rho-1)(\delta-1)$ isolated vertices, here $p\ and\ q$ are prime

Here, $Z_{\rho\delta}$ graph has $\rho\delta-(\rho-1)(\delta-1)$ non-isolated vertices and $\binom{\rho}{2}+\binom{\delta}{2}-\binom{1}{2}$ edges. The degree of Vertices are $(\rho-1),(\delta-1)$, $\binom{\rho-1}{2},\binom{\delta-1}{2}$, the degree of vertices differ according to the multiples of $0,\rho,\delta$ and $\rho\delta$

Now, First Zagreb Index, $M_1(Z_{\rho\delta}) = (\rho-1)\big((\rho-1)+(\delta-1)+(\rho-1)\big) + (\delta-1)\big((\rho-1)+(\delta-1)+(\delta-1)\big) + \binom{\rho-1}{2}\big((\rho-1)+(\rho-1)\big) + \binom{\delta-1}{2}\big((\delta-1)+(\delta-1)\big)$

$$= 2(\rho-1)^2 + 2(\delta-1)^2 + 2(\rho-1)(\delta-1) + (\rho-1)^2(\rho-2) + (\delta-1)^2(\delta-2)$$

Second Zagreb Index, $M_2(Z_{\rho\delta}) = (\rho-1)\big((\rho-1)+(\delta-1)(\rho-1)\big) + (\delta-1)\big((\rho-1)+(\delta-1)(\delta-1)\big) + \binom{\rho-1}{2}\big((\rho-1)(\rho-1)\big) + \binom{\delta-1}{2}\big((\delta-1)(\delta-1)\big)$

$$= (\rho-1)^3 + (\delta-1)^3 + (\rho-1)^2(\delta-1) + (\rho-1)(\delta-1)^2 + \frac{(\rho-1)^3(\rho-2)}{2} + \frac{(\delta-1)^3(\delta-2)}{2}$$

Randic Index, $R(Z_{\rho\delta}) = (\rho-1)\frac{1}{\sqrt{((\rho-1)+(\delta-1))(\rho-1)}} + (\delta-1)\frac{1}{\sqrt{((\rho-1)+(\delta-1))(\delta-1)}} + \binom{\rho-1}{2}\frac{1}{\sqrt{(\rho-1)(\rho-1)}} + \binom{\delta-1}{2}\frac{1}{\sqrt{(\delta-1)(\delta-1)}}$

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$$= \frac{\rho-1}{\sqrt{\rho(\rho-3)+\delta(\rho-1)+2}} + \frac{\delta-1}{\sqrt{\delta(\delta-3)+\rho(\delta-1)+2}} + \frac{(\rho-2)}{2} + \frac{(\delta-2)}{2}$$

Sum Connectivity Index, $SCI(Z_{\rho\delta}) = (\rho-1)\frac{1}{\sqrt{(\rho-1)+(\delta-1)+(\rho-1)}} + (\delta-1)\frac{1}{\sqrt{(\rho-1)+(\delta-1)+(\delta-1)}} + \binom{\rho-1}{2}\frac{1}{\sqrt{(\rho-1)+(\rho-1)}} + \binom{\delta-1}{2}\frac{1}{\sqrt{(\delta-1)+(\delta-1)}}$

$$= \frac{\rho-1}{\sqrt{2\rho+\delta-3}} + \frac{\delta-1}{\sqrt{\rho+2\delta-3}} + \frac{(\rho-1)(\rho-2)}{2\sqrt{2(\rho-1)}} + \frac{(\delta-1)(\delta-2)}{\sqrt{2(\delta-1)}}$$

Harmonic Index, $H(Z_{\rho\delta}) = (\rho-1)\frac{2}{(\rho-1)+(\delta-1)+(\rho-1)} + (\delta-1)\frac{2}{(\rho-1)+(\delta-1)+(\delta-1)} + \binom{\rho-1}{2}\frac{2}{(\rho-1)+(\rho-1)} + \binom{\delta-1}{2}\frac{2}{(\delta-1)+(\delta-1)}$

$$= \frac{2(\rho-1)}{2\rho+\delta-3} + \frac{2(\delta-1)}{\rho+2\delta-3} + \frac{(\rho-2)}{2} + \frac{(\delta-2)}{2}$$

Geometric Arithmetic Index, $GA(Z_{\rho\delta}) = (\rho-1)\frac{2\sqrt{\rho(\rho-3)+\delta(\rho-1)+2}}{(\rho-1)+(\delta-1)+(\rho-1)} + (\delta-1)\frac{2\sqrt{\delta(\delta-3)+\rho(\delta-1)+2}}{(\rho-1)+(\delta-1)+(\delta-1)} + \binom{\rho-1}{2}\frac{2\sqrt{(\rho-1)^2}}{(\rho-1)+(\rho-1)} + \binom{\delta-1}{2}\frac{2\sqrt{(\delta-1)^2}}{(\delta-1)+(\delta-1)}$

$$= \frac{2(\rho-1)\sqrt{\rho(\rho-3)+\delta(\rho-1)+2}}{2\rho+\delta-3} + \frac{2(\delta-1)\sqrt{\delta(\delta-3)+\rho(\delta-1)+2}}{\rho+2\delta-3} + \frac{(\rho-1)(\rho-2)}{2} + \frac{(\delta-1)(\delta-2)}{2}$$

Inverse Sum Indeg Index, $ISI(Z_{\rho\delta}) = (\rho-1)\frac{\rho(\rho-3)+\delta(\rho-1)+2}{(\rho-1)+(\delta-1)+(\rho-1)} + (\delta-1)\frac{\delta(\delta-3)+\rho(\delta-1)+2}{(\rho-1)+(\delta-1)+(\delta-1)} + \binom{\rho-1}{2}\frac{(\rho-1)^2}{(\rho-1)+(\rho-1)} + \binom{\delta-1}{2}\frac{(\delta-1)^2}{(\delta-1)+(\delta-1)}$

$$= \frac{(\rho-1)(\rho(\rho-3)+\delta(\rho-1)+2)}{2\rho+\delta-3} + \frac{(q-1)(\delta(\delta-3)+\rho(\delta-1)+2)}{\rho+2\delta-3} + \frac{(\delta-2)(\rho-1)^2}{4} + \frac{(\delta-2)(\delta-1)^2}{4}$$

Atom Bond Connectivity Index, $ABC(Z_{\rho\delta}) = (\rho-1)\sqrt{\frac{2\rho+\delta-5}{\rho(\rho-3)+\delta(\rho-1)+2}} + (\delta-2)\sqrt{\frac{2\delta+\rho-5}{\delta(\delta-3)+\rho(\delta-1)+2}} + \binom{\rho-1}{2}\sqrt{\frac{2(\rho-1)-2}{(\rho-1)^2}} + \binom{\delta-1}{2}\sqrt{\frac{2(\delta-1)-2}{(\delta-1)^2}}$

$$= (\rho-1)\sqrt{\frac{2\rho+\delta-5}{\rho(\rho-3)+\delta(\rho-1)+2}} + (q-1)\sqrt{\frac{\rho+2\delta-5}{\delta(\delta-3)+\rho(\delta-1)+2}} + \frac{(\rho-2)\sqrt{2(\rho-2)}}{2} + \frac{(\delta-2)\sqrt{2(\delta-2)}}{2}$$

Sombor Index, $SO(Z_{\rho\delta}) = (\rho-1)\sqrt{\left((\rho-1)+(\delta-1)\right)^2 + (\rho-1)^2} + (\delta-1)\sqrt{\left((\rho-1)+(\delta-1)\right)^2 + (\delta-1)^2} + \binom{\rho-1}{2}\sqrt{(\rho-1)^2+(\rho-1)^2} + \binom{\delta-1}{2}\sqrt{(\delta-1)^2+(\delta-1)^2}$

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$$= (\rho-1)\sqrt{2\rho^2+\delta^2-6\rho-4\delta+2\rho\delta+5} + (\delta-1)\sqrt{\rho^2+2\delta^2-6\delta-4\rho+2\rho\delta+5} + \frac{(\rho-1)^2(\rho-2)}{\sqrt{2}} + \frac{(\delta-1)^2(\delta-2)}{\sqrt{2}}$$

Forgotten Index, $F(Z_{\rho\delta}) = (\rho-1)\left(\left((\rho-1)+(\delta-1)\right)^2+(\rho-1)^2\right)+(\delta-1)\left(\left((\rho-1)+(\delta-1)\right)^2+(\delta-1)^2\right)+\binom{\rho-1}{2}((\rho-1)^2+(\rho-1)^2)+\binom{\delta-1}{2}((\delta-1)^2+(\delta-1)^2)$

$$= (\rho-1)(2\delta^2+\delta^2-6\rho-4\delta+2\rho\delta+5) + (\delta-1)(\delta^2+2\rho^2-6\delta-4\rho+2\rho\delta+5)+(\rho-1)^3(\rho-2) + (\delta-1)^3(\delta-2)$$

Theorem: 2.3

Let $Z_{\rho^\kappa\delta}$ be a gcd sharing graph then

$$M_1(Z_{\rho^\kappa\delta}) = \rho^{\kappa-1}(\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1) + \rho^{2(\kappa-1)}(\delta-1)(2\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-2) + \rho^{2(\kappa-1)}(\rho-1)(\rho^{\kappa-1}\delta+2\rho^\kappa-\rho^{\kappa-1}-2) + \rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)(\rho^{\kappa-1}\delta-1) + \rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)(\rho^\kappa-1)$$

$$M_2(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)^2 + \rho^{2(\kappa-1)}(\delta-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1) + \rho^{2(\kappa-1)}(\rho-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^\kappa-1) + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2}(\rho^{\kappa-1}\delta-1)^2 + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{2}(\rho^\kappa-1)^2$$

$$R(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)} + \frac{\rho^{2(\kappa-1)}(\delta-1)}{\sqrt{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}} + \frac{\rho^{2(\kappa-1)}(\rho-1)}{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^\kappa-1)} + \frac{p^{\kappa-1}(q-1)\left(p^{\kappa-1}(q-1)-1\right)}{2\sqrt{(\rho^{\kappa-1}\delta-1)^2}} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{2\sqrt{(\rho^\kappa-1)^2}}$$

$$SCI(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2\sqrt{2(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)}} + \frac{\rho^{2(\kappa-1)}(\delta-1)}{\sqrt{(2\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-2)}} + \frac{\rho^{2(\kappa-1)}(\rho-1)}{\sqrt{(\rho^{\kappa-1}\delta+2\rho^\kappa-\rho^{\kappa-1}-2)}} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2\sqrt{2(\rho^{\kappa-1}\delta-1)}} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{2\sqrt{2(\rho^\kappa-1)}}$$

$$H(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)} + \frac{2\rho^{2(\kappa-1)}(\delta-1)}{(2\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-2)} + \frac{2\rho^{2(\kappa-1)}(p-1)}{(\rho^{\kappa-1}\delta+2\rho^\kappa-\rho^{\kappa-1}-2)} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2(\rho^{\kappa-1}\delta-1)} + \frac{\rho^{\kappa-1}(p-1)(\rho^{\kappa-1}(p-1)-1)}{2(\rho^\kappa-1)}$$

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$$GA(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{4} + \frac{\rho^{2(\kappa-1)}(\delta-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}}{(2\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-2)} + \frac{\rho^{2(\kappa-1)}(\rho-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^\kappa-1)}}{(\rho^{\kappa-1}\delta+2\rho^\kappa-\rho^{\kappa-1}-2)} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{4} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{4}$$

$$ISI(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)}{4} + \frac{\rho^{2(\kappa-1)}(\delta-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}{(2\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-2)} + \frac{\rho^{2(\kappa-1)}(\rho-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^\kappa-1)}{(\rho^{\kappa-1}\delta+2\rho^\kappa-\rho^{\kappa-1}-2)} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{4} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(p-1)-1)}{4}$$

$$ABC(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)\sqrt{2(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-2)}}{2(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)} + \rho^{2(\kappa-1)}(\delta-1)\sqrt{\frac{(2\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-4)}{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}} + \rho^{2(\kappa-1)}(\rho-1)\sqrt{\frac{(\rho^{\kappa-1}\delta+2\rho^\kappa-\rho^{\kappa-1}-4)}{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)(\rho^\kappa-1)}} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)\sqrt{2(\rho^{\kappa-1}\delta-2)}}{2(\rho^{\kappa-1}\delta-1)} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)\sqrt{2(\rho^\kappa-2)}}{2(\rho^\kappa-1)}$$

$$SO(Z_{\rho^\kappa\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)}{\sqrt{2}} + \rho^{2(\kappa-1)}(\delta-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)^2+(\rho^{\kappa-1}\delta-1)^2} + \rho^{2(\kappa-1)}(\rho-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)^2+(\rho^\kappa-1)^2} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)(\rho^{\kappa-1}\delta-1)}{\sqrt{2}} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)(\rho^\kappa-1)}{\sqrt{2}}$$

$$F(Z_{\rho^\kappa\delta}) = \rho^{\kappa-1}(\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)^2 + \rho^{2(\kappa-1)}(\delta-1)\left((\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)^2+(\rho^{\kappa-1}\delta-1)^2\right) + \rho^{2(\kappa-1)}(\rho-1)\left((\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)^2+(\rho^\kappa-1)^2\right) + \rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)(\rho^{\kappa-1}\delta-1)^2 + \rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)(\rho^\kappa-1)^2$$

Proof

Let $Z_{\rho^\kappa\delta}$ be a gcd sharing graph with $\rho^{\kappa-1}(\rho-1)(\delta-1)$ isolated vertices, here ρ,δ are prime and $\kappa \geq 2$. Here, $Z_{\rho^\kappa\delta}$ graph has $\rho^\kappa q - \rho^{\kappa-1}(\rho-1)(\delta-1)$ non-isolated vertices and $\binom{\rho^{\kappa-1}\delta}{2} + \binom{\rho^\kappa}{2} - \binom{\rho^{\kappa-1}}{2}$ edges. The degree of Vertices are $\binom{\rho^{\kappa-1}}{2}$, $\rho^{2(\kappa-1)}(\delta-1)$, $\rho^{2(\kappa-1)}(\rho-1)$, $\binom{\rho^{\kappa-1}(\delta-1)}{2}$, $\binom{\rho^{\kappa-1}(\rho-1)}{2}$, the degree of vertices differ according to the multiples of $0, \rho, \delta$ and $\rho\delta$

First Zagreb Index, $M_1(Z_{\rho^\kappa\delta}) = \binom{\rho^{\kappa-1}}{2}\left((\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)+(\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)\right) + \rho^{2(\kappa-1)}(\delta-1)\left((\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)+(\rho^{\kappa-1}\delta-1)\right) + \rho^{2(\kappa-1)}(\rho-1)\left((\rho^{\kappa-1}\delta+\rho^\kappa-\rho^{\kappa-1}-1)+(\rho^\kappa-1)\right) +$

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$$\binom{\rho^{\kappa-1}(\delta-1)}{2}(\rho^{\kappa-1}\delta-1+\rho^{\kappa-1}\delta-1)+\binom{\rho^{\kappa-1}(\rho-1)}{2}(\rho^{\kappa}-1+\rho^{\kappa}-1)$$

$$=\rho^{\kappa-1}(\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1) + \rho^{2(\kappa-1)}(\delta-1)(2\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-2) + \rho^{2(\kappa-1)}(\rho-1)(\rho^{\kappa-1}\delta+2\rho^{\kappa}-\rho^{\kappa-1}-2) + \rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)(\rho^{\kappa-1}\delta-1) + \rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)(\rho^{\kappa}-1)$$

Second Zagreb Index, $M_2(Z_{\rho^{\kappa}\delta})=\binom{\rho^{\kappa-1}}{2}\left((\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)\right)+\rho^{2(\kappa-1)}(\delta-1)\left((\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)\right)+\rho^{2(\kappa-1)}(\rho-1)\left((\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)\right)+\binom{\rho^{\kappa-1}(\delta-1)}{2}(\rho^{\kappa-1}\delta-1)(\rho^{\kappa-1}\delta-1))+\binom{\rho^{\kappa-1}(\rho-1)}{2}\left(\left((\rho^{\kappa}-1)\right)(\rho^{\kappa}-1)\right)\right)$

$$=\frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2 + \rho^{2(\kappa-1)}(\delta-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1) + \rho^{2(\kappa-1)}(\rho-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1) + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2}(\rho^{\kappa-1}\delta-1)^2 + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{2}(\rho^{\kappa}-1)^2$$

Randic Index, $R(Z_{\rho^{\kappa}\delta})= \binom{\rho^{\kappa-1}}{2}\frac{1}{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)}}+\rho^{2(\kappa-1)}(\delta-1)\frac{1}{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}}+\rho^{2(\kappa-1)}(\rho-1)\frac{1}{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(p^{\kappa}-1)}}+\binom{p^{\kappa-1}(q-1)}{2}\frac{1}{\sqrt{\rho^{\kappa-1}\delta-1)(\rho^{\kappa-1}\delta-1)}}+\binom{\rho^{\kappa-1}(\rho-1)}{2}\frac{1}{\sqrt{(\rho^{\kappa}-1)(\rho^{\kappa}-1)}}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)}+\frac{\rho^{2(\kappa-1)}(\delta-1)}{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}} + \frac{\rho^{2(\kappa-1)}(\rho-1)}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)} + \frac{p^{k-1}(q-1)\left(p^{k-1}(q-1)-1\right)}{2\sqrt{(\rho^{\kappa-1}\delta-1)^2}} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{2\sqrt{(\rho^{\kappa}-1)^2}}$$

Sum Connectivity Index, $SCI(Z_{\rho^{\kappa}\delta}) = \binom{\rho^{\kappa-1}}{2}\frac{1}{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)}}+\rho^{2(\kappa-1)}(\delta-1)\frac{1}{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa-1}\delta-1)}}+p^{2(k-1)}(p-1)\frac{1}{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa}-1)}}+\binom{\rho^{\kappa-1}(\delta-1)}{2}\frac{1}{\sqrt{(\rho^{\kappa-1}\delta-1)+(\rho^{\kappa-1}\delta1)}}+\binom{\rho^{\kappa-1}(\rho-1)}{2}\frac{1}{\sqrt{(\rho^{\kappa}-1)+(\rho^{\kappa}-1)}}$

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$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2\sqrt{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)}} + \frac{\rho^{2(\kappa-1)}(\delta-1)}{\sqrt{(2\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-2)}} + \frac{\rho^{2(\kappa-1)}(\rho-1)}{\sqrt{(\rho^{\kappa-1}\delta+2\rho^{\kappa}-\rho^{\kappa-1}-2)}} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2\sqrt{2(\rho^{\kappa-1}\delta-1)}} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{2\sqrt{2(\rho^{\kappa}-1)}}$$

Harmonic Index, $H\left(Z_{\rho^{\kappa}\delta}\right) = \binom{\rho^{\kappa-1}}{2}\frac{2}{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)} + \rho^{2(\kappa-1)}(\delta-1)\frac{2}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa-1}\delta-1)} + \rho^{2(\kappa-1)}(p-1)\frac{2}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa}-1)} + \binom{\rho^{\kappa-1}(\delta-1)}{2}\frac{2}{2\rho^{\kappa-1}\delta-1)} + \binom{\rho^{\kappa-1}(p-1)}{2}\frac{2}{(\rho^{\kappa}-1)+(\rho^{\kappa}-1)}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)} + \frac{2\rho^{2(\kappa-1)}(\delta-1)}{(2\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-2)} + \frac{2\rho^{2(\kappa-1)}(p-1)}{(\rho^{\kappa-1}\delta+2\rho^{\kappa}-\rho^{\kappa-1}-2)} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2(\rho^{\kappa-1}\delta-1)} + \frac{\rho^{\kappa-1}(p-1)(\rho^{\kappa-1}(p-1)-1)}{2(\rho^{\kappa}-1)}$$

Geometric Arithmetic Index, $GA\left(Z_{\rho^{\kappa}\delta}\right) = \binom{\rho^{\kappa-1}}{2}\frac{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2}}{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)} + \rho^{2(\kappa-1)}(\rho-1)\frac{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa-1}\delta-1)} + \rho^{2(\kappa-1)}(\rho-1)\frac{\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)}}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa}-1)} + \binom{\rho^{\kappa-1}(\delta-1)}{2}\frac{\sqrt{(\rho^{\kappa-1}\delta-1)^2}}{2(\rho^{\kappa-1}\delta-1)} + \binom{\rho^{\kappa-1}(\rho-1)}{2}\frac{\sqrt{(\rho^{\kappa}-1)^2}}{(\rho^{\kappa}-1)+(\rho^{\kappa}-1)}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{4} + \frac{\rho^{2(\kappa-1)}(\delta-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}}{(2\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-2)} + \frac{\rho^{2(\kappa-1)}(\rho-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)}}{(\rho^{\kappa-1}\delta+2\rho^{\kappa}-\rho^{\kappa-1}-2)} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{4} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)}{4}$$

Inverse Sum Indeg Index, $ISI\left(Z_{\rho^{\kappa}\delta}\right) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}\frac{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2}{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)} + \rho^{2(\kappa-1)}(\delta-1)\frac{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa-1}\delta-1)} + \rho^{2(\kappa-1)}(\rho-1)\frac{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)+(\rho^{\kappa}-1)} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2}\frac{(\rho^{\kappa-1}\delta-1)^2}{2(\rho^{\kappa-1}\delta-1)} + \binom{\rho^{\kappa-1}(\rho-1)}{2}\frac{(\rho^{\kappa}-1)^2}{(\rho^{\kappa}-1)+(\rho^{\kappa}-1)}$

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$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)}{4} + \frac{\rho^{2(\kappa-1)}(\delta-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}{(2\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-2)} + \frac{\rho^{2(\kappa-1)}(\rho-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)}{(\rho^{\kappa-1}\delta+2\rho^{\kappa}-\rho^{\kappa-1}-2)} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{4} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(p-1)-1)}{4}$$

Atom Bond Connectivity Index, $ABC(Z_{\rho^{\kappa}\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}\sqrt{\frac{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)-2}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2}} + \rho^{2(\kappa-1)}(\delta-1)\sqrt{\frac{(2\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-4)}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}} + \rho^{2(\kappa-1)}(\rho-1)\sqrt{\frac{(\rho^{\kappa-1}\delta+2\rho^{\kappa}-\rho^{\kappa-1}-4)}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)}} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)}{2}\sqrt{\frac{2(\rho^{\kappa-1}\delta-1)-2}{(\rho^{\kappa-1}\delta-1)^2}} + \left(\frac{\rho^{\kappa-1}(\rho-1)}{2}\right)\sqrt{\frac{(\rho^{\kappa}-1)+(\rho^{\kappa}-1)-2}{(\rho^{\kappa}-1)^2}}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)\sqrt{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-2)}}{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)} + \rho^{2(\kappa-1)}(\delta-1)\sqrt{\frac{(2\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-4)}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta-1)}} + \rho^{2(\kappa-1)}(\rho-1)\sqrt{\frac{(\rho^{\kappa-1}\delta+2\rho^{\kappa}-\rho^{\kappa-1}-4)}{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)(\rho^{\kappa}-1)}} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)\sqrt{2(\rho^{\kappa-1}\delta-2)}}{2(\rho^{\kappa-1}\delta-1)} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)\sqrt{2(\rho^{\kappa}-2)}}{2(\rho^{\kappa}-1)}$$

Sombor Index, $SO(Z_{\rho^{\kappa}\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}\sqrt{2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2} + \rho^{2(\kappa-1)}(\delta-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2+(\rho^{\kappa-1}\delta-1)^2} + \rho^{2(\kappa-1)}(\rho-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2+(\rho^{\kappa}-1)^2} + \left(\frac{\rho^{\kappa-1}(\delta-1)}{2}\right)\sqrt{2(\rho^{\kappa-1}\delta-1)^2} + \left(\frac{\rho^{\kappa-1}(\rho-1)}{2}\right)\sqrt{2(\rho^{\kappa}-1)^2}$

$$= \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)}{\sqrt{2}} + \rho^{2(\kappa-1)}(\delta-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2+(\rho^{\kappa-1}\delta-1)^2} + \rho^{2(\kappa-1)}(\rho-1)\sqrt{(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2+(\rho^{\kappa}-1)^2} + \frac{\rho^{\kappa-1}(\delta-1)(\rho^{\kappa-1}(\delta-1)-1)(\rho^{\kappa-1}\delta-1)}{\sqrt{2}} + \frac{\rho^{\kappa-1}(\rho-1)(\rho^{\kappa-1}(\rho-1)-1)(\rho^{\kappa}-1)}{\sqrt{2}}$$

Forgotten Index, $F(Z_{\rho^{\kappa}\delta}) = \frac{\rho^{\kappa-1}(\rho^{\kappa-1}-1)}{2}(2(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2) + \rho^{2(\kappa-1)}(\delta-1)((\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1)^2+(\rho^{\kappa-1}\delta-1)^2) + \rho^{2(\kappa-1)}(\rho -$

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$$
\begin{aligned}
&1)\left(\left(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1\right)^2+\left(\rho^{\kappa}-1\right)^2\right)+\\
&\binom{\rho^{\kappa-1}(\delta-1)}{2}\left(2\left(\rho^{\kappa-1}\delta-1\right)^2\right)+\binom{\rho^{\kappa-1}(\rho-1)}{2}\left(2\left(\rho^{\kappa}-1\right)^2\right)\\
= \; & \rho^{\kappa-1}\left(\rho^{\kappa-1}-1\right)\left(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1\right)^2\\
&+\rho^{2(\kappa-1)}(\delta-1)\left(\left(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1\right)^2\right.\\
&\left.+\left(\rho^{\kappa-1}\delta-1\right)^2\right)\\
&+\rho^{2(\kappa-1)}(\rho-1)\left(\left(\rho^{\kappa-1}\delta+\rho^{\kappa}-\rho^{\kappa-1}-1\right)^2+\left(\rho^{\kappa}-1\right)^2\right)\\
&+\rho^{\kappa-1}(\delta-1)\left(\rho^{\kappa-1}(\delta-1)-1\right)\left(\rho^{\kappa-1}\delta-1\right)^2\\
&+\rho^{\kappa-1}(\rho-1)\left(\rho^{\kappa-1}(\rho-1)-1\right)\left(\rho^{\kappa}-1\right)^2
\end{aligned}
$$

Table 1: Correlation Coefficient of the Degree Based Topological Indices

n	$n-\phi(n)$	M_1	M_2	R	SCI	H	GA	ISI	ABC	SO	F
4	2	2	1	1	0.707	1	1	0.5	0	1.414	2
6	4	18	19	1.893	1.894	1.8	3.825	4.15	2.937	13.912	44
8	4	36	54	2	2.449	2	6	9	1.333	25.455	108
9	3	12	12	1.5	1.5	1.5	3	3	1.06	13.202	24
10	6	90	181	2.841	3.863	2.722	8.72	21.722	6.935	64.652	382
12	8	164	572	3.218	6.147	3.809	9.777	38.734	11.701	155.85	940
14	8	266	799	3.803	6.347	3.673	21.643	65.259	11.902	189.667	1640
15	7	108	220	3.394	4.593	3.3	11.746	25.6	8.105	78.263	488
16	8	392	1372	4	7.483	4	28	98	1.979	277.185	2744
18	12	822	3477	4.166	12.313	5.767	23.559	139.806	22.585	591.162	4700
20	12	908	3058	5.264	11.868	5.707	24.601	108.628	23.507	649.19	7064
Correlation		**0.914524**	**0.89647**	**0.958977**	**0.985076**	**0.99795**	**0.867662**	**0.921662**	**0.917507**	**0.923339**	**0.8789**

Our computational analysis reveals a distinct correlation among the topological indices. From our results six indices- Zagreb Index, Sum Connectivity Index, Harmonic Index, Inverse Sum Indeg Index, Atom Bond Connectivity Index, Sombor Index shows very strong correlation $(r > 0.9)$ indicates high redundancy, allowing one index to represent the cluster for efficient analysis and three indices- Randić Index, Geometric Arithmetic Index, Forgotten Index shows strong correlation $(r > 0.8)$ provides complementary insights, capturing unique structural nuances like degree disparities and hub dominance. This pattern suggests that while all indices reflect underlying graph structure, their sensitivity to specific architectural features varies according to their mathematical construction.

Conclusion

In this paper, we compute degree-based topological indices including the Zagreb Index, Randić Index, Sum Connectivity Index, Harmonic Index, Geometric Arithmetic Index, Inverse Sum Indeg Index, Atom Bond Connectivity Index, Sombor Index and Forgotten Index for greatest common divisor (GCD) sharing graphs corresponding to certain prime powers, products of primes and product of prime powers with correlation of the indices. Our work establishes rigorous mathematical foundations for understanding these algebraic graphs through topological descriptors while providing benchmark results for future investigations extending to entropy-based complexity measures and graph neural network applications in structured algebraic domains.

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