Recurrence to Vanishing Sets in Matrix-Driven Toral Dynamics

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Abstract

We studied recurrence phenomena on the d-dimensional torus generated by linear and nonlinear dynamical mechanisms, with emphasis on sets defined through infinitely many close returns. First, we analyzed fractal subsets of the torus arising from coordinate-wise expanding maps associated with real parameters greater than one, where recurrence is governed by a general decay function. Using geometric and measure-theoretic techniques, we determined the Hausdorff dimension of these sets and identified how it depends on the expansion rates and the rate of decay of the target sizes.

We then introduced a unified framework that connects shrinking target problems with quantitative recurrence for toral transformations induced by real, non-singular matrices. Within this setting, we considered sequences of anisotropic target neighborhoods whose sizes vary independently along each coordinate direction. The targets were allowed to move dynamically according to uniformly Lipschitz vector-valued functions, leading to non-stationary and non-symmetric recurrence conditions. For this general class of moving hyperrectangular targets, we established precise dimension formulae for the associated limsup sets. These results reveal a robust relationship between matrix-driven toral dynamics, geometric shrinking mechanisms, and the fractal structure of recurrent orbits.

1 Introduction

Let $(\mathcal{X}, \rho)$ be a compact metric space and let $F: \mathcal{X} \to \mathcal{X}$ be a measurable transformation. Assume that there exists a Borel probability measure ν on $\mathcal{X}$ which is invariant and ergodic with respect to F. Classical results in ergodic theory ensure that for any measurable set $\mathcal{A} \subset \mathcal{X}$ with $\nu(\mathcal{A}) > 0$, the set

$$\mathcal{E}(\mathcal{A}) := \{x \in \mathcal{X}: F^k(x) \in \mathcal{A} \quad \text{forinfinitelymany} k \in \mathbb{N}\}$$

has full ν-measure. This expresses the recurrent nature of almost all orbits in measure-preserving systems [1].

A natural refinement of this observation arises when the target set is allowed to vary along the orbit. Given a function $\varphi: \mathbb{N} \to \mathbb{R}^+$ and a reference point $y \in \mathcal{X}$, one may study the limsup set

$$\mathcal{S}(\varphi, y) := \{x \in \mathcal{X}: \rho(F^k(x), y) < \varphi(k) \quad \text{forinfinitelymany} k \in \mathbb{N}\},$$

which captures the frequency of visits of trajectories to shrinking neighborhoods centered at y. Problems of this type are commonly referred to as shrinking target problems and play a central role in modern metric dynamics.

Another viewpoint focuses on self-recurrence, where the centers of the shrinking neighborhoods depend on the initial point itself. This leads to the quantitative recurrence set [2].

$$\mathcal{R}(\varphi) := \{x \in \mathcal{X} : \rho(F^k(x), x) < \varphi(k) \quad \text{for infinitely many } k \in \mathbb{N}\}.$$

Despite their different formulations, the sets $\mathcal{S}(\varphi, y)$ and $\mathcal{R}(\varphi)$ often exhibit parallel geometric and dimensional behavior in a wide range of dynamical systems.

In this paper, we investigate such recurrence phenomena in the setting of toral dynamics generated by linear actions. Let

$$\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$$

denote the d-dimensional torus endowed with the metric induced by the Euclidean norm. Let A be a non-singular $d \times d$ matrix with real entries, which induces a transformation $F_A : \mathbb{T}^d \to \mathbb{T}^d$ defined by [3].

$$F_A(x) = Ax (\mathrm{mod} \mathbb{Z}^d).$$

We write F_A^k for the k-th iterate of this map.

To describe anisotropic shrinking behavior, let $\boldsymbol{\phi}(k) = (\phi_1(k), \ldots, \phi_d(k))$ be a vector of positive functions on $\mathbb{N}$. For $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$, we define the rectangular neighborhood [4].

$$\mathcal{Q}(z, \boldsymbol{\phi}(k)) := \{x \in \mathbb{T}^d : |x_i - z_i| < \phi_i(k) \text{ for all } 1 \le i \le d\},$$

as well as the multiplicative neighborhood

$$\mathcal{M}(z, \phi(k)) := \left\{x \in \mathbb{T}^d : \prod_{i=1}^d |x_i - z_i| < \phi(k)\right\}.$$

The corresponding multiplicative recurrence set is given by

$$\mathcal{H}(A, \phi) := \{x \in \mathbb{T}^d : F_A^k(x) \in \mathcal{M}(x, \phi(k)) \quad \text{for infinitely many } k \in \mathbb{N}\},$$

which is closely related to problems arising in multiplicative Diophantine approximation [5].

To unify shrinking target problems and quantitative recurrence, we introduce a moving-target framework. Let $\{g_k\}_{k \ge 1}$ be a sequence of vector-valued maps on $\mathbb{T}^d$ satisfying a uniform Lipschitz condition: there exists $L > 0$ such that

$$\| g_k(u) - g_k(v) \| \le L \| u - v \| \qquad \text{for all } u, v \in \mathbb{T}^d \text{ and all } k \in \mathbb{N}.$$

We then define the modified shrinking target set

$$\mathcal{W}(A, \boldsymbol{\phi}, \{g_k\}) := \{x \in \mathbb{T}^d : F_A^k(x) \in \mathcal{Q}(g_k(x), \boldsymbol{\phi}(k)) \quad \text{for infinitely many } k \in \mathbb{N}\}.$$

The primary aim of this work is to determine the Hausdorff dimension of the sets $\mathcal{H}(A, \phi)$ and $\mathcal{W}(A, \boldsymbol{\phi}, \{g_k\})$ under suitable assumptions on the matrix A and the decay rates of the functions ϕ_i. Our results reveal that the resulting dimension depends only on the spectral properties of A and the asymptotic behavior of the shrinking functions, and is independent of the particular choice of the Lipschitz sequence $\{g_k\}$ [6].

The paper is organized as follows. Section 2 collects preliminary material and auxiliary results. Section 3 addresses multiplicative recurrence sets. Section 4 is devoted to the modified shrinking target framework and the proof of the main dimension formulae. Extensions to broader classes of matrix actions are discussed in Section 5.

2 Preliminaries

This section collects the basic tools and auxiliary results that will be used throughout the paper. We begin with a brief overview of expanding maps on the unit interval and their digit expansions, then introduce symbolic partitions, geometric properties of cylinders, and several dimension-theoretic lemmas [7,8].

2.1 Expanding interval maps and digit representations

Let $\alpha > 1$ be a real number. We define the expanding transformation
$$S_\alpha: [0,1) \longrightarrow [0,1), \qquad S_\alpha(x) = \alpha x - \lfloor \alpha x \rfloor,$$
where $\lfloor\cdot\rfloor$ denotes the integer part. The pair $([0,1), S_\alpha)$ will be referred to as an α–expanding dynamical system.

A fundamental property of this system is that every point $x \in [0,1)$ admits a unique representation as a finite or infinite series of the form [9].
$$x = \sum_{k=1}^{\infty} \frac{d_k(x)}{\alpha^k}, \tag{2}$$
where the digits are defined recursively by
$$d_k(x) := \lfloor \alpha \ \ S_\alpha^{k-1}(x) \rfloor, \qquad k \geq 1.$$
The sequence
$$\mathbf{d}(x) = (d_1(x), d_2(x), \dots)$$
is called the α–expansion of x. Occasionally, we write $x = (d_1(x), d_2(x), \dots)$ to emphasize the symbolic nature of the expansion.

Each digit satisfies [10].
$$d_k(x) \in \mathcal{D}_\alpha := \{0,1, \dots, \lceil \alpha - 1 \rceil\}.$$
However, not every sequence in $\mathcal{D}_\alpha^{\mathbb{N}}$ corresponds to the expansion of a point in $[0,1)$. This motivates the following notion.

2.2 Admissible words and symbolic complexity

A finite or infinite word $(b_1, b_2, \dots)$ over the alphabet $\mathcal{D}_\alpha$ is called *admissible* if there exists a point $x \in [0,1)$ whose α–expansion begins with this word. For $m \geq 1$, we denote by $\mathcal{W}_\alpha^m$ the set of all admissible words of length m.

The combinatorial complexity of admissible words is controlled by the following classical estimate [11].

Let $\alpha > 1$. Then for every integer $m \geq 1$,
$$\alpha^m \leq \#\mathcal{W}_\alpha^m \leq \frac{\alpha^{m+1}}{\alpha - 1},$$
where $\#$ denotes cardinality.

2.3 Cylinder sets and geometric properties

Given an admissible word $\mathbf{b} = (b_1, \dots, b_m) \in \mathcal{W}_\alpha^m$, we define the associated *cylinder set* of order m by
$$\mathcal{I}_{m,\alpha}(\mathbf{b}) := \{x \in [0,1) : d_k(x) = b_k \text{for} 1 \leq k \leq m\}.$$
Each cylinder is a left-closed, right-open interval in $[0,1)$ and satisfies
$$|\mathcal{I}_{m,\alpha}(\mathbf{b})| \leq \alpha^{-m},$$
where $|\cdot|$ denotes interval length.

A cylinder $\mathcal{I}_{m,\alpha}(\mathbf{b})$ is said to be *maximal* if
$$|\mathcal{I}_{m,\alpha}(\mathbf{b})| = \alpha^{-m}.$$

Maximal cylinders play a key role in our construction of covers and Cantor-type subsets.

For every $m \geq 1$, among any $m + 2$ consecutive cylinders of order m, there exists at least one maximal cylinder [12].

2.4 Approximation within cylinders

Let $\{h_k\}_{k\geq 1}$ be a sequence of functions on the circle $\mathbb{T}=[0,1)$ satisfying a uniform Lipschitz condition. That is, there exists a constant $L>0$ such that [14].

$$|h_k(u)-h_k(v)|\leq L|u-v| \qquad \text{for all} u,v\in\mathbb{T} \text{and} k\geq 1. \tag{3}$$

Let $\mathcal{I}_{m,\alpha}(\mathbf{b})$ be a maximal cylinder of order m and let $\varepsilon>0$. Then there exists a point $z_{m,\mathbf{b}}\in\mathcal{I}_{m,\alpha}(\mathbf{b})$ such that

$$|S_\alpha^m(z_{m,\mathbf{b}})-h_m(z_{m,\mathbf{b}})|<\varepsilon.$$

2.5 Bi-Lipschitz invariance of dimension

We recall a standard fact concerning the stability of Hausdorff dimension under bi-Lipschitz mappings.

Let $E\subset\mathbb{R}^k$ and let $\Phi: E\to\mathbb{R}^k$ be a bi-Lipschitz map. That is, there exist constants $0<a\leq b<\infty$ such that [15].

$$a|x-y|\leq|\Phi(x)-\Phi(y)|\leq b|x-y| \qquad \text{for all} x,y\in E.$$

Then

$$\dim_H\Phi(E)=\dim_H E.$$

2.6 Mass transference for rectangular targets

Let $\mathbb{T}^d$ denote the d–dimensional torus equipped with Lebesgue measure λ_d. For each $n\geq 1$ and $1\leq i\leq d$, let $z_{i,n}\in\mathbb{T}$ and $r_{i,n}>0$ with $r_{i,n}\to 0$ as $n\to\infty$.

Consider the limsup set [16].

$$\mathcal{L}:=\limsup_{n\to\infty}\prod_{i=1}^d B(z_{i,n},r_{i,n}),$$

where $B(z,r)$ denotes the ball of radius r centered at z.

Assume that for some exponents $a_i\geq 0$,

$$\lambda_d\left(\limsup_{n\to\infty}\prod_{i=1}^d B(z_{i,n},r_{i,n}^{a_i})\right)=\lambda_d(\mathbb{T}^d).$$

Then the Hausdorff dimension of $\mathcal{L}$ admits the lower bound

$$\dim_H\mathcal{L}\geq\min_{q\in\mathcal{Q}}s(q),$$

where $\mathcal{Q}=\{u_i+v_i: 1\leq i\leq d\}$ and

$$s(q)=1+\sum_{i\in I_1(q)}\left(1-\frac{v_i}{q}\right)+\sum_{i\in I_3(q)}\frac{u_i}{q},$$

with the index sets defined by

$$I_1(q)=\{i: u_i\geq q\}, \qquad I_2(q)=\{i: u_i+v_i\leq q\}\backslash I_1(q), \qquad I_3(q)=\{1,\dots,d\}\backslash(I_1(q)\cup I_2(q)).$$

2.7 Geometric coverings for multiplicative constraints

Finally, we introduce a geometric construction that will be crucial for estimating Hausdorff dimension from above. For $\eta > 0$, define [17].

$$\mathcal{K}_\eta := \{y = (y_1, \ldots, y_d) \in [0,1]^d : \prod_{i=1}^{d} y_i \le \eta\}.$$

Let $d \ge 2$ and $s \in (d-1, d)$. For sufficiently small $\eta > 0$, the set $\mathcal{K}_\eta$ admits a cover by d–dimensional cubes $\{Q\}$ such that

$$\sum_Q |Q|^s \ll \eta^{\ s-d+1},$$

where $|Q|$ denotes the side length of Q and the implied constant is independent of η. With the notation introduced above, the following inclusion holds:

$$\mathcal{E}_n(\delta) \subset \bigcup_{Q \in \mathcal{Q}} \quad \bigcup_{\mathbf{w}_1 \in \mathcal{W}_{\alpha_1}^n, \ldots, \mathbf{w}_d \in \mathcal{W}_{\alpha_d}^n} \prod_{i=1}^{d} \mathcal{E}_{n,\alpha_i}(\mathbf{w}_i; a_Q, b_Q), \tag{4}$$

where $\mathcal{Q}$ is the family of cubes provided by Lemma 3.7.

Proof. Let $x = (x_1, \ldots, x_d) \in \mathcal{E}_n(\delta)$. By definition,

$$(|S_{\alpha_1}^n(x_1) - x_1|, \ldots, |S_{\alpha_d}^n(x_d) - x_d|) \in \mathcal{K}_\delta.$$

Lemma 3.7 guarantees the existence of a cube $Q = [a_Q, b_Q]^d \in \mathcal{Q}$ such that

$$a_Q \le |S_{\alpha_i}^n(x_i) - x_i| \le b_Q \qquad \text{forall} 1 \le i \le d.$$

For each coordinate x_i, the digit expansion determines a unique admissible word $\mathbf{w}_i \in \mathcal{W}_{\alpha_i}^n$ such that $x_i \in \mathcal{I}_{n,\alpha_i}(\mathbf{w}_i)$. Hence, [18].

$$x \in \prod_{i=1}^{d} \{y_i \in [0,1) : a_Q \le |S_{\alpha_i}^n(y_i) - y_i| \le b_Q\} = \prod_{i=1}^{d} \mathcal{E}_{n,\alpha_i}(\mathbf{w}_i; a_Q, b_Q),$$

which proves the claimed inclusion.

2.8 Proof of Theorem 1.1

Recall that the multiplicative recurrence set is defined by

$$\mathcal{H}_d(A, \phi) := \{x \in \mathbb{T}^d : \prod_{i=1}^{d} |S_{\alpha_i}^n(x_i) - x_i| \le \phi(n) \quad \text{forinfinitelymany} n \in \mathbb{N}\}.$$

Let

$$\mathcal{E}_n(\phi) := \{x \in \mathbb{T}^d : \prod_{i=1}^{d} |S_{\alpha_i}^n(x_i) - x_i| < \phi(n)\}.$$

Then

$$\mathcal{H}_d(A, \phi) = \limsup_{n\to\infty} \mathcal{E}_n(\phi).$$

As usual in dimension theory, the proof is divided into two parts: an upper bound and a matching lower bound [19].

Upper bound

Fix $s \in (d-1, d)$. For n sufficiently large, we apply Lemma 3.7 with $\delta = \phi(n)$ to obtain a collection $\mathcal{Q}$ of cubes such that

$$\sum_{Q \in \mathcal{Q}} |Q|^s \ll \phi(n)^{\ s-d+1}. \tag{5}$$

Combining Lemmas ?? and 3.7, each set $\mathcal{E}_{n,\alpha_i}(\mathbf{w}_i; a_Q, b_Q)$ can be covered by at most four intervals of length $(b_Q - a_Q)\alpha_i^{-n}$. Consequently,

$$\prod_{i=1}^{d} \mathcal{E}_{n,\alpha_i}(\mathbf{w}_i; a_Q, b_Q)$$

can be covered by at most 4^d rectangles with side lengths $|Q|\alpha_i^{-n}$.

Since $\alpha_i^{-n} \geq \alpha_d^{-n}$ for all i, each such rectangle can be covered by balls of diameter $\alpha_d^{-n}|Q|$. The number of balls required is bounded by

$$C \quad \prod_{i=1}^{d} \frac{\alpha_i^{-n}}{\alpha_d^{-n}} = C \quad \prod_{i=1}^{d} \alpha_i^{-n} \alpha_d^{n}.$$

Let $\varepsilon > 0$. For n sufficiently large, $\alpha_d^{-n}|Q| < \varepsilon$. By the definition of Hausdorff measure,

$$\mathcal{H}_\varepsilon^s(\mathcal{H}_d(A,\phi)) \ll \sum_n \sum_{Q\in\mathcal{Q}} \prod_{i=1}^{d} \#\mathcal{W}_{\alpha_i}^n \cdot \prod_{i=1}^{d} \alpha_i^{-n}\alpha_d^n \cdot (\alpha_d^{-n}|Q|)^s.$$

Using Lemma 3.2 and (5), we deduce that [20]

$$\mathcal{H}^s(\mathcal{H}_d(A,\phi)) = 0 \qquad \text{whenever} \qquad s > d - 1 + \frac{\log\alpha_d}{\log\alpha_d + \tau},$$

where

$$\tau := \liminf_{n\to\infty} \frac{-\log\phi(n)}{n}.$$

Hence,

$$\dim_H \mathcal{H}_d(A,\phi) \leq d - 1 + \frac{\log\alpha_d}{\log\alpha_d + \tau}.$$

Lower bound

Let

$$\mathcal{H}_1(\alpha_d,\phi) := \{x \in \mathbb{T} : |S_{\alpha_d}^n(x) - x| \leq \phi(n) \text{forinfinitelymany} n\}.$$

Standard one-dimensional results yield

$$\dim_H \mathcal{H}_1(\alpha_d,\phi) = \frac{\log\alpha_d}{\log\alpha_d + \tau}.$$

For any $x_d \in \mathcal{H}_1(\alpha_d,\phi)$, we have

$$[0,1)^{d-1} \times \{x_d\} \subset \mathcal{H}_d(A,\phi).$$

Applying the slicing lemma for Hausdorff dimension, we obtain

$$\dim_H \mathcal{H}_d(A,\phi) \geq (d-1) + \dim_H \mathcal{H}_1(\alpha_d,\phi) = d - 1 + \frac{\log\alpha_d}{\log\alpha_d + \tau}.$$

Combining the upper and lower bounds completes the proof of Theorem 1.1. [21].

3 Proof of Theorem 1.2 and Corollary 1.4

In this section we analyze the modified shrinking target set

$$\mathcal{W}(A,\boldsymbol{\phi},\{g_n\}) = \left\{x \in \mathbb{T}^d : |S_{\alpha_i}^n(x_i) - g_n^{(i)}(x)| \leq \phi_i(n) \text{forall} 1 \leq i \leq d \text{infinitelyoften}\right\},$$

where $A = \operatorname{diag}(\alpha_1,\ldots,\alpha_d)$ with $\alpha_i > 1$. The proofs of the upper and lower bounds are presented separately in the following subsections, leading to the proof of Corollary 1.4 as a direct consequence [22].

3.1 Upper bound for the Hausdorff dimension

Let $\mathcal{C}(\boldsymbol{\phi})$ denote the set of accumulation points

$$\mathbf{t} = (t_1,\ldots,t_d) \qquad \text{ofthesequence} \qquad \left\{\left(\frac{-\log\phi_1(n)}{n},\ldots,\frac{-\log\phi_d(n)}{n}\right)\right\}_{n\geq 1}.$$

We begin by assuming that $\mathcal{C}(\boldsymbol{\phi})$ consists of a single point $\mathbf{t}$, that is,

$$\lim_{n\to\infty} \frac{-\log\phi_i(n)}{n} = t_i, \qquad 1 \leq i \leq d.$$

For each $n \geq 1$ and each admissible word $\mathbf{w}_i \in \mathcal{W}_{\alpha_i}^n$, define the one–dimensional target set

$$\mathcal{J}_{n,i}(\mathbf{w}_i) := \left\{x \in \mathcal{I}_{n,\alpha_i}(\mathbf{w}_i) : |S_{\alpha_i}^n(x) - g_n^{(i)}(x)| < \phi_i(n)\right\}.$$

By construction, the modified shrinking target set can be written as

$$\mathcal{W}(A, \boldsymbol{\phi}, \{g_n\}) = \limsup_{n\to\infty} \bigcup_{\mathbf{w}_1,\dots,\mathbf{w}_d} \mathcal{J}_{n,1}(\mathbf{w}_1) \times \cdots \times \mathcal{J}_{n,d}(\mathbf{w}_d), \tag{6}$$

where the union is taken over all admissible words $\mathbf{w}_i \in \mathcal{W}_{\alpha_i}^n$.

Let $x, x' \in \mathcal{J}_{n,i}(\mathbf{w}_i)$. Using the uniform Lipschitz property of $\{g_n\}$, we obtain [23].

$$2\phi_i(n) \geq |S_{\alpha_i}^n(x) - g_n^{(i)}(x)| + |S_{\alpha_i}^n(x') - g_n^{(i)}(x')|$$

$$\geq |S_{\alpha_i}^n(x) - S_{\alpha_i}^n(x')| - |g_n^{(i)}(x) - g_n^{(i)}(x')| \geq (\alpha_i^n - L)|x - x'|.$$

For n sufficiently large, this yields the diameter estimate

$$|\mathcal{J}_{n,i}(\mathbf{w}_i)| \leq 4 \;\; \phi_i(n) \;\; \alpha_i^{-n}, \qquad\qquad 1 \leq i \leq d. \tag{7}$$

Fix $N \geq 1$. From (6), we have

$$\mathcal{W}(A, \boldsymbol{\phi}, \{g_n\}) \subset \bigcup_{n=N}^{\infty} \mathcal{D}_n,$$

where

$$\mathcal{D}_n := \bigcup_{\mathbf{w}_1,\dots,\mathbf{w}_d} \mathcal{J}_{n,1}(\mathbf{w}_1) \times \cdots \times \mathcal{J}_{n,d}(\mathbf{w}_d).$$

Using (7) and Lemma 3.2, one constructs a cover of $\mathcal{D}_n$ by hyperrectangles whose side lengths are comparable to $\phi_i(n)\alpha_i^{-n}$. Standard covering arguments then imply that

$$\dim_H \mathcal{W}(A, \boldsymbol{\phi}, \{g_n\}) \leq \min_{1\leq i\leq d} \lambda_i(\mathbf{t}),$$

where the functions $\lambda_i(\mathbf{t})$ are defined explicitly in terms of $\mathbf{t}$ and the eigenvalues α_i.

3.2 Lower bound for the Hausdorff dimension

We now establish the complementary lower bound

$$\dim_H \mathcal{W}(A, \boldsymbol{\phi}, \{g_n\}) \geq \sup_{\mathbf{t}\in\mathcal{C}(\boldsymbol{\phi})} \min_{1\leq i\leq d} \lambda_i(\mathbf{t}). \tag{8}$$

Let

$$\overline{\mathcal{W}}_{\alpha_i}^n := \{\mathbf{u} \in \mathcal{W}_{\alpha_i}^n : \mathcal{I}_{n,\alpha_i}(\mathbf{u}) \quad \text{ismaximal}\}.$$

For each maximal cylinder $\mathcal{I}_{n,\alpha_i}(\mathbf{w}_i)$, Lemma 3.4 provides a point $z_{n,\mathbf{w}_i} \in \mathcal{I}_{n,\alpha_i}(\mathbf{w}_i)$ such that [24].

$$|S_{\alpha_i}^n(z_{n,\mathbf{w}_i}) - g_n^{(i)}(z_{n,\mathbf{w}_i})| < \frac{1}{2}\phi_i(n).$$

For n sufficiently large and for any $x \in \mathcal{I}_{n,\alpha_i}(\mathbf{w}_i)$ satisfying $|x - z_{n,\mathbf{w}_i}| \leq \frac{1}{4}\alpha_i^{-n}\phi_i(n)$, we obtain

$$|S_{\alpha_i}^n(x) - g_n^{(i)}(x)| \leq (\alpha_i^n + L)|x - z_{n,\mathbf{w}_i}| + \frac{1}{2}\phi_i(n) \leq \phi_i(n).$$

Hence each $\mathcal{J}_{n,i}(\mathbf{w}_i)$ contains a ball of radius comparable to $\alpha_i^{-n}\phi_i(n)$.

It follows that [25].

$$\mathcal{W}(A, \boldsymbol{\phi}, \{g_n\}) \supset \limsup_{n\to\infty} \prod_{i=1}^{d} \bigcup_{\mathbf{w}_i\in\overline{\mathcal{W}}_{\alpha_i}^n} B\big(z_{n,\mathbf{w}_i}, c \;\; \alpha_i^{-n}\phi_i(n)\big),$$

for a suitable constant $c > 0$. Using Lemma 3.3 together with the mass transference principle (Lemma 3.6), we obtain the desired lower bound (8) [26].

3.3 Proof of Corollary 1.4

Assume now that

$$\phi_1 = \phi_2 = \cdots = \phi_d = \phi.$$

Then every accumulation point $\mathbf{t} \in \mathcal{C}(\boldsymbol{\phi})$ has the form $\mathbf{t} = (t, \ldots, t)$. If

$$\tau := \liminf_{n\to\infty} \frac{-\log\phi(n)}{n} < \infty,$$

the set $\mathcal{C}(\boldsymbol{\phi})$ is bounded, and Theorem 1.2 yields

$$\dim_H \mathcal{W}(A, \phi, \{g_n\}) = \min_{1\le i\le d} \lambda_i(\tau),$$

where

$$\lambda_i(t) = \sum_{\alpha_j > \alpha_i e^t} 1 + \sum_{\alpha_j \le \alpha_i} \frac{\log\alpha_i}{\log\alpha_i + t} + \sum_{\alpha_i e^t \ge \alpha_j > \alpha_i} \frac{\log\alpha_j}{\log\alpha_i + t}.$$

Since each $\lambda_i(t)$ is decreasing in t, this expression simplifies to the explicit formula stated in Corollary 1.4.

If $\tau = \infty$, we compare ϕ with the exponential functions $\phi_M(n) = e^{-Mn}$. For M sufficiently large, [27].

$$\mathcal{W}(A, \phi, \{g_n\}) \subset \mathcal{W}(A, \phi_M, \{g_n\}),$$

which implies

$$0 \le \dim_H \mathcal{W}(A, \phi, \{g_n\}) \le \min_{1\le i\le d} \lambda_i(M).$$

Letting $M \to \infty$ yields

$$\dim_H \mathcal{W}(A, \phi, \{g_n\}) = 0,$$

and the proof of the corollary is complete.

4 Proof of Theorem 1.6

Recall that the modified shrinking target set is given by

$$\mathcal{W}(A, \phi, \{g_n\}) = \{x \in \mathbb{T}^d : A^n(x) \in B(g_n(x), \phi(n)) \text{forinfinitelymany} n \in \mathbb{N}\}.$$

Assume that A is an integer matrix diagonalizable over $\mathbb{Z}$ with eigenvalues $1 < \alpha_1 \le \cdots \le \alpha_d$. Then there exist a diagonal integer matrix Λ and a toral automorphism $\Phi: \mathbb{T}^d \to \mathbb{T}^d$ such that

$$\Phi \circ A = \Lambda \circ \Phi.$$

Moreover, Φ is bi-Lipschitz: there exist constants $0 < m \le M < \infty$ such that

$$m|x - y| \le |\Phi(x) - \Phi(y)| \le M|x - y|, \qquad x, y \in \mathbb{T}^d. \tag{9}$$

Define the auxiliary sets [28].

$$\mathcal{S} := \{x \in \mathbb{T}^d : A^n(x) \in B(g_n(\Phi(x)), \frac{M}{m}\phi(n)) \text{i.m.} n\},$$

and

$$\mathcal{S}_1 := \{y \in \mathbb{T}^d : \Lambda^n(y) \in B(\Phi(g_n(y)), \frac{M^2}{m}\phi(n)) \text{i.m.} n\}.$$

Since $\{g_n\}$ is uniformly Lipschitz and Φ is bi-Lipschitz, the sequence $\{\Phi \circ g_n\}$ is also uniformly Lipschitz. By Corollary 1.4, we obtain

$$\dim_H \mathcal{S} = \dim_H \mathcal{S}_1 = \min_{1\le i\le d} \left(\frac{i\log\alpha_i - \sum_{\alpha_j > \alpha_i e^\tau} (\log\alpha_j - \log\alpha_i - \tau) + \sum_{j>i} \log\alpha_j}{\tau + \log\alpha_i} \right),$$

where $\tau = \liminf_{n\to\infty} (-\log\phi(n))/n$.

We claim that

$$\mathcal{S}_1 \subset \Phi(\mathcal{S}) \subset \mathcal{S}.$$

Indeed, if $y = \Phi(x)$ with $x \in \mathcal{S}$, then using $\Phi \circ A = \Lambda \circ \Phi$ and (9), we obtain

$$|\Lambda^n(y) - \Phi(g_n(y))| \le M|A^n(x) - g_n(\Phi(x))| \le \frac{M^2}{m}\phi(n)$$

for infinitely many n, hence $y \in \mathcal{S}_1$. Conversely, if $y = \Phi(x) \in \mathcal{S}_1$, the lower bound in (9) implies [29].

$$|A^n(x) - g_n(\Phi(x))| \le \frac{M}{m}\phi(n),$$

so $x \in \mathcal{S}$ and $y \in \Phi(\mathcal{S})$.

Therefore,

$$\dim_H \mathcal{W}(A, \phi, \{g_n\}) = \dim_H \Phi(\mathcal{S}) = \dim_H \mathcal{S},$$

and the proof is complete.

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