Analysing the conduction of heat inporous medium via Caputo fractionaloperatorwithSumudutransfo rm

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Abstract. In this article, we analyse the fractional Cattaneo heat equa-tion for studying the conduction of heat in porous medium. This equationis also used in studying extended irreversible thermodynamics, material, plasma,cosmologicalmodel,computationalbiology,anddiffusionthe- ory in crystalline solids. The Sumudu adomian decomposition technique, whichiscombinationofSumudutransformandanumericaltechnique, isappliedforgettingnumericalsolution.Theexistenceanduniqueness is analysed by using the fixed point theorem and the highest error of the designed technique is also analysed. Finally, the accuracy of the designed numerical method is presented by solving two examples and the findings are compared with the existing method.

Keywords: FractionalCattaneoheatequation(FCHE),Caputoderiva- tive,Sumudutransform,ExistenceandUniquenessAnalysis,ErrorAnal- ysis.

1 Introduction

The Fractional calculus (FC) has developed into avital tool for mod- ellingandsolvingtheeventsinengineeringandsciences.Itisutilised in many different scientific fields. Additionally, because of its abil- ity to be remembered and passed on, it is quite useful in replicat- ing and understanding natural occurrences. The fractional Caputo derivative [1] is one of the best fractional derivatives available in the literature. Various fractional mathematical models were explored andexaminedbynumerousresearchers,likethefractionaldiffusion- waveequationstudiedin[2],thefractionalBraintumourmodel[3], Whitham–Broer–Kaup equation with fractional order is presented in[4],theproblemofoilspillisanalysedin[5],theexistenceand uniqueness for fractional Cauchy reaction diffusion equations is analysed in[6], the time fractional wave equations is studied in[7], the 2019-nCOV outbreaks is studied in [8], the COVID-19 model is analysed through singular and non-singular fractional operators in[9], the fractional hosta parasitoid population dynamical model is studied in[10], the fractional model for population dynamics of two interactingspeciesisstudiedin[11],thefractionalRiccatidifferential equationisstudiedin[12],theCoupledSchrodinger-KdVequationis analysed with Caputo-Katugampola Type Memory in[13], the SIR modelisstudiedviaMorgan-Voyceseriesin[14],thesingularlyperturbedVolterraintegro-differentialequationswithdelayissolvedin [15]andthegeneralizedequalwidthwaveequationisinvestigated in [16]. There are many numerical techniques available in the literature for solving fractional order differential equations such as; the generalized Adams-Bashforth-Moultan method [17], the optimal control technique [18], the Taylor series expansion method [19], the Haar wavelet method[20],thehomotopyperturbationSumudutransformmethod [21]andthesine-gordonexpansionmethod[22].

Thefractionaldiffusionmodelshavebeenusedtodefineanomalous diffusion in a variety of ways [23,24,25,26]. As a result, anticipatingheattransportbehaviourhascaughtscientists'attention. The heat conduction model based of Fouriera's law is sufficient forthemajorityofapplications.However,thismodelviolatestherules of physics since the thermal disturbances propagate at an infinitepace[27,28,29].Forinstance,fractionalorderderivativesdescribe theFick'slawsoffractionalorder,whichareavailable.Thismethod is used to produce the Cattaneo equation, which is used to explain fractional order diffusion models and heat and mass transport [30]. Thelawofheatconditionwasgivenby Fourier[31],firsttimein 1822. Even though it was simply a hypothesis law, it gave the two centuriesofthermalconductionstudythatfollowedaframework and made it possible to characterise heat transport routes. Fourier's law was updated by Cattaneo [32] in 1948 by include the idea of relaxation time. Using Oldroyd's upper convected derivative [33] and modifying the rule given by the Cattaneo in 2009, Christov [34] createdanequationwithaonlyoneparameteroftemperature.

WeconsiderthefollowingfractionalCattaneoheatequation(FCHE)

$$ {}_0^cD_t^\alpha u(x,y,t)+\beta u_t(x,y,t)-\Delta u(x,y,t)=f(x,y,t), \quad (x,y,t)\in\Omega\times(0,T] \tag{1}$$

with

$$u(x,y,0)=g(x,y), \quad \frac{\partial u(x,y,0)}{\partial t}=\varphi(x,y),(x,y)\in\Omega \tag{2}$$

where $u(x,y,t)$ is the heat distribution function, β is a constant, $f(x,y,t)$ is the given source term, Ω is bounded domain, ${}_0^cD_t^\alpha$ is the Caputo fractional operator of order α with $1<\alpha\leq 2$, $g(x,y)$, and $\varphi(x,y)$ are given smooth functions.

The main aim of this research is to examine the FCHE with Caputofractionaloperator.TheSumuduadomiandecompositiontechnique (SADT) is used to get numerical solution. The existence and uniqueness is analysed by using the fixed point theorem, also the maximum error of the SADT is investigated. The uniqueness of this studyisthatitprovidesanaccuratepredictionregardingthemethod ofconductionofheatforaddressingthermo-elasticissuesinporous medium. The main achievement of this work is to design a very effective technique for solution of FCHE that allows for the detailed investigation of the heat conduction process in porous media. This study's findings will be useful for studying extended irreversible thermodynamics,material,plasma,cosmologicalmodel,computational biology, and diffusion theory in crystalline solids.

2 BasicdefinitionsofFractionalCalculus

Def.2.1.[1]TheCaputoderivativeof$y(t)$isgivenby

$$\mathrm{D}^\alpha y(t)=I^{m-\alpha}D^m y(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-f)^{m-\alpha-1}y^m(f)df$$

where$m-1<\alpha\leq m$.

Def.2.2.[35]TheSumudutransform(ST)ofagivenfunction$y(t)$ isdefinedas

$$ST[y(t)]=\int_0^\infty e^{-t}y(vt)dt, v\in(-t_1,t_2).$$

Def.2.3.[35]TheSToftheCaputofractionalorderderivativeis given by

$$ST[D^{\alpha}_{t}y(t)]=v^{-\alpha}ST[y(t)]-\sum_{k=0}^{m-1}v^{(-\alpha+k)}y^{(k)} \quad (0)$$

Theorem2.1.[1]If${}^{c}_{0}D^{\alpha}_{t}y(t)=e(t)$,thenitsuniquesolutionis defined as

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-f)^{\alpha-1}e(f)df$$

where$0<\alpha\leq 1$.

3 ExistenceandUniquenessAnalysisofthe FCHE

The FCHE isgiven byequation (1)can bewrittenin theform

$$ {}^{c}_{0}D^{\alpha}_{t}=\psi(x,y,t,u), \qquad (3)$$

where$\psi(x,y,t,u)=\Delta u(x,y,t)-\beta u_t(x,y,t)+f(x,y,t)$.
Equation(3)canbeconvertedintotheVolteratypeintegralequation usingtheorem(2.1):

$$u(x,y,t)-u(x,y,0)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(x,y,s,u)ds. \qquad (4)$$

Next, we have to prove that $\psi(x,y,t,u)$ satisfy Lipschitz condition.
Theorem3.1.[3] The function$\psi(x,y,t,u)$ in the given Voltera equationsatisfiesboththecontractionif$0<\eta\leq 1$,where$\eta=\delta^2-\beta\mu$andtheLipschitzcondition.
Proof: Let$u(x,y,t)$isbounded.So,wehave

$$\begin{aligned}\|\psi(x,y,t,u)-\psi(x,y,t,\rho)\| &= \|u_{xx}(x,y,t)-\beta u_t(x,y,t)\\ &\quad+f(x,y,t)-\rho_{xx}(x,y,t)\\ &\quad+\beta\rho_t(x,y,t)-f(x,y,t)\|\\ &=\|(u_{xx}-\rho_{xx})-\beta(u_t-\rho_t)\|,\\ &\leq\delta^2\|(u-\rho)\|-\beta\mu\|(u-\rho)\|\end{aligned}$$

$$\|\psi(x,y,t,u)-\psi(x,y,t,\rho)\|\leq(\delta^2-\beta\mu)\|(u-\rho)\| \tag{5}$$

Letting$\eta=(\delta^2-\beta\mu)$,then

$$\|\psi(x,y,t,u)-\psi(x,y,t,\rho)\|\leq\eta\|(u-\rho)\|$$

Thus, $\psi(x,y,t,u)$ satisfies Lipschitz and contraction condition if $0<\eta\leq1$.So,theiterativeformulafortheexistenceofabovecondition is represented as

$$u_{n+1}(x,y,t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\psi(x,y,s,u_n)ds, \tag{6}$$

withinitialconditionas$u(x,y,0)=u(x,y,t_0)$.
Thetwoconsecutivetermsaredifferby

$$\psi_n(x,y,t)=u_n(x,y,t)-u_{n+1}(x,y,t)$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\{\psi(x,y,s,u_{n-1})-\psi(x,y,t,u_{n-2})\}ds \tag{7}$$

Itcanbeobservedthat

$$u_n(x,y,t)=\sum_{i=0}^{n}\psi_i(x,y,t) \tag{8}$$

so,fromequation(7),wehave

$$\|\psi_n(x,y,t)\|=\|u_n(x,y,t)-u_{n-1}(x,y,t)\|. \tag{9}$$

UsingTriangularinequality,equation(6)becomes

$$\|\psi_n(x,y,t)\|\leq\frac{1}{\Gamma(\alpha)}\eta\left\|\int_0^t(t-s)^{\alpha-1}\psi_{n-1}(x,y,s)ds\right\|. \tag{10}$$

Theorem3.2.[3]Thesolutionoftheequation(3)existifthere exists a $t_0$which satisfy

$$\frac{1}{\Gamma(\alpha)}\eta t_0^{\alpha}\leq1.$$

Proof: Let $u(x,y,t)$ is be bounded and Lipschitz function then using equation (10), we get

$$\|\psi_n(x,y,t)\| \leq \|u_n(x,y,t)\| \left(\frac{1}{\Gamma(\alpha)}\eta.t^{\alpha}\right)^{n}. \tag{11}$$

Thus,itisestablishedthatthegivensolutionexistsandiscontinu- ous.

$$u(x,y,t)-u(x,y,0)=u_n(x,y,t)-\gamma_n(x,y,t), \tag{12}$$

here,weconsiderthat

$$\begin{aligned}\|\gamma_n(x,y,t)\| &= \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\psi(x,y,s,u_n)-\psi(x,y,t,u_{n-1})\}ds\right\|,\\ &\leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}\{\psi(x,y,s,u_n)-\psi(x,y,t,u_{n-1})\}ds\right\|,\\ &\leq \frac{1}{\Gamma(\alpha)}\|\eta u_n(x,y,t)-u_{n-1}(x,y,t)t.\|\end{aligned}$$

Inthesamewayatt_0,wehave

$$\|\gamma_n(x,y,t)\| \leq \left(\frac{1}{\Gamma(\alpha)}t_0^{\alpha}\right)^{n+1}\eta^{n+1}M. \tag{13}$$

As$n\to\infty$,wecanclearlyseethat$\|\gamma_n(x,u,t)\|\to 0$.

Theorem3.3.[3]TheFCHEwillhaveauniquesolutionifthe followingconditionholds

$$\left(1-\frac{1}{\Gamma(\alpha)}\eta t^{\alpha}\right)>0.$$

Proof: Let$w(x,y,t)$isanothersolutionof(3),then

$$\|u(x,y,t)-w(x,y,t)\| = \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\psi(x,y,s,u)-\psi(x,y,t,w))ds\right\|,$$

$$\|u(x,y,t)-w(x,y,t)\| \leq \frac{1}{\Gamma(\alpha)}\eta\|u(x,y,t)-w(x,y,t)\| \tag{14}$$

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Now,onsimplifying(14),wehave

$$\left\| u(x,y,t)-w(x,y,t) \right\| \left(1-\frac{1}{\Gamma(\alpha)}\eta t^{\alpha}\right) \leq 0$$

hence,if

$$1-\frac{1}{\Gamma(\alpha)}\eta t^{\alpha} > 0. \tag{15}$$

then,$u(x,y,t)=w(x,y,t)$.
Hence,theFCHEpossessauniquesolution.

4 Application of the designed technique tothe FCHE

Inthispartofthearticle,weapplytheproposedtechniquetothe FCHE and we consider the following FCHE

$${}^{c}_{0}D^{\alpha}_{0}u(x,y,t)+\beta u_t(x,y,t)-\Delta u(x,y,t)=f(x,y,t) \tag{16}$$

where$(x,y,t)\in\Omega(0,T],1<\alpha\leq 2,$with

$$u(x,y,0)=g(x,y), \quad \frac{\partial u(x,y,0)}{\partial t}=\phi(x,y),(x,y)\in\Omega. \tag{17}$$

Now,weapplySTonequation(16),wehave

$$ST^{c}_{0}D^{\alpha}_{t}u(x,y,t)=ST[\Delta u(x,y,t)-\beta u_t(x,y,t)+f(x,y,t)].$$

Fromdefinition(2.3)andequation(17),wehave

$$\frac{1}{v^{\alpha}}ST[u(x,y,t)]-\frac{1}{v^{\alpha}}u(x,y,0)-\frac{1}{v^{\alpha-1}}\frac{\partial u(x,y,\vartheta)}{\partial t}=ST[\Delta u(x,y,t)-\beta u_t(x,y,t)+f(x,y,t)]$$

$$\frac{1}{v^{\alpha}}ST[u(x,y,t)]=\frac{1}{v^{\alpha}}u(x,y,0)+\frac{1}{v^{\alpha-1}}\frac{\partial u(x,y,\vartheta)}{\partial t}+ST[\Delta u(x,y,t)-\beta u_t(x,y,t)+f(x,y,t)]$$

$$ST[u(x,y,t)]=g(x,y) + v\varphi(x,y) + v^{\alpha}ST[\Delta u(x,y,t) - \beta u_t(x,y,t) + f(x,y,t)].$$

ApplyingSTinverseonaboveequation,wehave

$$u(x,y,t)=g(x,y)+t\varphi(x,y,t)+ST^{-1}v^{\alpha}[ST\Delta u(x,y,t) - \beta u_t(x,y,t)+f(x,y,t)] . \tag{18}$$

ApplyingtheAdomiandecompositionmethodtoequation(18),so we take

$$u(x,y,t)=\sum_{n=0}^{\infty} p^n u(x,y,t).$$

Hence,weget

$$\sum_{n=0}^{\infty} p^n u(x,y,t)=g(x,y)+t\varphi(x,y)+ST^{-1}v^{\alpha}ST \sum_{n=0}^{\infty} p^n \Delta u_n(x,y,t)-\beta \sum_{n=0}^{\infty} p^n (u_n)_t(x,y,t)+f(x,y,t) . \tag{19}$$

Comparingthecoefficientoflikepowersof$p$ofequation(19),we have

$$u_0(x,y,t)=g(x,y)+t\varphi(x,y), \tag{20}$$

$$u_1(x,y,t)=ST^{-1}\{v^{\alpha}ST[\Delta u_0(x,y,t)-\beta(u_0)_t(x,y,t)+f(x,y,t)]\}, \tag{21}$$

$$u_2(x,y,t)=ST^{-1}\{v^{\alpha}ST[\Delta u_1(x,y,t)-\beta(u_1)_t(x,y,t)+f(x,y,t)]\}, \tag{22}$$

$$u_n(x,y,t)=ST^{-1}(v^{\alpha}ST[\Delta u_{n-1}(x,y,t)-\beta(u_{n-1})_t(x,y,t) +f(x,y,t)]), \tag{23}$$

Thefinalsolutionisgivenby

$$u(x,y,t)=\lim_{k\to\infty}\sum_{n=0}^{k}u_n(x,y,t). \tag{24}$$

5 ErrorAnalysisoftheSADT

ThemaximumerroroftheSADTisexaminedinthissection.
Theorem5.1.[3]If$\exists$, $0<\epsilon<1$suchthat$\|u_{n+1}(x,y,t)\|\leq\epsilon\|u_n(x,y,t)\|$,forall$n$andiftheseries $\sum_{n=0}^{r}u_n(x,y,t)$istheapproximatesolution$u(x,y,t)$,thenthemaximumerrorisgivenby

$$\left\|u(x,y,t)-\sum_{n=0}^{r}u_n(x,y,t)\right\|\leq\frac{\epsilon^{r+1}}{(1-\epsilon)}\|u_0(x,y,t)\|.$$

Proof: Wehave

$$\begin{aligned}\left\|u(x,y,t)-\sum_{n=0}^{r}u_n(x,y,t)\right\| &= \left\|\sum_{n=r+1}^{\infty}u_n(x,y,t)\right\|\\ &\leq\sum_{n=r+1}^{\infty}\|u_n(x,y,t)\|\\ &\leq\sum_{n=r+1}^{\infty}\epsilon^m\|u_0(x,y,t)\|\\ &\leq(\epsilon)^{r+1}[1+(\epsilon)^1+(\epsilon)^2+\cdots]\|u_0(x,y,t)\|\\ &\leq\frac{\epsilon^{r+1}}{1-\epsilon}\|u_0(x,y,t)\|.\end{aligned}$$

Henceproved.

6 NumericalSimulation

Two examples of the FCHE have been solve in this section to validate thedesignedtechnique
Application6.1. ConsiderFCHEgiveninthearticle[36]

$${}_0^cD_t^\alpha u(x,y,t)+u_t(x,y,t)-\Delta u(x,y,t)=\left(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}-2t^3+3t^2\right)e^{2x+y} \tag{25}$$

$(x,y,t)\in\Omega\times(0,T]$, and the analytical solution of equation (25) at $\alpha=2$ is $u(x,y,t)=t^3e^{x+y}$.

Solution. Here $\beta=1$, $f(x,y,t)=\left(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}-2t^3+3t^2\right)e^{x+y}$, and from the analytical solution we can find the initial conditions. Now, we apply SADT to equation (25) so, we get

$$u_0(x,y,t)=0,$$

$$u_1(x,y,t)=e^{x+y}\left[6\frac{t^3}{\Gamma(4)}-2\frac{t^{\alpha+3}}{\Gamma(\alpha+4)}+3\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}\right],$$

$$u_2(x,y,t)=e^{x+y}\left(6\frac{t^3}{\Gamma(4)}-\frac{3}{2}\frac{t^{\alpha+3}}{\Gamma(\alpha+4)}+\frac{9}{4}\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}-4\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}\right.$$

$$\left.+\frac{3(\alpha+2)}{(\alpha+3)}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{(9\alpha+24)}{(\alpha+3)}\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}\right),$$

$$u_3(x,y,t)=e^{x+y}\left(6\frac{t^3}{\Gamma(4)}-42\frac{t^{\alpha+3}}{\Gamma(\alpha+4)}+6\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}-3\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}\right.$$

$$-8\frac{t^{3\alpha+3}}{\Gamma(3\alpha+4)}+\frac{(9\alpha+18)}{(4\alpha+12)}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{(21\alpha+15)}{(2\alpha+8)}\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}$$

$$+\frac{18\alpha+48}{\alpha+3}+\frac{8\alpha+12}{2\alpha+4}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}-\frac{3(2\alpha+1)(\alpha+2)}{(2\alpha+2)(\alpha+3)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\left.+\frac{(6\alpha+12)}{(\alpha+3)}-\frac{(9\alpha+24)(2\alpha+2)}{(\alpha+3)(2\alpha+3)}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right).$$

The final solution is given by

$$u(x,y,t)=u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+\ldots \tag{26}$$

Table1. Maximumabsoluteerrorfor$(x,y)\in[0.01,0.1]$anddistinctvalueof$t$andα forEx.6.1.

t	α= 2	α=1.95
0.01	$1.901724e^{-06}$	$2.023245e^{-06}$
0.02	$2.115769e^{-05}$	$2.279847e^{-05}$
0.03	9.413964^{-05}	$2.023245e^{-04}$
0.04	$2.856856e^{-04}$	$3.085486e^{-04}$
0.05	$7.015689e^{-04}$	$7.553783e^{-04}$
0.06	$1.505322e^{-03}$	$1.613395e^{-03}$
0.07	$2.939414e^{-03}$	$3.132917e^{-03}$
0.08	$5.350787e^{-03}$	$5.667438e^{-03}$
0.09	$9.220735e^{-03}$	$9.701331e^{-03}$
0.10	$1.519915e^{-02}$	$1.519915e^{-02}$

Table2. TheComparativeanalysisofmaximumabsoluteerrorfortheSADTand existing methods for distinct value of$t$andα, for Ex. 6.1.

t	α=1.45		α=1.65	
Methods	**SADT**	**RBFPUMethod[36]**	**SADT**	**RBFPUMethod[36]**
1/40	$1.2520e^{-04}$	$1.4587e^{-04}$	$7.6894e^{-04}$	$1.9204e^{-04}$
1/80	$1.0872e^{-05}$	$5.0793e^{-05}$	$6.3571e^{-06}$	$7.6507e^{-05}$
1/160	$1.0045e^{-06}$	$1.7519e^{-05}$	$5.7788e^{-07}$	$3.0822e^{-05}$
1/320	$9.6980e^{-08}$	$6.1028e^{-06}$	$5.6512e^{-08}$	$1.2305e^{-05}$
1/640	$9.7131e^{-09}$	$2.1131e^{-06}$	$5.8764e^{-09}$	$4.8922e^{-06}$
1/1280	$1.0060e^{-09}$	$7.2381e^{-07}$	$6.4297e^{-10}$	$1.9304e^{-06}$

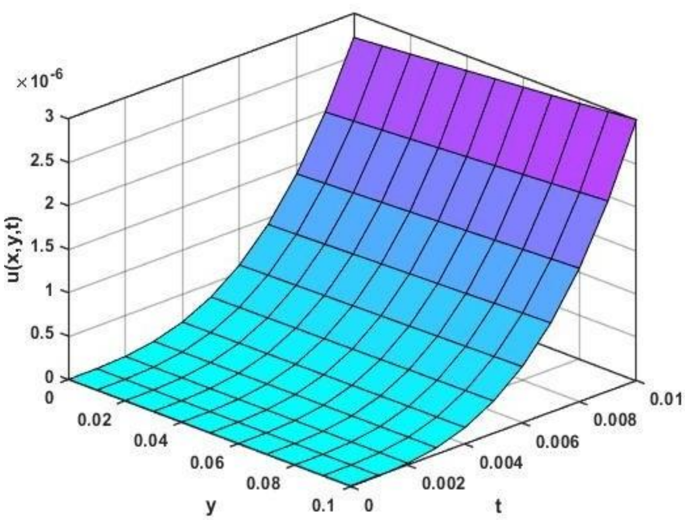

Fig.1. Approximatesolution$u(x,y,t)$at$x$=0.1andα=2,forEx.6.1.

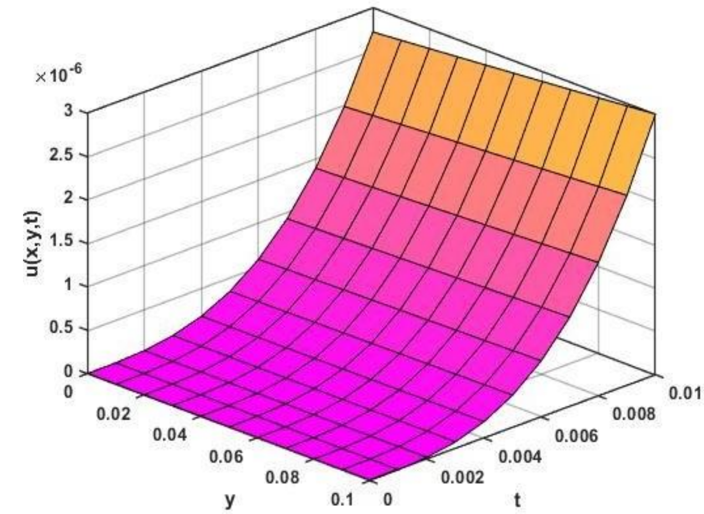

Fig.2. Analytical solution$u(x,y,t)$at$x=0.1$and$\alpha=2$,forEx.6.1.

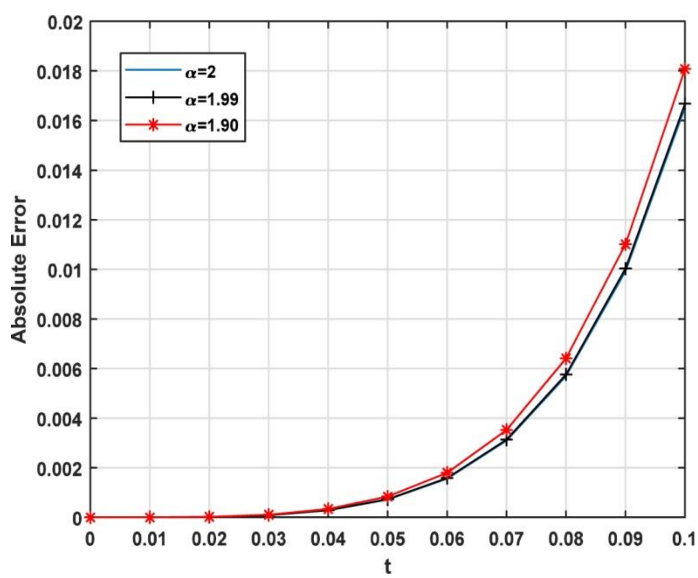

Fig.3. AbsoluteerrorofSADTat$x$=0.1anddistinctvalueofα,forEx.6.1.

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Application 6.2. Consider another form of FCHE given in the article [36]

$$ {}_0^cD_t^\alpha u(x,y,t)+u_t(x,y,t)-\Delta u(x,y,t) =\Big(\frac{6t^2}{\Gamma(4-\alpha)} +(2+\alpha)t^{1+\alpha}+2\pi^2t^{2+\alpha}\Big)\sin(\pi x)\sin(\pi y) \tag{27} $$

Where $(x,y,t)\in\Omega\times(0,T]$, and the analytical solution of (27) is $u(x,y,t) =\sin(\pi x)\sin(\pi y)t^{\alpha+2}$.

Solution. Here $f(x,y,t)= \frac{6t^2}{\Gamma(4-\alpha)} +(2+\alpha)t^{1+\alpha}+2\pi^2t^{2+\alpha}$ $\sin(\pi x)\sin(\pi y)$, $\beta =1$, and from the analytical solution we can find theinitialconditions.Now,weapplytheproposedtechnique,SADT, to equation (27) so, we get

$u_0(x,y,t)=0$,

$$ u_1(x,y,t)= \frac{12}{\Gamma(4-\alpha)\Gamma(\alpha+3)}t^{\alpha+2} +(2+\alpha)\Gamma(\alpha+2) \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} +2\pi^2\Gamma(\alpha+2) \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} \sin(\pi x)\sin(\pi y) $$

$$ u_2(x,y,t)= \frac{-24}{\Gamma(4-\alpha)} +2\pi^2\Gamma(\alpha+3) \times \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \times\frac{(\alpha+2)(2\alpha+1)\Gamma(\alpha+2)}{(2\alpha+2)}+(2+\alpha)\Gamma(\alpha+2)-\frac{12(\alpha+2)}{(\alpha+3)\Gamma(4-\alpha)} \times\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{-2\pi^2\Gamma(2\alpha+2)\Gamma(\alpha+3)}{(2\alpha + 3)} -2(2+\alpha)\Gamma(\alpha+2) \times\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} -4\pi^2 \Gamma(\alpha+3) \frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{12}{\Gamma(4-\alpha)\Gamma(\alpha+3)}t^{\alpha+2} \times\sin(\pi x)\sin(\pi y), $$

$$ u_3(x,y,t)= \frac{-2\pi^2(2\alpha+2)\Gamma(\alpha+3)}{(2\alpha+3)} - \frac{24\pi^2(\alpha+2)}{(\alpha+3)\Gamma(4-\alpha)} - \frac{24(2\alpha + 2)}{(2\alpha+3)\Gamma(4-\alpha)} +2\pi^2 (2+\alpha) \frac{\Gamma(\alpha+2)}{\Gamma(2)} \times\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\Big(\frac{-24}{\Gamma(4-\alpha)}+ $$

$$2\pi^2\Gamma(\alpha+3))\ \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \quad -\frac{12(\alpha+2)(2\alpha+1)}{(\alpha+3)\Gamma(4-\alpha)(2\alpha+2)} +$$

$$\frac{(\alpha+2)(2\alpha+1)\Gamma(\alpha+2)}{(2\alpha+2)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\ \frac{48\pi^2}{\Gamma(4-\alpha)} - 4\pi^2\ \Gamma(\alpha+3)$$

$$\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{t^{4\alpha}\Gamma(4}{\alpha+1)}\ \frac{2\pi^2(2\alpha+1)(\alpha+2)\Gamma(\alpha+2)}{(2\alpha+2)}$$

$$-\frac{2\pi^2\Gamma(\alpha+3)(2\alpha+2)(3\alpha+1)}{\Gamma(3\alpha+2)(2\alpha+3)} - \frac{2\Gamma(\alpha+2)(\alpha+2)(3\alpha+1)}{(3\alpha+2)} +$$

$$(2+\alpha)\Gamma(\alpha+2) + \quad \frac{12(\alpha+2)}{(\alpha+3)\Gamma(4-\alpha)}\ \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{12}{\Gamma(4-\alpha)\Gamma(\alpha+3)}\ \frac{t^{\alpha+2}}{}$$

$$+8\pi^4\ \Gamma(\alpha+2)\ \frac{t^{4\alpha+2}}{\Gamma(4\alpha+3)} + \frac{-4\pi^4(2\alpha+2)\Gamma(\alpha+3)}{(2\alpha+3)} - \frac{4\pi^2\Gamma(2\alpha+3)(3\alpha+2)}{(3\alpha+3)}$$

$$+4\pi^2(2+\alpha)\ \frac{\Gamma(\alpha+2)}{\Gamma(2)}\ \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\ \times\sin(\pi x)\sin(\pi y)$$

Thefinalsolutionisgivenby

$$u(x,y,t)= \quad u_0(x,y,t)+u_1(x,y,t)+ \quad u_2(x,y,t)+u_3(x,y,t)+\ldots \tag{28}$$

Table3. Maximumabsoluteerrorfor$(x,y)\in[0.01,0.1]$anddistinctvalueof$t$andα forEx.6.2.

t	$\alpha=2$	$\alpha=1.95$
0.01	$2.745706e^{-07}$	$5.304699e^{-07}$
0.02	$4.393655e^{-06}$	$7.142152e^{-06}$
0.03	2.224556^{-05}	$3.269201e^{-05}$
0.04	$7.031549e^{-05}$	$9.620824e^{-05}$
0.05	$1.716896e^{-04}$	$2.222602e^{-04}$
0.06	$3.560595e^{-04}$	$4.405737e^{-04}$
0.07	$6.597259e^{-04}$	$7.857576e^{-04}$
0.08	$1.125603e^{-03}$	$1.297095e^{-03}$
0.09	$1.803226e^{-03}$	$2.018378e^{-03}$
0.10	$2.748749e^{-03}$	$2.997764e^{-03}$

Table4. TheComparativeanalysisofmaximumabsoluteerrorfortheSADTand existing methods for distinct value of$t$andα, for Ex. 6.2.

t	α=1.45		α=1.65	
Methods	**SADT**	**RBFPUMethod[36]**	**SADT**	**RBFPUMethod[36]**
1/40	$1.6721e^{-07}$	$1.9018e^{-04}$	$8.6385e^{-08}$	$8.5648e^{-04}$
1/80	$1.9760e^{-08}$	$5.7019e^{-05}$	$6.1827e^{-09}$	$73.9328e^{-04}$
1/160	$2.2424e^{-09}$	$1.7025e^{-05}$	$4.3288e^{-10}$	$1.7791e^{-04}$
1/320	$2.4825e^{-10}$	$4.9817e^{-06}$	$3.0108e^{-11}$	$8.1217e^{-05}$
1/640	$2.7049e^{-11}$	$1.4812e^{-06}$	$2.0899e^{-12}$	$3.6646e^{-05}$
1/1280	$2.9160e^{-12}$	$4.3706e^{-07}$	$1.4497e^{-13}$	$1.6309e^{-05}$

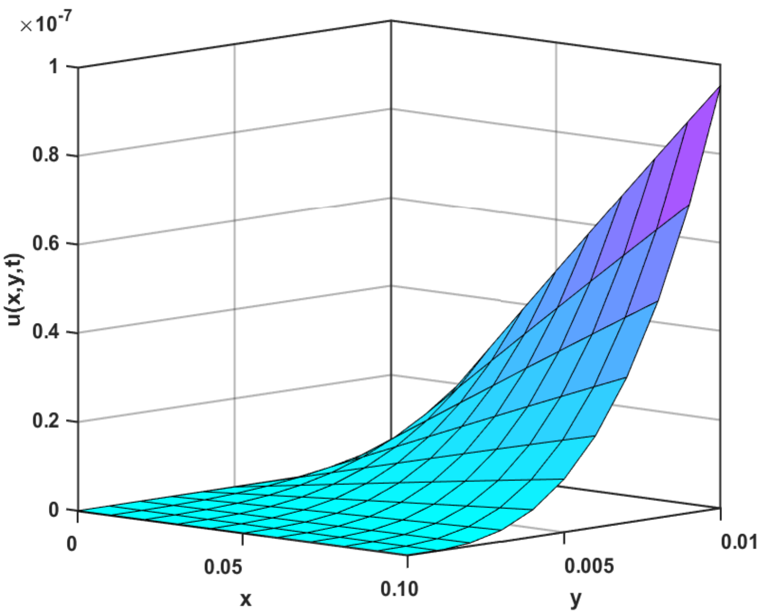

Fig.4. Approximatesolution$u(x,y,t)$at$t$=0.1andα=2,forEx. 6.2.

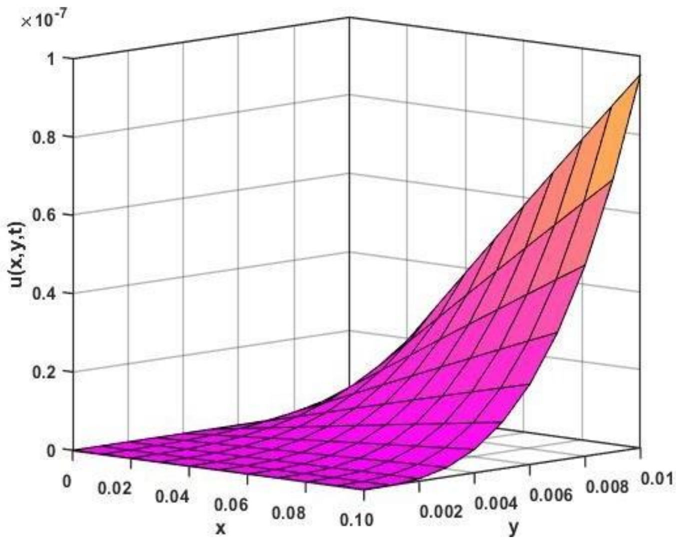

Fig.5. Analyticalsolution$u(x,y,t)$at$t$=0.1,for Ex.6.2.

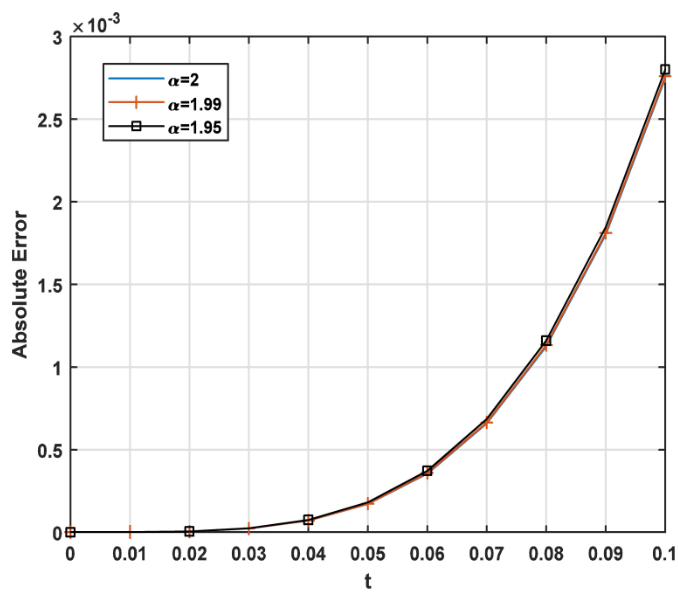

Fig.6. AbsoluteerrorofSADTatx=0.1anddistinctvalueofα,forEx.6.2.

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7 Simulationresultsanddiscussion

This article uses SADT to solve two FCHE examples. For Ex. 6.1 and Ex. 6.2, the maximum absolute errors are found for distinct valueof α and t. The results are shown in tables 1 and 3, for Ex. 6.1 and Ex. 6.2. It is evident that the SADT has relatively small maximum absolute errors. The current techniques for Ex. 6.1 and Ex. 6.2 are compared with the maximum absolute errors of the SADT in Tables 2 and 4. The results show that the SADT is more accurate than the existing techniques,with a maximum absolute errors that is lower than the RBF-PU approach.
TheexactsolutionandtheapproximatesolutionbytheSADTare representedbygraphsat y=0.1and distinct valueofx,t,αin Fig. (1)−(2) for Ex. 6.1 and in Fig. (4)−(5) for Ex. 6.2. Figure 3 illus-tratestheabsolute,whereasFig.6illustratestheabsoluteforEx. 6.2. It is evident that the precise answer and the numerical solutionare almost identical in nature. Therefore, it is evident from the fig-ures and tables that the suggested method, SADT, is a very effective andefficientmethodforsolvingthefractionalmodels.

8 Conclusions

In thisarticle, we investigate the FCHE foranalysing theconduction ofheatinporousmediumviaCaputofractionalorderoperator.The existence and uniqueness is analysed by using the fixed point theo-rem and the highest errors of the designed method, SADT, is also analysed. Two different cases are solved, and the proposed approach, SADT, yields more accurate results than the existing methods; also, themaximumabsoluteerrorsforSADTaresmallerthanthoseofthe existingmethod.Thisresearchwillbebeneficialforhandlingtheis-suesofthermo-elasticinporousmediasinceitprovidesanaccurate forecastabouttheheatconductionandamathematicalexplanation of how it works. Hence, we can conclude that the SADT is a highly efficienttechniquethatcanbeusedtofindthesolutionofnon-linear fractional order models arising in real life phenomena.

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