Meshfree Full Approximation Scheme-Full multilevel algorithm for Convection-dominated problems

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Abstract

This paper presents an efficient full approximation scheme-full multilevel (FAS-FML) algorithm for the radial basis function-based finite difference (RBF-FD) method. The algorithm produces an accurate solution by solving the discretized equations from the coarsest level to the coarser level and then to the finer level to desire the finest level node points. Convection-dominated problems are taken to demonstrate the validity of the algorithm. The proposed algorithm saves a minimum 66% of computation time for the convection-dominated problems than the general RBF-FD method. The convergence conditions of the method were verified numerically. Outcomes prove that the developed algorithm concurs with exact solutions in the measure of accuracy and efficiency.

Keywords: FAS-FML algorithm, meshfree method, Convection dominated problems, CPU time, convergence of the method.

AMS Subject classifications: 65Y04; 65Y15; 35Q35

1. INTRODUCTION

Many practical applications in which convection plays a significant role, such as modeling of shallow waters, magneto-hydrodynamics, electromagnetism, meteorology, weather forecasting, oceanography, gas dynamics, oil recovery simulation, and many more. Convection-dominated flows may be the cause of unphysical oscillations. This is why efficient and accurate numerical methods are required for solving these problems. Over the last several decades, meshfree methods have gained a great deal of attention from both the scientific and engineering research communities for the interpolation of multivariate scattered data. These methods have the advantage of coping with changes in the geometry of the domain of interest compared with traditional discretization methods such as FEM, FVM, and FDM. The automatic generation of a good-quality mesh poses a significant problem in the analysis of practical engineering systems. An obvious benefit of the meshfree methods is that they are free from mesh creation, and mesh creation is still a time-consuming effort in complex domains. The mesh generation cost is automatically terminated because of arbitrary node points used to discretize the domain of meshfree methods. So all research work related to meshfree methods is associated between mathematics and interdisciplinary applications. Recently, researchers have concentrated more on mesh-free methods such as the Galerkin method, the smoothed particle hydrodynamics method, the local radial basis function (RBF-FD) method, and the finite point method to find an approximation solution of differential equations. RBF methods are meshfree in nature and depend on only the distance between the arbitrary scattered points in the domain. So the methods can be used for arbitrary-shaped bodies with higher dimensions in different geometry [1-5]. Inverse multiquadrics, Gaussian, and multiquadrics are commonly used RBFs for solving differential equations. At the same time, the methods are not computationally efficient [6-11]. A numerical scheme is said to be efficient with respect to other numerical schemes if the scheme achieves the same level of accuracy in less computation time. These methods are inefficient due to the calculation of the square root in the function, the inverse of matrices during the calculation of derivatives, and the need the need to append more numbers of points for higher order accuracy. The band width of matrices will increase due to the increased number of points, which will slow down the procedure of finding solutions. Therefore, the RBF method became inefficient, although it achieved good order of accuracy. Some researchers [12-13] combine mesh-free methods with mesh-based methods to make an efficient scheme.

Hosseini and Hashemi [14] developed a local RBF collocation method for the Burgers' equations. It was also applied to Navier-Stokes equations in curvilinear geometry with unbounded domain [4]. The meshfree RBF-FD method has been successfully used in some applications [15-18]. Kolodziej et al. [19] implemented RBF with the method of fundamental solutions (MFS) to solve a heat source problem for arbitrary domains. Sun et al. [20] developed compact support radial basis functions to solve coupled radiative and conductive heat transfer problems. Zhao et al. [21] solved the convection-diffusion-reaction equation on implicit surfaces using the RBF-FD method by transforming it into a diffusion-type equation. Recently, Barik and Sekhar [11] proposed an efficient RBF-FD method without the help of other mesh-based methods. In our work, a mesh-free RBF-FD FAS-FML algorithm has been developed for convection-dominated problems.

2. Methodology for meshfree FAS-FML RBF-FD method

Radial basis functions are well known for approximation of the multivariate scattered data. The RBF interpolation is defined by the linear combination of translated radial basis functions. The RBF interpolation is defined by

$$w(x) \approxeq s(x) = \sum_{i=1}^{m} \lambda_i \Phi(||x - x_i||) + \sum_{j=1}^{n} \beta_j P_j(x), x \epsilon \mathbb{R}^d \tag{1}$$

where $||x - x_i||$ is the Euclidean norm of the difference between x and x_i, $P_j(x), j = 1,2, \ldots N$ is a basis for $\Pi_n(\mathbb{R}^d)$ and $s(x_i) = w(x_i)$. Orthogonal conditions have considered to solve the resultant linear system. λ_i and β_j can be determined by solving the following symmetric linear system.

$$\left(\begin{array}{c|c} C & P \\ \hline P^T & 0 \end{array}\right) \left(\begin{array}{c} \lambda \\ \hline \beta \end{array}\right) = \left(\begin{array}{c} \mathrm{G} \\ \hline 0 \end{array}\right) \tag{2}$$

where $C = [c_{ki}]_{nxn}$ and $P = [P_j(x_i)]_{mxN}, i = 1,2, \ldots n, k = 1,2, \ldots, N$

The Lagrange form of RBF interpolant is defined in the form

$$s(x) = \sum_{i=1}^{m} \Psi_i(x) w(x_i) \tag{3}$$

where $\Psi_i(x)$ satisfies the Dirac delta conditions.
The derivative of a function w at a given point x_k is denoted by $l(w(x_k))$ and defined by the help of neighborhood points of x_k (say m_k nodes $x_1, x_2, \dots, x_{m_k}$) is

$$l(w(x_k)) \approxeq \sum_{i=1}^{m_k} d_{ki} w(x_i),\ k = 1,2,\dots,m \tag{4}$$

By using (3)

$$l\big(w(x_k)\big) \approxeq l\big(s(x_k)\big) = \sum_{i=1}^{m_k} l(\Psi_i(x_k)) w(x_i), \tag{5}$$

From (4) and (5)

$$d_{ki} = l\big(\Psi_i(x_k)\big),\ i = 1,2,\dots,m_k,\ \ k = 1,2,\dots,m.$$

The weights are calculated by solving the linear system:

$$\left(\begin{array}{c|c} C & P \\ \hline P^T & 0 \end{array}\right)_i \left(\begin{array}{c} \mathbf{D} \\ \hline \mu \end{array}\right)_i = \left(\begin{array}{c} (l(B(\mathbf{x})))^T \\ \hline 0 \end{array}\right)_i \tag{6}$$

Where C is as (2), and $B(x) = [\Phi_1 \Phi_2 \dots \Phi_m | P_1(x) P_2(x) \dots P_n(x)]$.

By solving $\quad C'W = G \quad$ (7)

where $C' = [C_1', C_2', \dots, C_N']^T$ and each C_k' $(k = 1,2,\dots,N)$ is a row vector for kth internal points. $W = [w_1, w_2, \dots, w_N]^T$ and G are the column vectors. N is the total number of internal points in the domain.

The RBF-FD method can be made effective and efficient by dividing the domain into different levels. Now we are presenting a multilevel FAS-FML algorithm based on the different levels.

2.1 Meshfree FAS-FML algorithm

Consider the system of linear equations $\mathrm{DU} = \mathrm{F}$ after discretized by RBF-FD method.
The variables used in flowchart are listed below.

$\Omega_k \rightarrow$ Domain of kth level mesh.

$P_{k-1}^{k} \rightarrow$ Prolongation operator from $(k-1)th$ level to kth level.

$D_{\Omega_k} \rightarrow$ Coefficient matrix of the system of linear equations in kth level.
$F_{\Omega_k} \rightarrow$ Right-hand side matrix of the system of linear equations in kth level.
$U_{\Omega_k} \rightarrow$Convergent solutions of the system of linear equations at kth level.
$\Theta_{\Omega_k} \rightarrow$ Starting Solution of system of equations at additional points in kth level.
$U^*_{\Omega_k} \rightarrow$ Starting solution at kth level.
$N_{\Omega_k} \rightarrow$Number of points in kth level mesh

Prolongation operator is required to pass the solution from coarsest level to coarser level and then finer level to desired finest level. Let U_{Ω_k} be the kth level accurate solution, $k = 1,2,\dots,m$.

The Prolongation operator denoted by

$$P^k_{k-1}: \Omega_{k-1} \rightarrow \Omega_k$$

and defined $$P^k_{k-1}\left(U_{\Omega_{k-1}}\right) = U_{\Omega_k}$$

Algorithm

Step-1 Solve the system of linear equations $D_{\Omega_1} U_{\Omega_1} = F_{\Omega_1}$ obtained from discretization of the partial differential equation using the RBF-FD method until you get the convergent solution of the system at the coarsest level (using Gauss-Seidel iterative method). Let the convergent solution be U_{Ω_1}.

Step-2 Prolongate the solution U_{Ω_1} from coarsest level to next level and calculate the solution at additional nodes Θ_{Ω_2} by RBF-FD interpolation.

Step-3 Solve the system of linear equations in the second level by considering $U^*_{\Omega_2} = U_{\Omega_1} + \Theta_{\Omega_2}$ as the starting solution for level 2 to get the convergent solution U_{Ω_2}.

Step-4 Prolongate the solution $U_{\Omega_{k-1}}$ from $(k-1)th$ level to kth level and calculate the solution at additional nodes (Θ_{Ω_k}) by RBF-FD interpolation.

Step-5 Solve the system of linear equations in kth level by considering $U^*_{\Omega_k} = U_{\Omega_{k-1}} + \Theta_{\Omega_k}$ as the starting solution for kth level to get the convergent solution U_{Ω_k}.

Step-6 If kth level is the desired finest level, then stop the process and U_{Ω_k} is a convergent solution; otherwise, add 1 to k and go to Step 4.

The above algorithm is summarized in the following flowchart (Fig. 1)

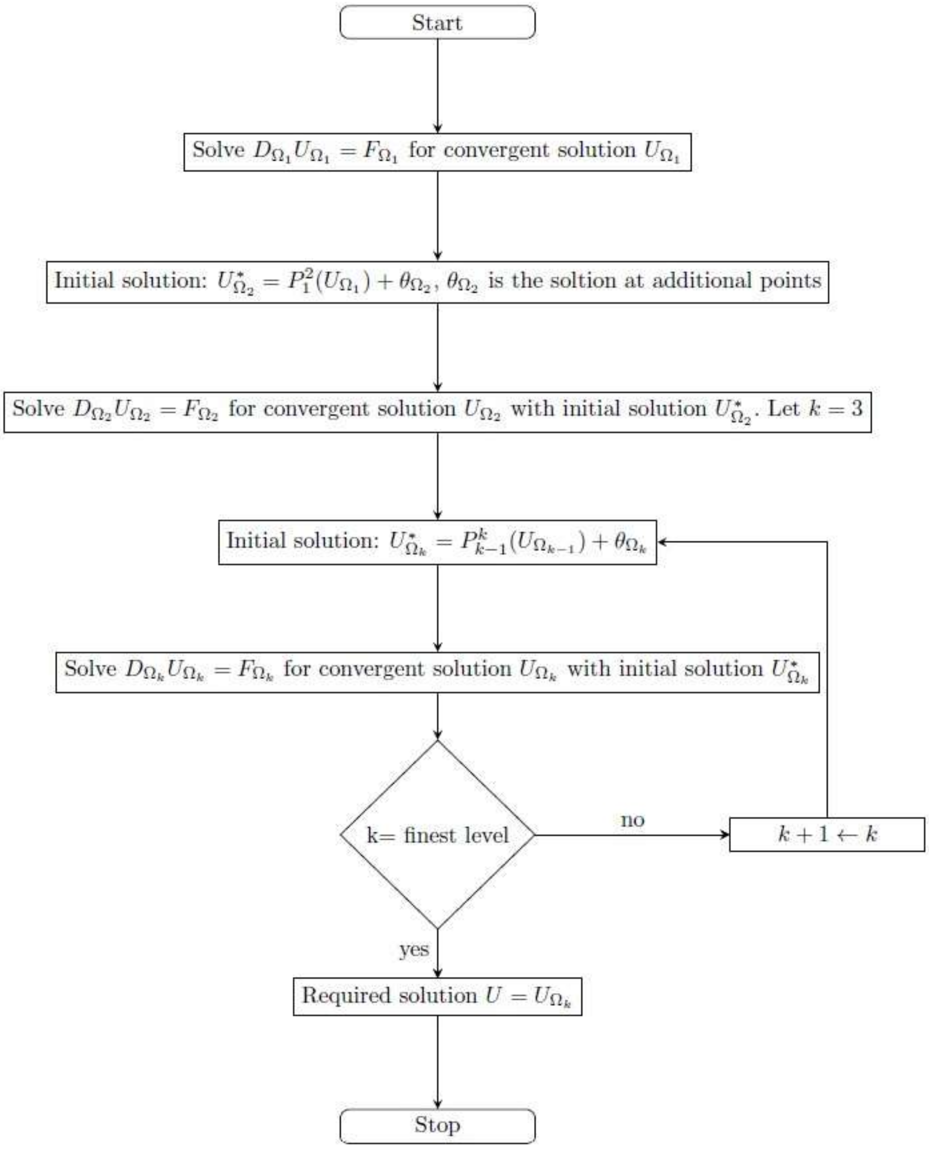

Fig. 1 Flowchart for implementation of multilevel FAS-FML RBF-FD method

3. Numerical Experiments

Convection dominated problems are presented to demonstration the efficiency of FAS-FML algorithm. Multiquadric RBF-FD is considered for our simulation. To verify the accuracy of the proposed method, we calculate the error using l^2 norm which is defined by

$$||error||_2 = \sqrt{\frac{\sum_{i=1}^{N}\{v_{exact}(x_i, y_i) - v_{appro}(x_i, y_i)\}^2}{N}}$$

where $v_{exact}(x_i, y_i)$ and $v_{appro}(x_i, y_i)$ are exact and approximation solution of velocity v at (x_i, y_i). N is the total number of node points. The order of convergence is computed to show the accuracy of the method. The shape parameter ϵ is finalized depending upon the less error with high rate of convergence. The forcing function $f(x,y)$ and boundary conditions are calculated by using the exact solution for each problem. 1089 points are considered the finest level for both problems. The solution obtained at the finest level was calculated directly in the RBF-FD method. To get the solution in FAS-FML, we have to divide the domain into a number of levels, such as coarsest, coarser, finer, and finest levels, etc., as discussed in the algorithm. We have taken three levels to get the solution at the finest level, such as $81, 289$, and 1089 in our present numerical experiment.

Problem 1

Consider the convection diffusion equation $v_x - r\left(v_{xx} + v_{yy}\right) = f(x,y), 0 \leqslant x, y \leqslant 1$,

with exact solution $v(x,y) = xy + \frac{e^{\frac{x-1}{r}} - e^{-\frac{1}{r}}}{1-e^{-1/a}}$.

The value of $r = 0.1$ is chosen to make the PDE a convection-dominated PDE. The derivatives of the function are calculated by considering central-type neighborhood points. That is, the derivatives v_x, v_y, v_{xx} and v_{yy} are calculated at the kth point with m_k neighborhood points using equations $(4) - (6)$ as follows:

$$v_x(x_k, y_k) \approxeq \sum_{i=1}^{m_k} b_{ki}^{x} v(x_i, y_i),\ v_y(x_k, y_k) \approxeq \sum_{i=1}^{m_k} b_{ki}^{y} v(x_i, y_i)$$

$$v_{xx}(x_k, y_k) \approxeq \sum_{i=1}^{m_k} b_{ki}^{xx} v(x_i, y_i),\ v_{yy}(x_k, y_k) \approxeq \sum_{i=1}^{m_k} b_{ki}^{yy} v(x_i, y_i)$$

where $b_{ki}^{x}, b_{ki}^{y}, b_{ki}^{xx}$ and b_{ki}^{yy} are similar to d_{ki} in the equation (4). The discretization of PDE for kth internal point with m_k neighborhood points is given below.

$$\sum_{i=1}^{m_k} (b_{ki}^{x} - r b_{ki}^{xx} - r b_{ki}^{yy}) v(x_i, y_i) = f(x_k, y_k)$$

The error and rate of convergence are calculated for different values of the shape parameter ϵ using the RBF-FD method and the proposed method tabulated in Tables 1 and 2, respectively. The shape parameter is finalized depending upon the less error with a high rate of convergence. So the optimized shape parameter is $\epsilon = 1.6$ with rate of convergence 3.3 for both the RBF-FD method and the proposed method. First, the problem was solved directly in the finest level with 1089 points by the RBF-FD method, which takes 0.063 seconds of CPU time to get an optimized solution and is presented in Table 3. In FAS-FML, we convert the domain into two levels, such as 289 and 1089, and solve the problem in the first level, 289 points, and extend it to the finest level, 1089 points. The optimal solution obtained at the finest level in 0.031 secs, which saved 51% of CPU time. Again, the domain is divided into three levels, such as $81, 289$, and 1089. First solve the problem in 81 points, extend it to 289 points, further extend it to the finest level 1089 points, and the optimal solution achieved in 0.016 secs, which saved 75% of CPU time. Further extensions of levels are not giving any advantage. It is consistent that the proposed method saves 75% of the computation time with the same accuracy. Hence the proposed method is very efficient to achieve the fast convergence. The FAS-FML solution (left) and exact solution (right) are plotted and presented in Figure 2. Again, we compare exact solutions with FAS-FML solutions by fixing x (left) and y (right) and presented in Figure 3. Velocity contours of FAS-FML solutions (left) and exact solutions (right) are plotted and presented in Figure 4. It concludes that FAS-FML solutions are in good agreement with RBF-FD solutions with less computation time.

Table 1: l^2 norm error and convergence rate for problem 1 by general RBF-FD method

Nodes	Error		
	$\varepsilon = 0.5$	$\varepsilon = 1.0$	$\varepsilon = 1.6$
81	1.5301116(−2)	1.4808579(−2)	1.4091323(−2)
Rate	−	−	−
289	4.2323233(−3)	4.0040668(−3)	3.9090998(−3)
Rate	1.8	1.9	1.8
1089	1.409993(−3)	1.6400000(−3)	3.9272758(−4)
Rate	1.6	1.3	3.3

Problem 2

Consider the second convection diffusion equation

$$v_x + v_y - r\left(v_{xx} + v_{yy}\right) = f(x,y), 0 \leqslant x, y \leqslant 1, r = 0.05$$

with exact solution $v(x,y) = e^{\frac{(x-1)(1-y)}{2r}}$

Table-2: l^2 norm error and convergence rate for problem 1 by proposed method

Nodes	Error		
	$\varepsilon = 0.5$	$\varepsilon = 1.0$	$\varepsilon = 1.6$
81	1.5301152(−2)	1.4808579(−2)	1.4091323(−2)
Rate	−	−	−
289	4.2281714(−3)	4.0013092(−3)	3.9143395(−3)
Rate	1.8	1.9	1.8
1089	1.3871070(−3)	1.6178401(−3)	3.8584103(−4)
Rate	1.6	1.3	3.3

Table-3 Effect of proposed method on computation time for Problem 1

No. of levels	finest level nodes	coarsest level nodes	CPU Time in Secs.	%
1	1089	1089	0.063	−
2	1089	289	0.031	51
3	1089	81	0.016	75

The derivatives v_x, v_y, v_{xx} and v_{yy} are calculated at ith point as follows:

$$v_x(x_i, y_i) \approxeq \sum_{j=1}^{n_i} b_{ij}^{x} v(x_j, y_j),\; v_y(x_i, y_i) \approxeq \sum_{j=1}^{n_i} b_{ij}^{y} v(x_j, y_j)$$

$$v_{xx}(x_i, y_i) \approxeq \sum_{j=1}^{n_i} b_{ij}^{xx} v(x_j, y_j),\; v_{yy}(x_i, y_i) \approxeq \sum_{j=1}^{n_i} b_{ij}^{yy} v(x_j, y_j)$$

where $b_{ij}^{x}, b_{ij}^{y}, b_{ij}^{xx}$ and b_{ij}^{yy} are similar to d_{ij} in the equation (4), n_i is the number of neighbourhood points for ith internal point.

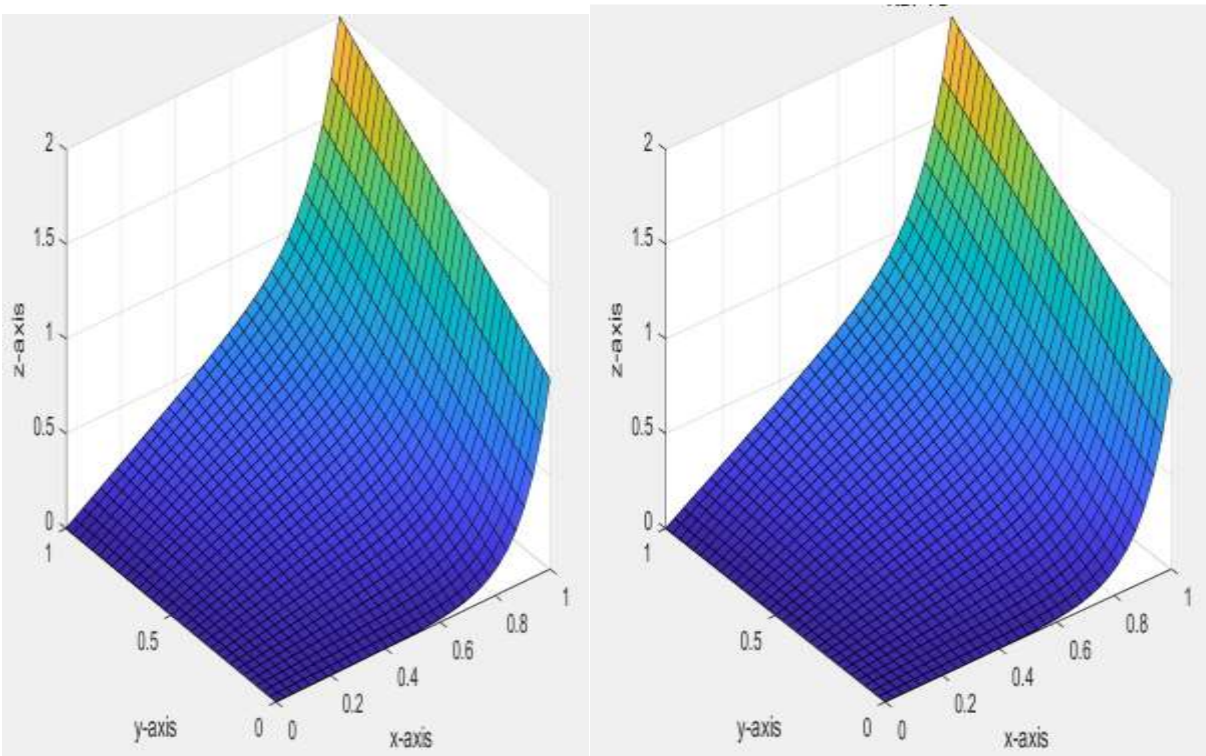

Fig.2: Comparison of numerical solution using FAS-FML method (left) with exact solution (right) of problem 1

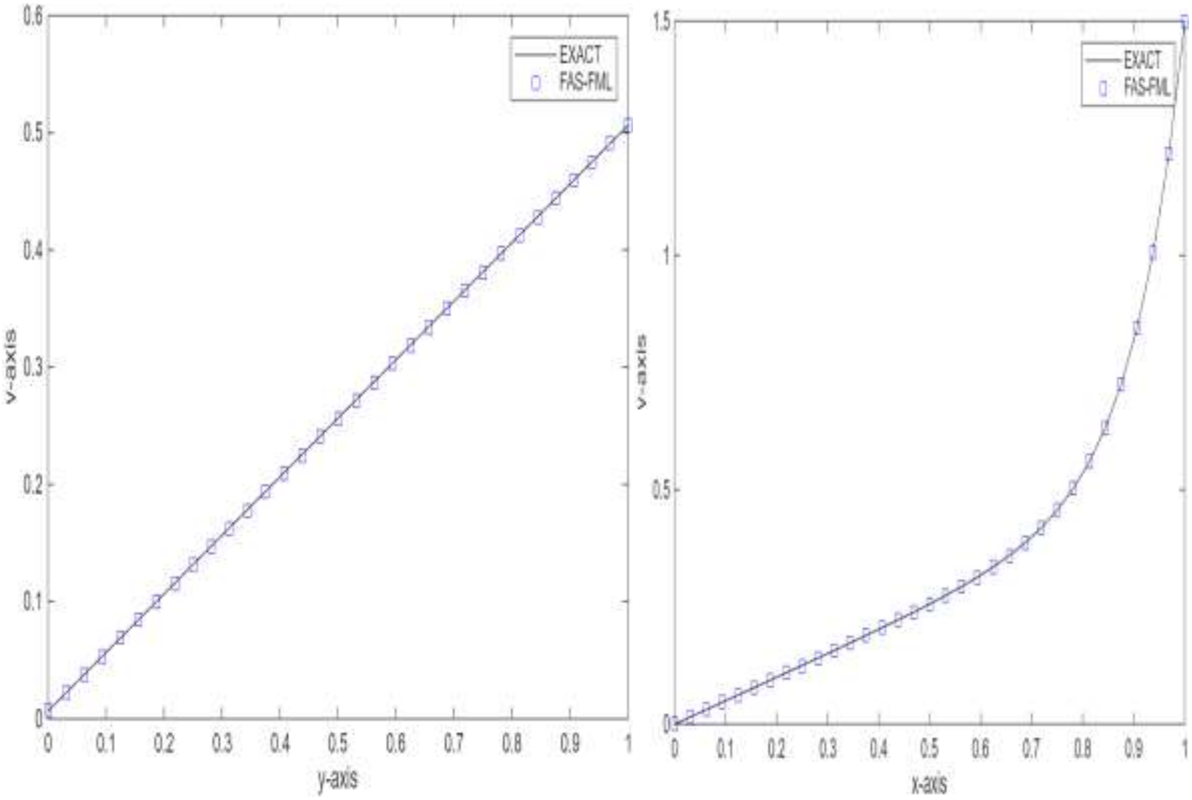

Fig.3: Comparison of solution using FAS-FML method with exact solution by fixing one variable of problem 1

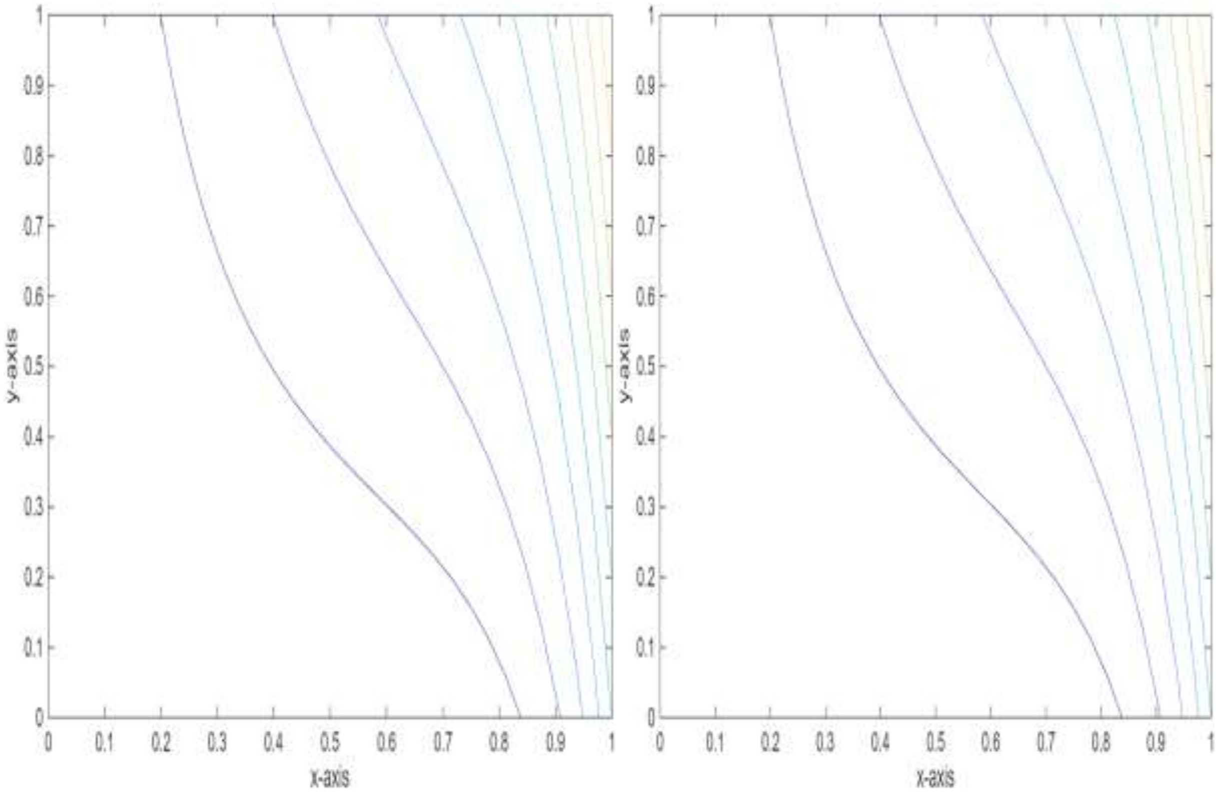

Fig.4: Velocity contour of proposed method (left) and exact solution (right) of problem 1

The discretization of PDE at ith internal point with n_i neighborhood points is given below.

$$\sum_{j=1}^{n_i}(b_{ij}^{x} + b_{ij}^{y} - rb_{ij}^{xx} - rb_{ij}^{yy})v(x_j, y_j) = f(x_i, y_i)$$

The l^2 norm error and converge rate are calculated for different values of shape parameter ϵ using the RBF-FD method and the proposed method tabulated in the Tables 4 and 5 respectively. The optimized shape parameter is $\epsilon = 1.6$ with rates of convergence of 3.17 and 3.36 for the RBF-FD method and FAS-FML method, respectively. The convergence rate is almost the same for both methods. The CPU time and percentage of saving CPU time in the proposed method with respect to the RBF-FD method are presented in Table 6. It is consistent that the proposed method saves 66% of the CPU time with the same order of accuracy. Hence, the FAS-FML method is very efficient to achieve fast convergence. The exact solution (right) and FAS-FML solution are plotted and presented in Figure 5. Again, we compare exact solutions with FAS-FML solutions by fixing x (left) and y (right) and presented in Figure 6. Velocity contours of FAS-FML (left) solutions and exact solutions (right) are plotted and presented in Figure 7. It concludes that FAS-FML solutions agree with RBF-FD solutions.

Table-4: l^2 norm error and convergence rate for problem 2 by general RBF-FD method

Nodes	Error		
	$\varepsilon = 0.5$	$\varepsilon = 1.2$	$\varepsilon = 1.6$
81	2.2042610(−2)	2.2911878(−2)	2.3702079(−2)
Rate	−	−	−
289	5.645358(−3)	5.8571575(−3)	6.0968017(−3)
Rate	1.96	1.97	1.96
1089	1.4342903(−3)	1.2111505(−3)	6.741553(−4)
Rate	1.98	2.27	3.17

Table-5: l^2 norm error and convergence rate for problem 2 by FAS-FML multilevel RBF-FD method

Nodes	Error		
	$\varepsilon = 0.5$	$\varepsilon = 1.2$	$\varepsilon = 1.6$
81	2.2042610(−2)	2.2911878(−2)	2.3702079(−2)
Rate	−	−	−
289	5.6452169(−3)	5.8571110(−3)	6.0969149(−3)
Rate	1.96	1.97	1.96
1089	1.4535420(−3)	1.1923186(−3)	5.9142604(−4)
Rate	1.96	2.30	3.36

Table-6: Effect of proposed method on computation time for Problem 2

No. of levels	finest level nodes	coarsest level nodes	CPU Time in Secs.	%
1	1089	1089	0.047	–
2	1089	289	0.031	34
3	1089	81	0.016	66

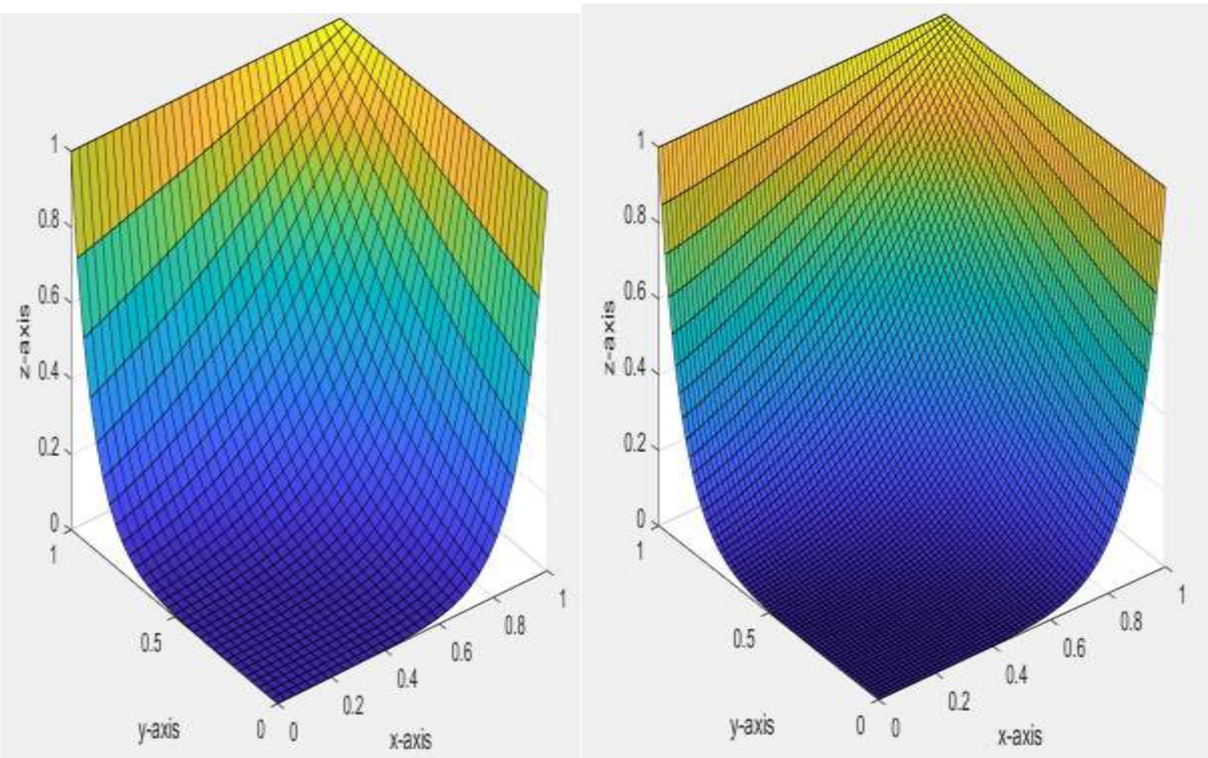

Fig.5: Comparison of numerical solution using multilevel FAS-FML method (left) with exact solution (right) of problem 2

4. Convergence Analysis

A necessary and sufficient condition for convergence of the Gauss-Seidel iterative method is verified by checking the spectral radius of the iterative matrix for both problems 1 and 2 and tabulated in Table 7. The spectral radius is less than one for both problems, which confirms that the Gauss-Seidel iterative method converged. Hence, our results through the proposed method are convergent.

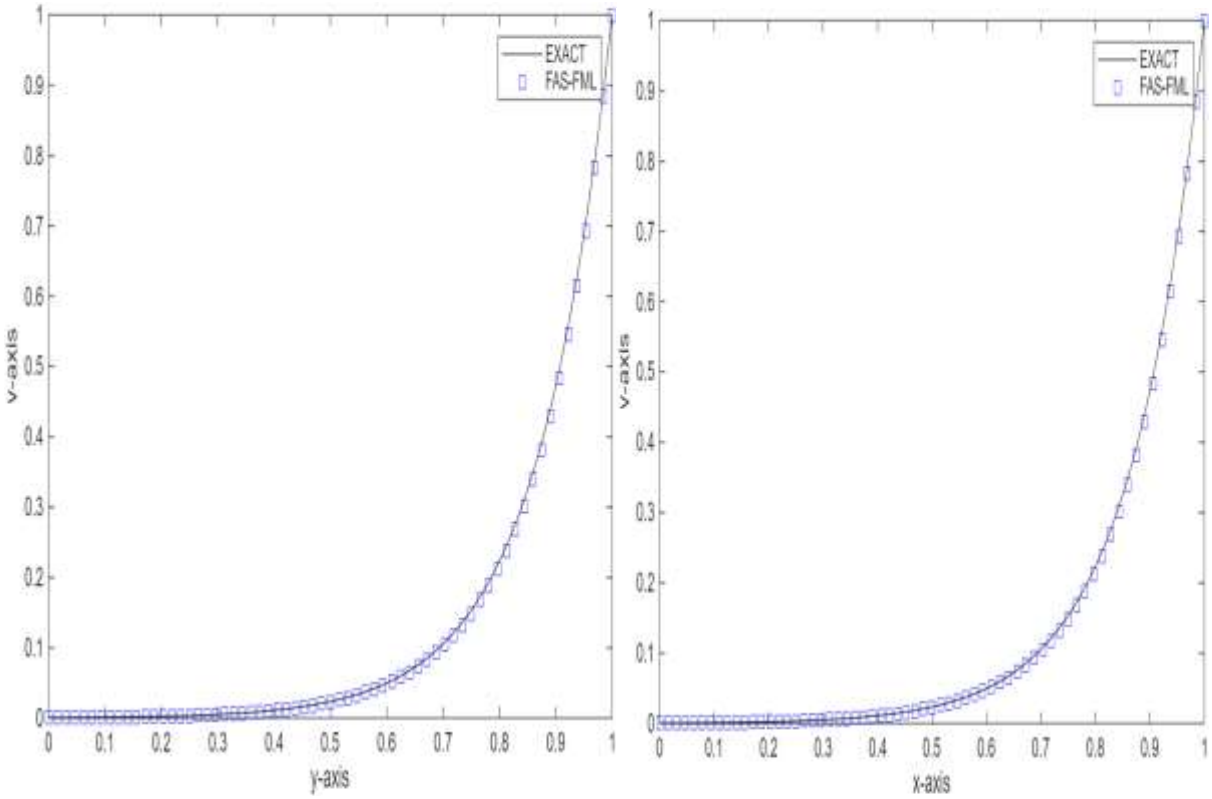

Fig.6: Comparison of solution using multilevel FAS-FML method with exact solution by fixing one variable of problem 2

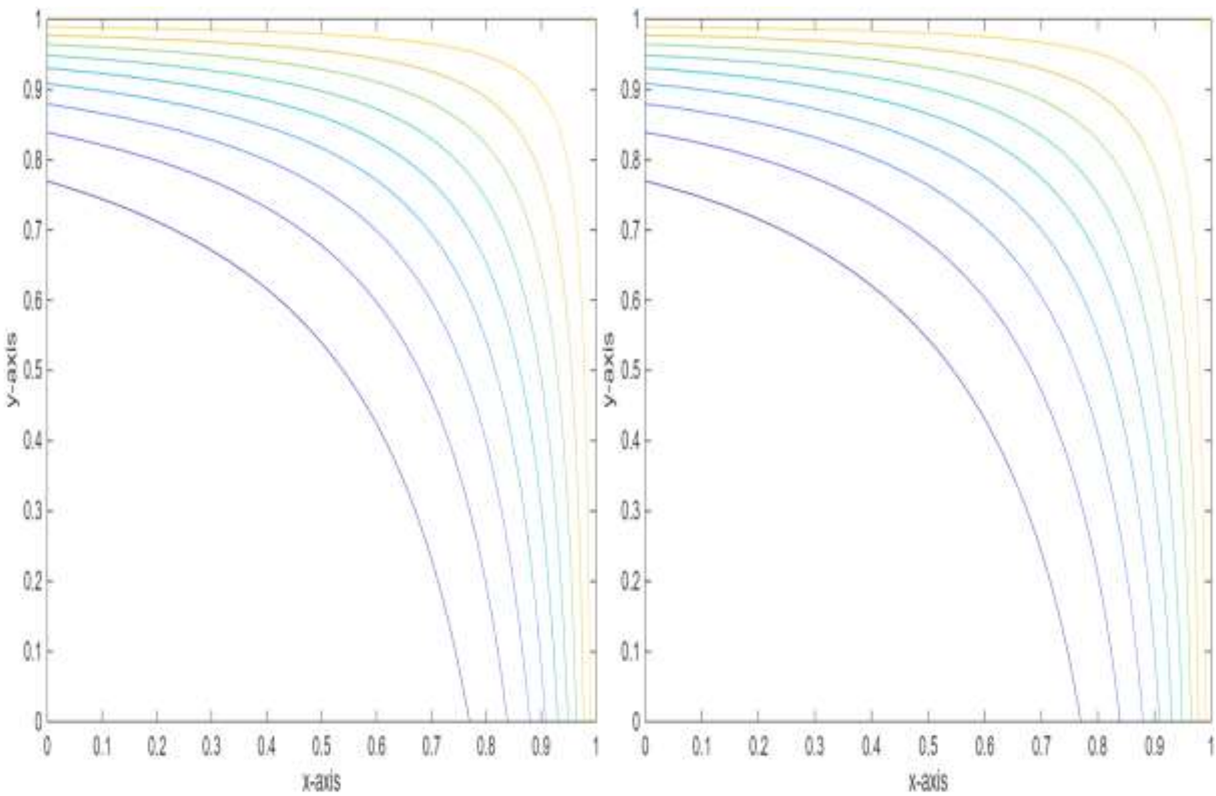

Fig.7: Velocity contour of proposed method (left) and exact solution (right) of problem 2

Table-7: Spectral radius of the iterative matrix for both problems

FAS-FML method	81	289	1089
Problem 1	0.8730232	0.946263	0.9852592
Problem 2	0.8852122	0.971192	0.9912832

5. Conclusions

This paper presented a FAS-FML algorithm for the RBF-FD meshfree method. The method provided accurate solutions by solving the discretized system of equations from coarsest level to coarser level and then finer level to desired finest level node points. The validity of the algorithms is illustrated through numerical experiments. The numerical results were compared with the exact solution. We also calculated the computing time of the proposed algorithm, and it saved at least 66% of CPU time for the convection-dominated problems than the general RBF-FD method. The convergence condition of the proposed method is also presented in the numerical experiments. Comparison of the proposed FAS-FML method with the RBF-FD method indicates that the FAS-FML RBF-FD method is an interesting approach in terms of accuracy and efficiency.

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Conflicts of interest

No Animals are involved in the research work.

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