Volume 8,Number 1 January 2006
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

A quarterly international publication of Eudoxus Press, LLC

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Journal of Computational Analysis and Applications

Editorial Board-Associate Editors

JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.1,5-24,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

Solitary Waves of Depression

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Abstract

It is shown that the regularized long-wave equation admits a family of solitary waves of depression. Some of these solitary waves are stable while others are unstable. The proof of stability and instability is based on the general theory of Grillakis, Shatah and Strauss. The results are illustrated by numerical simulation using a spectral discretization.

Keywords:
Model Equations, Solitary Waves, Stability, Dispersion.

1 Introduction

This article is focused on stability properties of traveling-wave solutions to the regularized long-wave equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.1}$$

which appears as a model equation for surface water waves. In particular, it is shown that there is a family of solitary waves of depression which contains both stable and unstable members. To put this into perspective, recall that equation (1.1) has positive solitary-wave solutions of the form

$$u(x,t) = 3(c-1)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}(x-ct)\right), \tag{1.2}$$

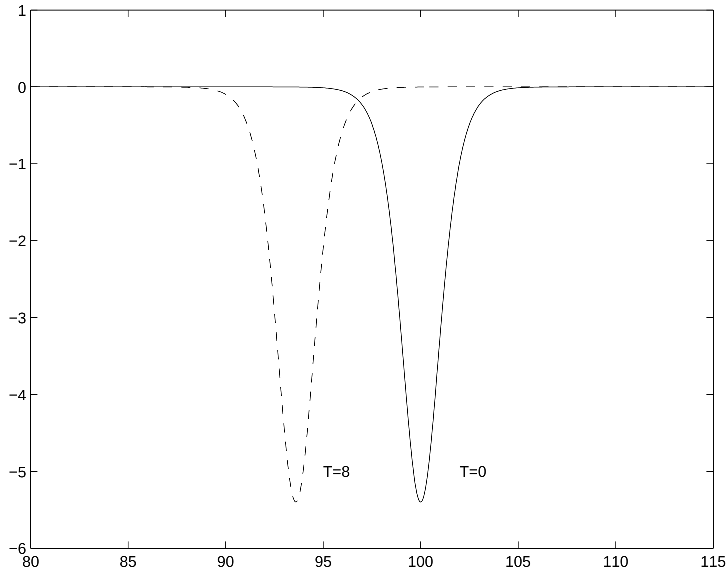

Figure 1: Solitary wave of depression with speed $c = -0.8$.

where $c > 1$ is the speed of the solitary wave. As can be seen from the formula, these solutions are strictly positive progressive waves which propagate without changing their profile over time. It is well known that these positive solitary waves are stable with respect to small perturbations. One of the first proofs of stability was given by Benjamin and Bona in [3, 6], where the concept of orbital stability was introduced. In fact, it was proved by Miller and Weinstein that these solitary waves are asymptotically stable [12].

The proof of stability and instability given in the present work relies on the very general theory of Grillakis, Shatah and Strauss [10], and subsequent work of Albert, Bona, Souganidis and Strauss [1, 9, 15]. Their method has been applied to a number of evolution equations, including the equation under study in this article [15]. However, in the existing literature, the focus has been on positive solitary-wave solutions, rather than on solitary waves of depression.

As is evident from (1.2), solitary waves are strictly negative when $c < 0$. Figure 1 shows a typical solitary wave of depression. It is apparent that the amplitude of the waves is of order 1 in this case. As will be explained in section 2, these solutions therefore do not fall into the regime of physical validity of the equation as a long-wave model. This concurs with the fact that solitary waves of depression do not occur on the surface of fluids unless surface tension is very strong [4].

Nevertheless, it will be shown in section 3 that most of the solitary waves of depression are observable in the sense that they are stable with respect to small perturbations. In particular, there is a critical value $c_0 = \frac{1}{6} - \frac{1}{12}\sqrt{10}$, such that the solitary wave is stable for $c < c_0$, and unstable for $c_0 < c < 0$. This situation is similar to the fact that for the generalized regularized long-wave equation

$$u_t + u_x + u^p u_x - u_{xxt} = 0,$$

where p is a positive integer, there exist both stable and unstable positive solitary-wave solution if $p \geq 4$. The generalized equation also admits solitary waves of depression, and their stability properties will be a topic of future study.

In section 4, numerical simulations are presented to illustrate the results of stability of instability obtained in section 3.

To close the introduction, we establish some notation to be used in the proof of stability and instability. For $1 \leq p < \infty$, the space $L^p = L^p(\mathbb{R})$ is the set of measurable real-valued functions of a real variable, for which the integral

$$\int_{-\infty}^{\infty} |f(x)|^p dx$$

is finite. For $s \geq 0$, the space $H^s = H^s(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ consisting of functions such that the integral

$$\int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi$$

is finite. Here the circumflex denotes the Fourier transform.

2 Long-wave models

Equation (1.1), which is also known as the Benjamin-Bona-Mahoney (BBM) equation, was introduced as a model for the propagation of long surface water waves of small amplitude in a narrow channel [5, 14]. Let us briefly recall the rationale behind using (1.1) as an alternative model instead of the related Korteweg-de Vries (KdV) equation

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{2.1}$$

Let (x, y, z) connote a standard Cartesian coordinate system with z the vertical direction and $z = 0$ located at the surface of a fluid in a long narrow channel of depth h. Consideration is given to waves on the surface whose primary direction of propagation is that of increasing values of x, which do not vary significantly in the y-direction, and for which the effects of surface tension and viscosity may be safely ignored. It is assumed that a typical wave amplitude is a, and a typical wavelength is λ, and that the quantities $\frac{h^2}{\lambda^2}$ and $\frac{a}{h}$ are of comparable magnitude. The function $u(x,t)$ describes the vertical deviation of the surface from its rest position at the point x at time t.

When the variables u, x and t are non-dimensional and scaled so that the dependent variable and its derivatives are of order one, (2.1) takes the revealing form

$$u_t + u_x + \epsilon u u_x + \epsilon u_{xxx} = O(\epsilon^2), \tag{2.2}$$

where ϵ is of order $\frac{h^2}{\lambda^2} \cong \frac{a}{h}$, and the $O(\epsilon^2)$ represents terms in the formal approximation which are of quadratic or higher order in ϵ. The KdV equation obtains by disregarding all terms of order ϵ^2 in (2.2). It also follows from (2.2) that

$$u_t + u_x = O(\epsilon), \tag{2.3}$$

and the small parameter ϵ appearing in the equation shows the dispersive term u_{xxx} and the nonlinear term uu_x to be corrections of the same order to the basic uni-directional hyperbolic equation $u_t + u_x = 0$. Under the assumption that differentiation does not alter the ϵ-order of the dependent variable, (2.3) implies that

$$u_{xxx} + u_{xxt} = O(\epsilon),$$

so that u_{xxx} may be replaced by $-u_{xxt}$ in (2.2) to obtain

$$u_t + u_x + \epsilon u u_x - \epsilon u_{xxt} = O(\epsilon^2).$$

Again, disregarding terms of order ϵ^2 and then rescaling, there appears the alternative model

$$u_t + u_x + u u_x - u_{xxt} = 0.$$

Now since this equation is given in the original variables, it appears that for solutions that are physically valid, u should be much smaller than 1. As was stated in the introduction, the solitary waves of depression have magnitude of order 1, so that they do not belong to the class of solutions that have a physical significance. This is also borne out by the fact that their velocity is negative, so that they are propagating to the left, whereas the derivation of equation (1.1) assumes right-moving waves.

Notwithstanding the size of initial data, it was proven that the initial-value problem associated to (1.1) is well posed in appropriate function classes. In particular, it was shown in [2] that the problem is globally well-posed in $H^1(\mathbb{R})$. For the proof of global well posedness, use is made of the invariant integral

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty} \left(u^2 + u_x^2\right)\, dx,$$

which is proven to be conserved as soon as the initial data are in $H^1(\mathbb{R})$. The equation has another invariant integral, namely

$$F(u) = \frac{1}{2}\int_{-\infty}^{\infty} \left(u^2 + \frac{1}{3}u^3\right)\, dx.$$

These two functionals are of critical importance in the proof of stability and instability given in the next section.

It will be convenient to recall an alternative formulation of the equation. Note that (1.1) can be rewritten as

$$(1 - \partial_{xx})u_t + \partial_x \left(u + \frac{1}{2}u^2\right) = 0.$$

Inverting the operator $1 - \partial_{xx}$, there appears the integral equation

$$u_t + \frac{\partial_x}{1 - \partial_{xx}} \left(u + \frac{1}{2}u^2\right) = 0. \tag{2.4}$$

Defining $J = -\frac{1}{2}\partial_x(1 - \partial_x^2)^{-1}$, it is plain that (2.4) can be written as

$$u_t = JF'(u).$$

This is the general form of an equation to which the theory in [10] is applicable.

3 Stability of solitary waves

A solitary-wave solution of (1.1) has the special form $u(x,t) = \phi(x - ct)$, where c is the speed of propagation of the solitary wave. It follows that ϕ satisfies the equation

$$-c\phi' + c\phi''' + \phi' + \phi\phi' = 0, \tag{3.1}$$

where ϕ' denotes the derivative of ϕ with respect to the variable $\eta = x - ct$. The equation (3.1) can be integrated once to yield

$$-c\phi + c\phi'' + \phi + \frac{1}{2}\phi^2 = 0. \tag{3.2}$$

It is elementary to check that

$$\phi(x) = 3(c-1)\text{sech}^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}\; x\right) \tag{3.3}$$

is a solution of this equation for all $c < 0$. Note also that equation (3.2) can be written in variational form in terms of the functionals E and F as

$$-cE'(\phi) + F'(\phi) = 0.$$

In light of the fact that the conserved integral E represents the H^1-norm, and that the initial-value problem is therefore globally well posed in H^1, the natural norm to use in the definition of stability is the H^1-norm. Accordingly, a viable definition is the following.

Definition. *A solitary-wave solution ϕ of (1.1) is stable if for every $\epsilon > 0$ there is $\delta > 0$ such that if $u \in C\big([0,\infty); H^1(\mathbb{R})\big)$ is a solution to (1.1) with $\|u(\cdot,0) - \phi\|_{H^1} \leq \delta$, then for every $t \in [0,\infty)$*

$$\inf_{s\in\mathbb{R}} \|u(\cdot,t) - \phi(\cdot - s)\|_{H^1} \leq \epsilon.$$

Otherwise, ϕ is called unstable.

Let us briefly explain why it is essential to consider the infimum over all translations. The expression (1.2) shows that solitary waves of larger amplitude travel at a higher speed. So in particular, two solitary waves which may differ ever so slightly in height will drift apart as time passes, even though their crests may have been perfectly aligned initially. As a consequence, the usual notion of Lyapunov stability is not appropriate for the problem at hand. Instead, the proper framework to study the stability of solitary waves is the stability in shape, or orbital stability. In fact, taking the infimum over all translations effectively measures the difference in shape of two wave profiles. With the appropriate notion of stability in place, the following theorem can be stated.

Theorem. Solitary-wave solutions of (1.1) are stable if $c < c_0 = \frac{1}{6} - \frac{1}{12}\sqrt{10}$, and unstable if $c_0 < c < 0$.

To prove the orbital stability of the solitary waves, use is made of the general theory of Grillakis, Shatah and Strauss [10]. To prove instability, their result cannot be applied directly, because the operator $J = -\frac{1}{2}\partial_x(1-\partial_x^2)^{-1}$ is not surjective. This difficulty has been surmounted however in the work of Souganidis and Strauss [15]. They consider a fairly general family of evolution equations which contains equation (1.1) as a special case. The only assumption used in their proof that does not hold in the present situation is the positivity of the solitary waves. This property is needed in one part of their proof (Theorem 2.3 in [15]) which is replaced here by Lemma 1. The statement is essentially the same though the proof is slightly more intricate.

We proceed to give an outline of the assumptions needed for the application of the theory in [10, 15] As was indicated before, equation (3.2) can be written in variational form as

$$-cE'(\phi_c) + F'(\phi_c) = 0,$$

where ϕ_c denotes a solitary wave with velocity c. The functional derivative of this relation is given by the linear operator

$$\mathcal{L}_c \quad = \quad c\partial_x^2 - c + \phi_c + 1.$$

Note that since $c < 0$, $c\partial_x^2 - c + 1$ is a positive operator. The following requirements on $\mathcal{L}_c$ have been shown to hold in [15] and [17] for a wide class of operators, including the operator at hand. Since the exact form of the function ϕ_c is known in this case, they could also be verified directly.

1. $\mathcal{L}_c$ has positive continuous spectrum bounded away from zero, a simple zero eigenvalue with eigenfunction ϕ_c', and one negative simple eigenvalue with corresponding eigenfunction χ_c.

2. The mapping $c \to \chi_c$ is continuous with values in $H^2(\mathbb{R})$, and $(1+|x|)\chi_c(x) \in L^1(\mathbb{R})$.

3. The mapping $c \to \phi_c$ is C^1 with values in $H^2(\mathbb{R})$, $\phi_c \in H^4(\mathbb{R})$, and $(1+|x|)\frac{\partial \phi}{\partial c}(x) \in L^1(\mathbb{R})$.

With these assumptions in place, the proof of stability and instability becomes essentially a special case of the results in [10, 15]. Accordingly, the stability of a solitary wave with speed c is determined by the convexity of the function $d(c) = -cE(\phi_c) + F(\phi_c)$. In particular, a solitary wave with speed c is stable if $d(c)$ is convex in a neighborhood of c, and it is unstable if $d(c)$ is concave in a neighborhood of c.

The only missing link is Theorem 2.3 in [15], which uses the strict positivity of the solitary wave. However, as mentioned previously, the following lemma replaces this theorem in the present case.

Lemma 1. Let c be fixed. If $d''(c) < 0$, then there exists a curve $\omega \mapsto \psi_\omega$ in a neighborhood of c, such that $\psi_c = \phi_c$, $E(\psi_\omega) = E(\phi_c)$ for all ω, and $F(\psi_\omega) < F(\phi_c)$ for $\omega \neq c$.

Proof: Consider the map $(\omega, s) \mapsto E(\phi_\omega + s\chi_c)$, where χ_c is the eigenfunction corresponding to the negative eigenvalue of the operator $\mathcal{L}_c$. Note that $(c, 0) \mapsto E(\phi_c)$. To obtain the mapping $\omega \mapsto \psi_\omega$, one may apply the implicit function theorem if it can be shown that

$$\frac{\partial}{\partial s}\left\{E(\phi_\omega + s\chi_c)\right\}\Big|_{\omega=c, s=0} = \int E'(\phi_c)\,\chi_c$$

is nonzero. The proof of this fact is relegated to the appendix. Once it is noted that this derivative is nonzero, the proof of the lemma follows the proof of Theorem 2.3 in [15] verbatim. $\square$

Since we are now exactly in a situation in which the theory in [10] and [15] can be applied, the convexity properties of the function $d(c)$ will be investigated.

Lemma 2. The function $d(c) = -cE(\phi_c) + F(\phi_c)$ is convex if $c < \frac{1}{6} - \frac{1}{12}\sqrt{10}$, and concave if $\frac{1}{6} - \frac{1}{12}\sqrt{10} < c < 0$.

Proof: Consider the first derivative

$$d'(c) = \left\langle -cE'(\phi_c) + F'(\phi_c), \frac{\partial \phi_c}{\partial c}\right\rangle - E(\phi_c) = -E(\phi_c).$$

By the formula for the solitary wave, it appears that

$$\begin{aligned} d'(c) &= -\frac{1}{2}\int_{-\infty}^{\infty} \left(\phi_c^2 + (\phi_c')^2\right)\, dx \\ &= -\,9(c-1)^2\sqrt{\frac{c}{c-1}}\int_{-\infty}^{\infty} \operatorname{sech}^4(x)\, dx \\ &\quad -\,9(c-1)^2\sqrt{\frac{c-1}{c}}\int_{-\infty}^{\infty} \operatorname{sech}^4(x)\, \tanh^2(x)\, dx. \end{aligned}$$

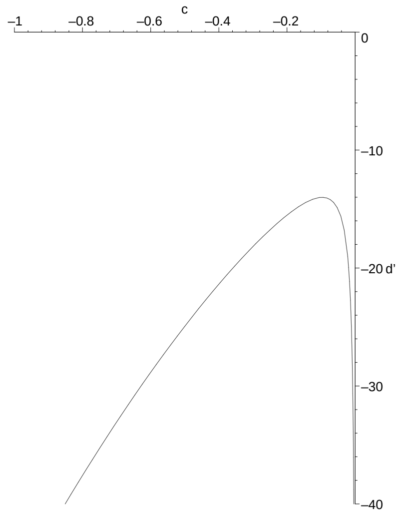

Figure 2: d'(c).

Evaluating the two integrals yields

$$d'(c) = -\ 12(c-1)^2\sqrt{\frac{c}{c-1}} \quad - \quad \frac{12}{5}(c-1)^2\sqrt{\frac{c-1}{c}}.$$

Elementary computations reveal that $d''(c)$ has a zero at $c = c_0 = \frac{1}{6} - \frac{1}{12}\sqrt{10}$, and that $d'(c)$ is increasing for $c < \frac{1}{6} - \frac{1}{12}\sqrt{10}$, and decreasing for $\frac{1}{6} - \frac{1}{12}\sqrt{10} < c < 0$. □

In connection with the theory in [10] and [15], this lemma provides a proof of Theorem 1. The numerical value of c_0 is approximately -0.097, as is also indicated in Figure 2.

4 Numerical simulation

In the following, a numerical study is presented to illustrate the results obtained in the previous section. To discretize equation (1.1), we use a Fourier-collocation method coupled with a 4-stage Runge-Kutta time integration scheme. Since the system of equations resulting from the spectral projection of (1.1) is not stiff, a high-order explicit time-stepping algorithm is the most viable candidate to match the extreme accuracy of the spectral discretization in the space variable.

For the purpose of numerical approximation, the problem is posed with periodic boundary conditions on the domain $x \in [0, L]$, where L varies between 200 and 500. It was shown by Pasciak [13] that solutions to the initial-value problem on the real line which have algebraic decay of some order maintain this property for all time. In particular, initial data with exponential decay will yield solutions that decay faster than any polynomial for positive times. For exponentially decaying initial data, it is therefore safe to assume that the solutions have sufficient decay, so that the tails lie below the computational accuracy of the computer if a sufficiently large domain is used. It was observed that $L = 500$ was more than sufficient for the computations shown in this paper.

The problem is translated to the interval $[0, 2\pi]$ by the scaling $u(x,t) = v(x/a, t)$, where $a = \frac{L}{2\pi}$. The initial-value problem is then

$$\left.\begin{array}{rcll} a^2 v_t + a v_x + a v v_x - v_{xxt} &=& 0, & x \in [0, 2\pi]\ , t \geq 0, \\ v(x,0) &=& u_0(ax), & \\ v(0,t) &=& v(2\pi, t), & t \geq 0. \end{array}\right\} \quad (4.1)$$

Let S_N be the subspace of $L^2(0, 2\pi)$ spanned by the set

$$\left\{ e^{ikx} \;\middle|\; k \in \mathbb{Z},\ -\frac{N}{2} \leq k \leq \frac{N}{2} - 1 \right\},$$

for N even. Instead of (4.1), we use the equivalent formulation as an integral equation as in (2.4), namely

$$v_t = -\frac{a\partial_x}{a^2 - \partial_x^2} \left(v + \frac{1}{2} v^2 \right).$$

The collocation approximation is defined as follows. Find a function v_N from $[0, T]$ to S_N, such that

$$\left.\begin{array}{rcl} \partial_t v_N(x_j) &=& K_N(v_N + \frac{1}{2} v_N^2)(x_j), \\ v_N(0) &=& I_N u_0(ax) \in S_N, \end{array}\right\} \quad (4.2)$$

at the collocation points $x_j = \frac{2\pi j}{N}$, for $j = 0, 1, 2, \ldots N-1$. Here I_N denotes the operator which gives the Nth degree trigonometric interpolant at the gridpoints x_j. We assume that the solution is written as the sum

$$v_N(x,t) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{v}_N(k,t) e^{ikx},$$

h	L^2 -error	Ratio
0.1000	7.8226e-05	
0.0500	4.4138e-06	17.723
0.0250	2.6056e-07	16.940
0.0125	1.5801e-08	16.490
0.0063	9.7229e-10	16.251
0.0031	6.0236e-11	16.142
0.0016	3.7116e-12	16.230
0.0008	2.1690e-13	17.112

Table 1: Regularized long-wave equation; error due to temporal discretization.

where the $\tilde{v}_N(k,t)$ can be thought of as the discrete Fourier coefficients of $v_N(x,t)$. K_N is defined generally via the discrete Fourier coefficients $\tilde{\psi}(k)$ of $\psi \in S_N$ as

$$(\widetilde{K_N\psi})(k) = a\frac{ik}{a^2+k^2}\tilde{\psi}(k),$$

where

$$\tilde{\psi}(k) = \frac{1}{N}\sum_{j=0}^{N-1}\psi(x_j)e^{-ikx_j},$$

for $-\frac{N}{2} \le k < \frac{N}{2} - 1$. The problem (4.2) is a system of N coupled ordinary differential equation for the discrete Fourier coefficients $\tilde{v}_N(k,t)$. This system is integrated using a four-stage explicit Runge-Kutta scheme with time step h.

No attempt has been made to prove the convergence of the discretization explained above. However, an experimental convergence study is presented to validate the numerical method. The norm used to calculate the error is the normalized discrete L^2-norm

$$\|v\|_{N,2}^2 = \frac{1}{N}\sum_{i=1}^{N}|v(x_i)|^2.$$

The L^2-error is then defined to be $\frac{\|v-v_N\|_{N,2}}{\|v\|_{N,2}}$.

To check the algorithm, we used the exact form (1.2) of the solitary waves with various values of c, both positive and negative. A representative result for the wave appearing in Figure 1 is given in Tables 1 and 2. In this calculation, the solution was approximated from $T = 0$ to $T = 8$ and the

N	L^2-error	Ratio
1024	4.921e-01	
2048	2.378e-01	2.07
4096	2.125e-02	11.19
8192	1.968e-04	107.69
16384	2.431e-08	8097.02
32768	1.335e-09	1.82

Table 2: Regularized long-wave equation; error due to spatial discretization.

size of the domain was $L = 200$. In the computations shown in Table 1, 4096 Fourier modes were used. The 4th-order convergence of the scheme is apparent up to $h = 0.0008$. Table 2 displays the spatial convergence rate for a calculation with time step $h = 0.001$. We observe exponential convergence before reaching the limit set by the size of the time step. Similar results obtain for all other trials.

In order to study the stability of solitary waves of depression, the exact formulation (3.3) for various values of $c < 0$ is used. Initial data are chosen as a perturbation of the solitary wave in the amplitude or the wavelength. Thus, typical initial data have the form

$$u_0(x) = A\,\phi_c(x) \tag{4.3}$$

where A represents the perturbation of the amplitude, or

$$u_0(x) = \phi_c(\gamma\,x) \tag{4.4}$$

where γ represents the perturbation of the wavelength.

Depending on the speed c of the perturbed solitary wave, the initial data evolve into a solitary wave of amplitude close to the perturbed solitary wave, or disintegrates. For solitary waves in the stable range of c, small perturbations always yield solutions that are close to the original solitary wave, as is to be expected. Even rather large perturbations can be used, but the resulting solitary waves generally have different speeds. In Figures 3, 4 and 5, a calculation is shown where a solitary wave with speed $c = -1$ is perturbed in the amplitude with $A = 0.67$ in (4.3). As can be seen in the figures, the initial wave profile sheds a dispersive tail and evolves into a solitary wave with $c \sim -0.38$ and with height close to the height of the initial data. In order to verify that the resulting waveform is close to a solitary wave, we measured the height, and compared it to a solitary wave of the

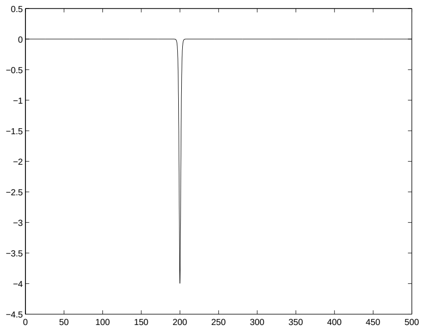

Figure 3:
Initial data: solitary wave with $c = -1$, perturbed with $A = 0.67$

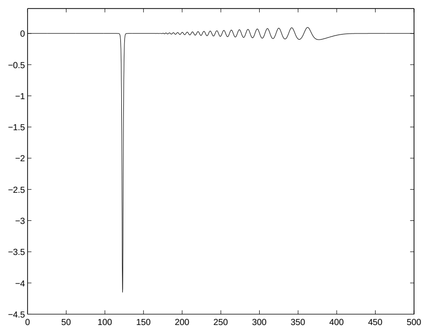

Figure 4: Resulting solitary wave with $c \sim -0.38$, and oscillatory tail at $T = 200$.

according height, translated to the minimum on the numerical grid. Table 3 shows the L^∞-error in shape between the evolving wave form and the corresponding solitary wave for the same calculation as shown in Figures 3, 4 and 5. It is better to use the L^∞-error for this comparison, because due to the finite grid size, there always exists a phase shift between the computed solitary wave and the fitted curve. The L^∞-error is defined analogously to the L^2-error by $\frac{\|v-v_N\|_{N,\infty}}{\|v\|_{N,\infty}}$, where

$$\|v\|_{N,\infty} = \max_{1\leq i\leq N} |v(x_i)|.$$

It is apparent in Table 3 that the error in shape diminishes over time. We also monitored discrete forms of the conserved integrals E and F, and it can be seen in Table 3, that their conservation was superior, thus adding confidence in the performed computations.

Experiments with solitary waves perturbed in wavelength as in (4.4) gave similar results. One interesting case is shown in Figures 6 and 7, where initial data were given by a solitary wave with speed $c = -0.5$, perturbed in the wavelength with $\gamma = 2$. It appears that the initial data evolve into a smaller negative solitary wave, a dispersive wavetrain and a positive solitary wave moving into the opposite direction.

An interesting point is that as the limit speed $c_0 = \frac{1}{6} - \frac{1}{12}\sqrt{10}$ for stability is approached, the perturbation of the solitary wave has to be smaller and smaller in order to observe stability. If a solitary wave with speed below, but close to c_0 is perturbed too much, it will disintegrate. In consequence, it seems that it would be difficult to determine the critical wavespeed c_0 purely

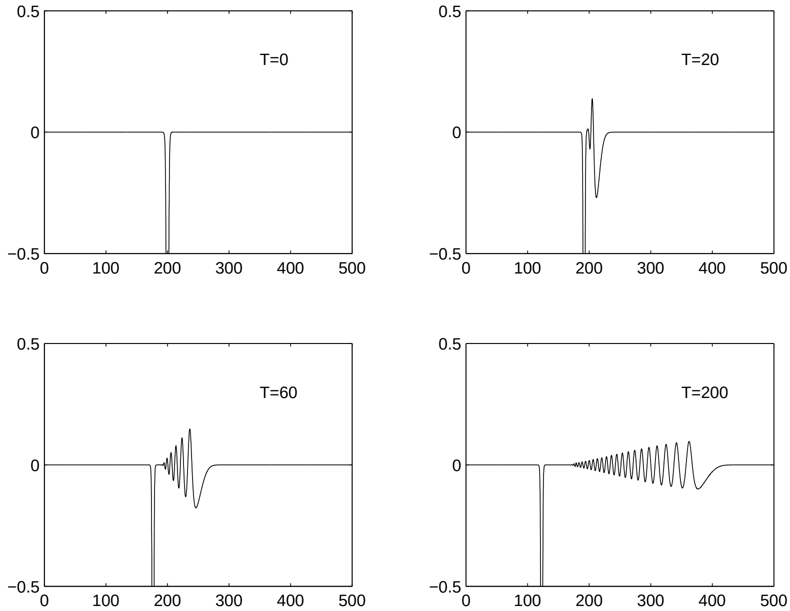

Figure 5: Close-up of the calculation in Figures 3 and 4.

through numerical experiments. A related question is whether there exists a functional relationship between the wavespeed and the maximal allowable perturbation in, say amplitude. Some computations have been made in the pursuit of establishing such a relation, but no conclusive evidence can be reported here.

In Figures 8 and 9, the evolution of the perturbation of an unstable solitary wave is depicted. In this particular case, the solitary wave had speed $c = -0.05$, and was perturbed in the amplitude with $A = 0.99$. To be sure, many different runs with varying perturbations were completed, and so long as $A < 1$, the solitary wave disintegrated completely. Again, the conserved integrals were monitored for the duration of the time evolution, and it was found that they were conserved well. In Figures 11 and 12, a computation of a solitary wave perturbed with $A = 0.99999$ is shown. It is apparent that perturbing an unstable solitary wave by lowering the amplitude ever so slightly results in the complete dispersion if the initial profile. This might be related to a result of Albert [1] which states that low-energy solutions of the generalized regularized long-wave equation

$$u_t + u_x + u^p u_x - u_{xxt} = 0, \tag{4.5}$$

t	L^∞-error	E	F
20	0.7701	42.2378	-2.0113
40	0.4340	42.2378	-2.0113
60	0.1997	42.2378	-2.0113
80	0.1058	42.2378	-2.0113
100	0.0527	42.2378	-2.0113
120	0.0272	42.2378	-2.0113
140	0.0170	42.2378	-2.0113
160	0.0280	42.2378	-2.0113
180	0.0199	42.2378	-2.0113
200	0.0039	42.2378	-2.0113

Table 3: Error in shape and conserved integrals at different times for the computations shown in Figures 3, 4 and 5.

disperse if $p > 4$. However, his result does not apply directly to the regularized long-wave equation proper.

The instability of solitary waves with speed above the critical speed seems to manifest itself in a completely different manner if the amplitude is raised, i.e. if $A > 1$. In this case, the initial profile develops into a stable solitary wave with speed below c_0, and a positive solitary wave, moving in the opposite direction. Such a case is depicted in Figures 13 and 14.

In closing, we would like to reiterate that the generalized regularized long-wave equation (4.5) also admits negative solitary waves. It will be interesting to study the stability of these waves, and to compare a possible instability to the instability of the positive solitary waves when $p \geq 4$.

Acknowledgements. This research was supported by the BeMatA program of the Research Council of Norway.

A Spectral Analysis of $\mathcal{L}_c$

In the proof of Lemma 1, it is used for the application of the implicit function theorem that the integral

$$\int E'(\phi_c)\,\chi_c = \frac{1}{c}\int F'(\phi_c)\chi_c = \frac{1}{c}\int \left(\phi_c + \frac{1}{2}\phi_c^2\right)\chi_c \tag{A.1}$$

is nonzero. Recall that ϕ_c is the solitary wave with speed c, and that χ_c is the eigenfunction corresponding to the sole negative eigenvalue of the linear

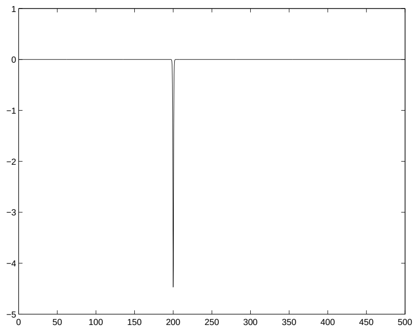

Figure 6:
Initial data: solitary wave with $c = -0.5$, perturbed in the wavelength with $\gamma = 2$.

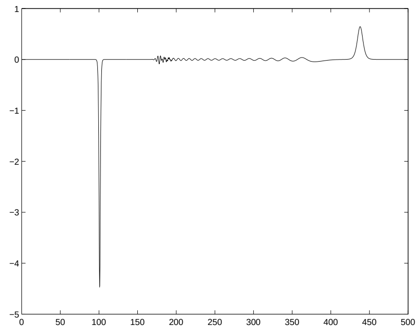

Figure 7: Resulting negative and positive solitary waves, separated by a dispersive wavetrain.

operator $\mathcal{L}_c$ given by

$$\mathcal{L}_c = c\partial_x^2 - c + \phi_c + 1.$$

In the present context, the exact form of the eigenfunction χ_c may be used to evaluated the integral (A.1). The spectral problem is of the form

$$\mathcal{L}_c \chi_c = \lambda_c \chi_c,$$

and it can be shown that ϕ_c' is the unique eigenfunction for the eigenvalue 0 (cf. [16]). Since ϕ_c' has exactly one zero, it follows from the general theory of second-order linear operators that 0 is the second eigenvalue from the left. Therefore, there is precisely one negative eigenvalue. In general, the eigenfunctions are given in terms of Gamma functions, but the case at hand is particularly simple. It can be checked that the lowest eigenvalue is $\lambda_c = \frac{5}{4}(c-1)$, and the corresponding eigenfunction is

$$\chi_c(x) = \operatorname{sech}^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}\; x\right).$$

Moreover, χ_c spans the eigenspace corresponding to λ_c. Using the expressions for ϕ_c and χ_c, the integral (A.1) can be evaluated as follows.

$$\begin{aligned}\frac{1}{c}\int_{-\infty}^{\infty}\left(\phi_c+\frac{1}{2}\phi_c^2\right)\chi_c\,dx &= \frac{1}{c}\,3(c-1)\int_{-\infty}^{\infty}\operatorname{sech}^5\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)\,dx\\ &\quad+\frac{1}{2c}\,9(c-1)^2\int_{-\infty}^{\infty}\operatorname{sech}^7\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)\,dx\\ &= \frac{1}{c}\,3(c-1)\left(2\sqrt{\frac{c}{c-1}}\right)\left[\frac{3}{8}\pi\;+\;\frac{3}{2}(c-1)\frac{5}{16}\pi\right].\end{aligned}$$

Thus it becomes obvious that this integral is nonzero for all negative c, and in particular for $c_0 < c < 0$.

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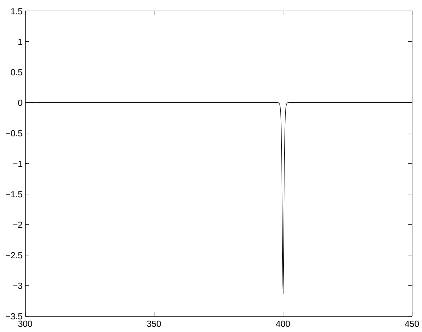

Figure 8:
Initial data: Perturbed unstable solitary wave with $c = -0.05$ and $A = 0.99$.

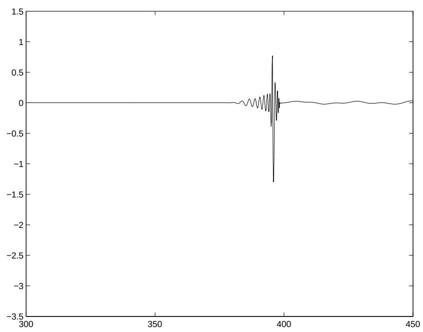

Figure 9: Resulting wave profile at $T = 160$.

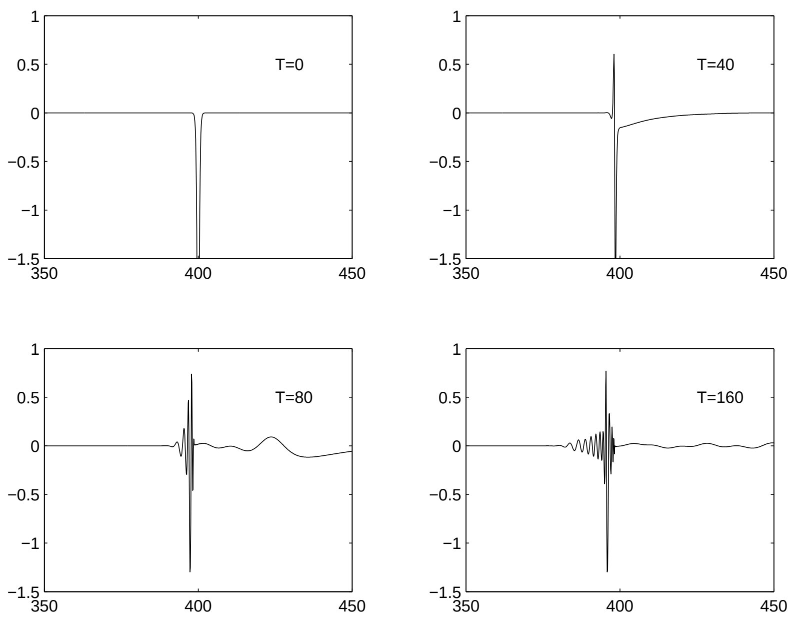

Figure 10: Perturbed unstable solitary wave with $c = -0.05$, close-up.

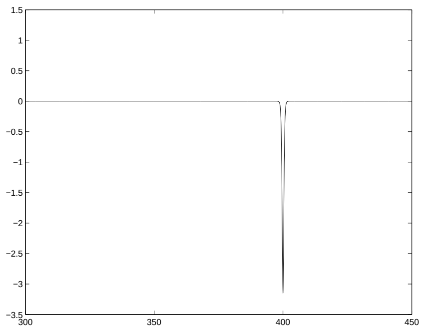

Figure 11:
Initial data: Perturbed unstable solitary wave with $c = -0.05$ and $A = 0.99999$.

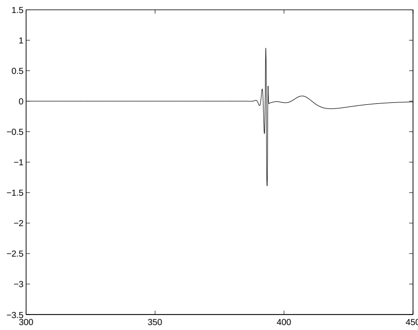

Figure 12: Resulting wave profile at $T = 160$.

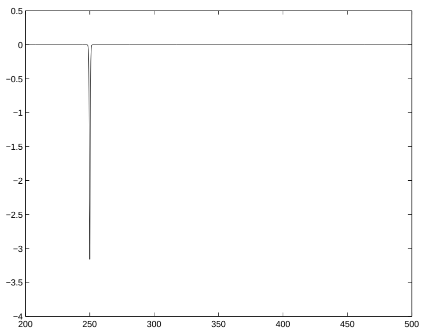

Figure 13:
Initial data: Perturbed unstable solitary wave with $c = -0.05$ and $A = 1.01$.

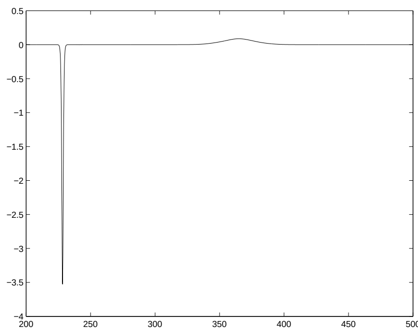

Figure 14: Resulting wave profile at $T = 160$. The negative solitary wave has a speed of approximately $c = -0.1754$.

JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.1,25-38,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

APPROXIMATION OF COMMON FIXED POINTS FOR A CLASS OF FINITE NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract: In this paper, we study the approximation problems of common fixed points of Halpern's iterative sequence for a class of finite nonexpansive mappings in Banach spaces without using the concept of Banach's limit. The main results generalize, extend and improve the corresponding results of Bauschke [1], Halpern [5], Shioji and Takahashi [10], Takahashi, Tamura and Toyoda [14], Wittmann [16], Xu [17], [18] and others.

2000 AMS Subject Classification: 47H05, 47H09, 49M05.

Key Words and Phrases: Nonexpansive mapping, Halpern's iterative sequence, common fixed point, Banach's limit.

Let X be a real Banach space and X^* be the dual space of X. Let J denote the *normalized duality mapping* from X into 2^{X^*} defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}$$

for all $x \in X$, where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between X and X^*.

The second author was supported from the Korea Research Foundation Grant (KRF-2000-DP0013).

Now we give some elementary definitions:

Definition 1. (1) A Banach space X is said to be *strictly convex* if

$$\frac{\|x+y\|}{2} < 1$$

for all $x, y \in S_X$, where $S_X = \{z \in X : \|z\| = 1\}$.

(2) For any ϵ with $0 \leq \epsilon \leq 2$, we define the *modulus* $\delta(\epsilon)$ *of convexity* of X by

$$\delta(\epsilon) = \inf \Big\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon \Big\}.$$

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for any $\epsilon > 0$.

(3) A Banach space X is said to be *smooth* if

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

exists for all $x, y \in S_X$. In this case, the norm of E is said to be *Gâteaux differentiable*

(4) A Banach space X is *uniformly smooth* if the limit

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

exists and is attained uniformly in $x, y \in S_X$.

(5) The norm of X is said to be *uniformly Gâteaux differentiable* if, for any $y \in S_X$,

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

exists uniformly for all $x \in S_X$.

Remark 1. (1) Banach space X is strictly convex if and only if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for all $x, y \in X$ and $0 < \lambda < 1$ implies that $x = y$.

(2) A uniformly convex Banach space X is strictly convex, but the converse is not true.

(3) If a Banach space X is (uniformly) smooth, then the normalized duality mapping J is single-valued. Moreover, if the norm of X is uniformly

Gâteaux differentiable, then the normalized duality mapping J is norm to weak* uniformly continuous on any bounded subsets of X.

Definition 2. Let C be a closed convex subset of a Banach space E and F be a subset of C.

(1) A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

(2) A mapping P of C onto F is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

for any $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$.

(3) A subset F of C is called a *nonexpansive retract* of C if there exists a nonexpansive retraction of C onto F.

Remark 2. (1) Let C be a nonempty closed convex subset of a Hilbert space H. Then a mapping P on H is the metric projection onto C if and only if, for any $x \in H$ and $y \in C$,

$$< x - Px, Px - y >\geq 0.$$

Thus, if P is the metric projection of H onto C, then P is sunny and nonexpansive.

(2) Let C be a nonempty convex subset of a smooth Banach space X. We call C a *retract* of X if there exists a continuous mapping $r : X \to C$ with $r(x) = x$ for all $x \in C$ and the mapping r is called a *retraction.* If $C_0 \subset C$ and P is a retraction of C onto C_0 such that

$$< x - Px, J(Px - y) >\geq 0$$

for all $x \in C$ and $y \in C_0$, then P is sunny and nonexpansive.

For a fixed $u \in C$ and each $t \in (0, 1)$, we can define a contractive mapping $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx \tag{1}$$

for all $x \in C$. Then, by Banach's contraction principle, there exists a unique fixed point $z_t \in C$ of T_t, that is, z_t is the unique solution of the equation

$$z_t = tu + (1 - t)Tz_t. \tag{2}$$

In [2], Browder proved that, if X is a Hilbert space, then z_t converges strongly to a fixed point of T as $t \to 0$ and, in [9], Reich extended Browder's result to the setting of uniformly smooth Banach spaces.

The fixed point z_t of T_t in (2) is defined implicitly, but we can devise explicitly an iterative method which converges in norm to a fixed point of T. In [5], Halpern studied initially such a method, which is called *Halpern's iterative sequence*, as follows:

Let $\{\alpha_n\}$ be a sequence in (0,1], u be a fixed anchor in C and $x_0 \in C$ be any initial value. Define a sequence $\{x_n\} \subset C$ in an explicit and iterative way by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0. \tag{H}$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies certain control conditions, two of which are

$$\alpha_n \to 0 \quad (n \to \infty), \tag{C1}$$

$$\sum_{n=0}^{\infty} \alpha_n = \infty \text{ or, equivalently, } \prod_{n=0}^{\infty} (1 - \alpha_n) = 0. \tag{C2}$$

In [7], Lions improves Halpern's control conditions by showing the strong convergence of the sequence $\{x_n\}$ if $\{\alpha_n\}$ satisfies (C1), (C2) and the following condition:

$$\frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}^2} \to 0 \quad (n \to \infty). \tag{C3}$$

Note that, for the natural and important choice $\{\frac{1}{n}\}$ of $\{\alpha_n\}$, the results of both Halpern and Lions don't work.

In [16], Wittmann overcame the problem mentioned above by proving the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies control conditions (C1) and (C2) and the following:

$$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{C4}$$

Recently, Xu [17] suggested the following control condition instead of the conditions (C3) or (C4):

(C5) $$\frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}} \to 0 \text{ or, equivalently, } \frac{\alpha_n}{\alpha_{n+1}} \to 1 \quad (n \to \infty)$$

and proved the strong convergence of Halpern's iterative sequence $\{x_n\}$ and, in [18], he also proved the strong convergence of the sequence $\{x_n\}$ by using the control conditions (C1) and (C2).

Very recently, in [3], Cho, Kang and Zhou considered the following control condition:

(C6) $$|\alpha_{n+1}-\alpha_n| \le \circ(\alpha_{n+1}) + \sigma_n,$$

where $\sum_{n=0}^{\infty} \sigma_n < \infty$, and proved some strong convergence theorems of the Halpern's iterative sequence $\{x_n\}$ for nonexpansive mappings in uniformly smooth Banach spaces. Their results improve the corresponding results of Lions [7], Wittmann [16], Xu [17], [18] and others. For further some examples and relations of the control conditions (C1)$\sim$(C6) on the sequence $\{\alpha_n\}$, see Cho, Kang and Zhou [3].

In this paper, we consider the new control condition to prove some strong convergence theorems of Halpern's iterative sequence for a class of finite nonexpansive mappings $T_1, T_2, \cdots, T_r$ of C into itself with $T_{n+r} = T_n$, where C is a subset of X, without using the concept of Banach's limit (see Remark 4):

(C7) $$|\alpha_{n+r}-\alpha_n| \le \circ(\alpha_{n+r}).$$

Now, we introduce several lemmas for our main results in this paper.

Lemma 1. ([15]) *Let $\{a_n\}$ be a real sequence of nonnegative numbers such that*

$$a_{n+1} \le (1-t_n)a_n + \circ(t_n), \quad n \ge 0,$$

where $t_n \in (0,1)$ with $\sum_{n=0}^{\infty} t_n = \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2. ([9]) *Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C be a closed convex subset of X and T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Let $x_0 \in C$ and z_t be a unique element of C which satisfies $z_t = tx_0 + (1-t)Tz_t$ and*

$0 < t < 1$. *Then* $\{z_t\}$ *converges strongly to a fixed point of* T *as* $t \to 0$. *Further, if* $Px_0 = \lim_{t\to 0} z_t$ *for each* $x_0 \in C$, *then*

$$< x_0 - Px_0, J(Px_0 - z) > \geq 0$$

for all $z \in F(T)$ *and* P *is a sunny nonexpansive retraction of* C *onto* $F(T)$.

Lemma 3. ([14]) Let E be a strictly convex Banach space and C be a closed convex subset of E. Let $S_1, S_2, \cdots, S_r$ be nonexpansive mappings of C into itself such that the set of common fixed points of $S_1, S_2, \cdots, S_r$ is nonempty. Let $T_1, T_2, \cdots, T_r$ be mappings of C into itself given by $T_i = (1-\lambda_i)I + \lambda_i S_i$ for any $0 < \lambda_i < 1$ and $i = 1, 2, \cdots, r$, where I denotes the identity mapping on C. Then $\{T_1, T_2, \cdots, T_r\}$ satisfies the following:

$$\bigcap_{i=1}^{r} F(T_i) = \bigcap_{i=1}^{r} F(S_i)$$

and

$$\begin{aligned}\bigcap_{i=1}^{r} F(T_i) &= F(T_r T_{r-1} \cdots T_1)\\ &= F(T_1 T_r \cdots T_2)\\ &= \cdots\\ &= F(T_{r-1} \cdots T_1 T_r).\end{aligned}$$

Now, we give our main results in this paper.

Theorem 4. Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C be a closed convex subset of X. Let $T_1, T_2, \cdots, T_r$ be nonexpansive mappings of C into itself such that the set $F = \bigcap_{i=1}^{r} F(T_i)$ of common fixed points of $T_1, T_2, \cdots, T_r$ is nonempty and satisfies that

$$\begin{aligned}\bigcap_{i=1}^{r} F(T_i) &= F(T_r T_{r-1} \cdots T_1)\\ &= F(T_1 T_r \cdots T_2)\\ &= \cdots\\ &= F(T_{r-1} \cdots T_1 T_r).\end{aligned}$$

Let $\{\alpha_n\}$ be a sequence in $(0,1)$ which satisfies $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and the control condition $(C7)$, that is, $|\alpha_{n+r}-\alpha_n| \le \circ(\alpha_{n+r})$. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_{n+1}x_0 + (1-\alpha_{n+1})T_{n+1}x_n, \quad n \ge 0, \end{cases}$$

where $T_{n+r} = T_n$. Then the sequence $\{x_n\}$ converges strongly to a point z in F. Further, if $Px_0 = \lim_{n\to\infty} x_n$ for each $x_0 \in C$, then P is a sunny nonexpansive retraction of C onto F.

Proof. We first show that

$$\lim_{n\to\infty} \|x_{n+r} - x_n\| = 0.$$

Since $F \neq \emptyset$, the sequences $\{x_n\}$ and $\{T_{n+1}x_n\}$ are bounded. Then there exists $L > 0$ such that

$$\|x_{n+r} - x_n\| \le L|\alpha_{n+r} - \alpha_n| + (1-\alpha_{n+r})\|x_{n+r-1} - x_{n-1}\|$$

for each $n \ge 1$. Therefore, by the control condition (C7), we have

$$\begin{aligned} &\|x_{n+r} - x_n\| \\ &\le L|\alpha_{n+r} - \alpha_n| + (1-\alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| \\ &\le \circ(\alpha_{n+r}) + (1-\alpha_{n+r})\|x_{n+r-1} - x_{n-1}\|. \end{aligned}$$

Thus, by Lemma 1, it follows that

$$\lim_{n\to\infty} \|x_{n+r} - x_n\| = 0.$$

Next, we prove

$$\lim_{n\to\infty} \|x_n - T_{n+r}\cdots T_{n+1}x_n\| = 0.$$

It suffices to show that

$$\lim_{n\to\infty} \|x_{n+r} - T_{n+r}\cdots T_{n+1}x_n\| = 0.$$

Since $x_{n+r} - T_{n+r}x_{n+r-1} = \alpha_{n+r}(x_0 - T_{n+r}x_{n+r-1})$ and $\lim_{n\to\infty} \alpha_n = 0$, we have $x_{n+r} - T_{n+r}x_{n+r-1} \to 0$. From

$$\begin{aligned}
&\|x_{n+r} - T_{n+r}T_{n+r-1}x_{n+r-2}\| \\
&\le \|x_{n+r} - T_{n+r}x_{n+r-1}\| + \|T_{n+r}x_{n+r-1} - T_{n+r}T_{n+r-1}x_{n+r-2}\| \\
&\le \|x_{n+r} - T_{n+r}x_{n+r-1}\| + \|x_{n+r-1} - T_{n+r-1}x_{n+r-2}\| \\
&= \|x_{n+r} - T_{n+r}x_{n+r-1}\| + \alpha_{n+r-1}\|x_0 - T_{n+r-1}x_{n+r-2}\|,
\end{aligned}$$

it follows that $x_{n+r} - T_{n+r}T_{n+r-1}x_{n+r-2} \to 0$. Similarly, we obtain the conclusion. Let z_t^n be a unique element of C which satisfies $0 < t < 1$ and

$$z_t^n = tx_0 + (1-t)T_{n+r}T_{n+r-1}\cdots T_{n+1}z_t^n.$$

From $F(T_{n+r}T_{n+r-1}\cdots T_{n+1}) = F$ and Lemma 2, we know that $\{z_t^n\}$ converges strongly to Px_0 of as $t \to 0$, where P is a sunny nonexpansive retraction of C onto F.

Next, we prove that

$$\limsup_{n\to\infty} < x_0 - Px_0, j(x_n - Px_0) > \le 0.$$

In fact, assume that $n = k \mod r$ for some $k \in \{0, 1, 2, \cdots, r-1\}$. Since

$$\begin{aligned}
&\|x_n - T_{n+r}\cdots T_{n+1}z_t^k\|^2 \\
&\le [\|x_n - T_{n+r}\cdots T_{n+1}x_n\| \\
&\quad + \|T_{n+r}\cdots T_{n+1}x_n - T_{n+r}\cdots T_{n+1}z_t^k\|]^2 \\
&\le \|x_n - T_{n+r}\cdots T_{n+1}x_n\|^2 \\
&\quad + 2\|x_n - z_t^k\|\|x_n - T_{n+r}\cdots T_{n+1}x_n\| + \|x_n - z_t^k\|^2,
\end{aligned}$$

$$\|x_n - T_{n+r}\cdots T_{n+1}x_n\| \to 0 \quad (n \to \infty),$$

$$(1-t)(x_n - T_{n+r}\cdots T_{n+1}z_t^k) = (x_n - z_t^k) - t(x_n - x_0),$$

$$\begin{aligned}
&(1-t)^2\|x_n - T_{n+r}\cdots T_{n+1}z_t^k\|^2 \\
&\ge \|x_n - z_t^k\|^2 - 2t < x_n - x_0, j(x_n - z_t^k) > \\
&= (1-2t)\|x_n - z_t^k\|^2 + 2t < x_0 - z_t^k, j(x_n - z_t^k) >,
\end{aligned}$$

we have

$$\begin{aligned}
&2t < x_0 - z_t^k, j(x_n - z_t^k) > \\
&\leq (1-t)^2\|x_n - T_{n+r}\cdots T_{n+1}z_t^k\|^2 - (1-2t)\|x_n - z_t^k\|^2 \\
&\leq (1-t)^2[\|x_n - T_{n+r}\cdots T_{n+1}x_n\|^2 \\
&\quad + 2\|x_n - z_t^k\|\|x_n - T_{n+r}\cdots T_{n+1}x_n\| + \|x_n - z_t^k\|^2] \\
&\quad - (1-2t)\|x_n - z_t^k\|^2 \\
&\leq t^2\|x_n - z_t^k\|^2 + (1-t^2)\|x_n - T_{n+r}\cdots T_{n+1}x_n\| \\
&\quad \times (\|x_n - T_{n+r}\cdots T_{n+1}x_n\| + 2\|x_n - z_t^k\|).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&< x_0 - z_t^k, j(x_n - z_t^k) > \\
&\leq \frac{t}{2}\|x_n - z_t^k\|^2 + \frac{(1-t)^2}{2t}\|x_n - T_{n+r}\cdots T_{n+1}x_n\| \\
&\quad \times (\|x_n - T_{n+r}\cdots T_{n+1}x_n\| + 2\|x_n - z_t^k\|).
\end{aligned} \tag{3}$$

Note that

$$\begin{aligned}
&< x_0 - Px_0, j(x_n - Px_0) > \\
&=< x_0 - Px_0, j(x_n - Px_0) - j(x_n - z_t^k) > \\
&\quad + < x_0 - Px_0, j(x_n - z_t^k) > \\
&=< x_0 - Px_0, j(x_n - Px_0) - j(x_n - z_t^k) > \\
&\quad + < x_0 - z_t^k, j(x_n - z_t^k) > + < z_t^k - Px_0, j(x_n - z_t^k) > .
\end{aligned} \tag{4}$$

Since X has a uniformly Gâteaux differentiable norm, we see that $j : X \to X^*$ is norm to weak* uniformly continuous on any bounded subsets of X. Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$| < x_0 - Px_0, j(x) - j(y) > | < \epsilon$$

for all $x, y \in B(0, s)$ with $\|x - y\| < \delta$, where $B(0, s) = \{z \in X : \|x\| \leq s\}$ and

$$s = \max\{\sup_{n\geq 0}\{\|x_n - Px_0\|\}, \sup_{n\geq 0}\{\|x_n - z_t^k\|\}\}.$$

On the other hand, since $z_t^k \to Px_0$ as $t \to 0$, it follows that, for the number $\delta > 0$, there exists $\mu > 0$ such that

$$\|z_t^k - Px_0\| < \delta \tag{5}$$

for $0 < t < \mu$ and so we have

$$| < x_0 - Px_0, j(x_n - Px_0) - j(x_n - z_t^k) > | < \epsilon \tag{6}$$

for all $0 < t < \mu$, $n \geq 0$ and $k \in \{1, 2, \cdots, r\}$. Thus it follows from (3)∼(6) that

$$\begin{aligned} &\limsup_{n\to\infty} < x_0 - Px_0, j(x_n - Px_0) > \\ &\leq \epsilon + \frac{t}{2} \limsup_{n\to\infty} \|x_n - z_t^k\|^2 + \|z_t^k - Px_0\| \limsup_{n\to\infty} \|x_n - z_t^k\|. \end{aligned} \tag{7}$$

Letting $t \to 0$ in (7) and noting $\|z_t^k - Px_0\| \to 0$ as $t \to 0$, we have

$$\limsup_{n\to\infty} < x_0 - Px_0, j(x_n - Px_0) > \leq \epsilon. \tag{8}$$

Since $\epsilon > 0$ is arbitrary, we have the desired conclusion.

Finally, we prove that the sequence $\{x_n\}$ converges strongly to Px_0. Let $\epsilon > 0$. From (8), there exists a positive integer n_0 such that

$$< x_0 - Px_0, j(x_n - Px_0) > < \frac{\epsilon}{2}$$

for all $n \geq n_0$. Since

$$(1 - \alpha_n)(T_n x_{n-1} - Px_0) = (x_n - Px_0) - \alpha_n(x_0 - Px_0),$$

we have

$$\begin{aligned} &(1 - \alpha_n)^2 \|T_n x_{n-1} - Px_0\|^2 \\ &\geq \|x_n - Px_0\|^2 - 2\alpha_n < x_0 - Px_0, j(x_n - Px_0) > \\ &\geq \|x_n - Px_0\|^2 - \alpha_n \epsilon \end{aligned}$$

for all $n \geq n_0$, which implies that

$$\begin{aligned} \|x_n - Px_0\|^2 &\leq (1 - \alpha_n)^2 \|T_n x_{n-1} - Px_0\|^2 + \alpha_n \epsilon \\ &\leq (1 - \alpha_n) \|x_{n-1} - Px_0\|^2 + \alpha_n \epsilon. \end{aligned}$$

Therefore, by Lemma 1, we have $\|x_n - Px_0\| \to 0$ as $n \to \infty$, that is, the sequence $\{x_n\}$ converges strongly to Px_0. This completes the proof.

Next, as an application of Theorem 4, we introduce the strong convergence theorems which are connected with the feasibility problem.

Using a nonlinear ergodic theorem, Crombez [4] considered the feasibility problem in the setting of Hilbert spaces. Let H be a Hilbert space, C_1, C_2, $\cdots$, C_r be closed convex subsets of H and I be the identity operator on H. Then the feasibility problem in the setting of Hilbert spaces may be stated as follows: The original (unknown) image z is known a priori to belong to the intersection C_0 of r well-defined sets C_1, C_2, $\cdots$, C_r in a Hilbert space, given only the metric projection P_i of H onto C_i $(i = 1, 2, \cdots, r)$, recover z by an iterative sequence. In [4], by using the weak convergence theorem by Opial [8], Crombez proved the following:

Theorem 5. *Let $T = \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i$ with $T_i = I + \lambda_i(P_i - I)$ for all $0 < \lambda_i < 1$ and $\alpha_i \geq 0$ for $i = 0, 1, 2, \cdots, r$ with $\sum_{i=0}^{r} \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0 = \bigcap_{i=1}^{r} C_i$ is nonempty. Then, starting from an arbitrary element $x \in H$, the sequence $\{T^n x\}$ converges weakly to an element of C_0.*

Later, Kitahara and Takahashi [6], Takahashi and Tamura [13] dealt with the feasibility problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banach spaces.

Using Lemma 3 and Theorem 4, we have the following:

Corollary 6. Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C be a closed convex subset of X. Let S_1, S_2, $\cdots$, S_r be nonexpansive mappings of C into itself such that the set $F = \bigcap_{i=1}^{r} F(S_i) \neq \emptyset$. Define a family of finite $\{T_1, T_2, \cdots, T_r\}$ by $T_i = (1 - \lambda_i)I + \lambda_i S_i$ for all $0 < \lambda_i < 1$ $(i = 1, 2, \cdots, r)$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ which satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and the control condition $(C7)$, that is, $|\alpha_{n+r} - \alpha_n| = \circ(\alpha_{n+r})$. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_{n+1} x_0 + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad n \geq 0, \end{cases}$$

where $T_{n+r} = T_n$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, $\cdots$, S_r. Further, if $Px_0 = \lim_{n\to\infty} x_n$ for each $x_0 \in C$, then P is a sunny nonexpansive retraction of C onto $\bigcap_{i=1}^{r} F(S_i)$.

Proof. By Lemma 3 and Theorem 4, the sequence $\{x_n\}$ converges strongly to a common fixed point of $S_1, S_2, \cdots, S_r$.

Corollary 7. Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C be a closed convex subset of X. Let $C_1, C_2, \cdots, C_r$ be nonexpansive retracts of C into itself such that the set $\bigcap_{i=1}^r C_i \neq \emptyset$. Define a family of finite $\{T_1, T_2, \cdots, T_r\}$ by $T_i = (1-\lambda_i)I + \lambda_i P_{C_i}$ for all $0 < \lambda_i < 1$,$(i = 1, 2, \cdots, r)$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ which satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and the control condition $(C7)$, that is, $|\alpha_{n+r} - \alpha_n| = \circ(\alpha_{n+r})$. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_{n+1}x_0 + (1-\alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \end{cases}$$

where $T_{n+r} = T_n$. Then the sequence $\{x_n\}$ converges strongly to a point z of $\bigcap_{i=1}^r C_i$. Further, if $Px_0 = \lim_{n\to\infty} x_n$ for each $x_0 \in C$, then P is a sunny nonexpansive retraction of C onto $\bigcap_{i=1}^r C_i$.

Proof. By Corollary 6 and $\bigcap_{i=1}^r C_i = \bigcap_{i=1}^r F(P_{C_i})$, the conclusion follows.

Remark 3. In 1992, Wittmann [16] dealt with the iterative process for $r = 1$ in a Hilbert space and Shioji and Takahashi [10] extended the result of Wittmann to the setting of Banach spaces. On the other hand, in 1996, Bauschke [1] dealt with the iterative process for finding a common fixed point of finite nonexpansive mappings in a Hilbert space (see also Lions [7]). Recently, in [14], Takahashi, Tamura and Toyoda obtained a strong convergence theorem which unifies the results by Bauschke [1], Shioji and Takahashi [10] and, using their result, they considered the problem of image recovery in the setting of Banach spaces.

Remark 4. The proof lines of our main result, Theorem 4, are different from those of Takahashi, Tamura and Toyoda [14]. To prove Theorem 4, we used the control condition (C7) and Weng's lemma (Lemma 1) instead of the condition $\sum_{n=1}^\infty |\alpha_{n+r} - \alpha_n| < \infty$ and the following Banach's limit, respectively. Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \cdots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. We call μ *Banach's limit* if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \cdots) \in l^\infty$. If μ is Banach's limit, then we have the following:

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for all $(a_0, a_1, \cdots) \in l^\infty$. Further, if $a_n \to p$ as $n \to \infty$, then $\mu_n(a_n) = p$ (see [11] for more details on Banach's limit).

Remark 5. All the results in this paper can be extended to the setting of more general Banach space, that is, X is a reflexive Banach space with a uniformly Gâteaux differentiable norm and every weakly compact convex subset of X has the fixed point property for nonexpansive mappings.

Remark 6. If the control condition (C7) is replaced by more general assumption that $x_{n+r} - x_n \to 0$ as $n \to \infty$, then all the conclusions of Theorem 4 with corollaries are still true.

Remark 7. We note that, if $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+r}}$ exists and (C2) holds, then the control condition used in Takahashi, Tamura and Toyoda [14] $\sum_{n=1}^\infty |\alpha_n - \alpha_{n+r}| < \infty$ implies the control condition (C7). In general, these control conditions are independent each other. For the details, refer to Cho, Kang and Zhou [3].

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.1,39-51,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

Double Scale Method for Solving Nonlinear Systems Involving Multiplications of Distributions

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Abstract

In this paper, we develop a double scale numerical method which may be of great significance in computing discontinuous solutions of certain systems of PDE's arising in application.

It is of basic importance, especially in industrial applications, to study the solutions which represent shock waves. Then there appear "multiplications of distributions" in the form $Y \cdot \delta$ of a product of the Heaviside and the Dirac functions; their intuitive treatment may lead to "ambiguous results". To remove this ambiguity, J. F. Colombeau developed a mathematical theory by introducing the differential algebra $\mathcal{G}(\mathbb{R} \times \mathbb{R}^+)$ of generalized functions where discontinuous functions are involved through classes of C^∞ functions. The aim of this paper is to show how this method can be implemented in new numerical methods. We apply our apply to a system modeling elasticity and a system modeling gas dynamics. We observe a complete agreement between the theory (algebraic solutions of Riemann problems) and the numerical results.

Key words. Generalized functions, nonconservative system, numerical scheme, double scale method
AMS subject classifications. 46F10, 65M06, 76L05, 73C50, 35Q99

1 Introduction

In many branches of physics and engineering there appear heuristic expressions in the form of production by distributions. Sometimes they originate from unjustified formal calculations from systems which a priori do not involve such products. But they can also arise directly from the laws of physics and there are cases in which physicists and engineers do not know other approaches ([1, 2, 8]).

These products of distributions are ambiguous. The problem is to resolve the ambiguity both by algebraic calculations in some simplified cases, and by numerical calculations in general. For this purpose, one first needs a rigorous mathematical setting in which these products make sense.

Such a mathematical framework, a nonlinear theory of generalized functions, has been developed by J. F. Colombeau [2, 8, 1, 3]. The important point for the understanding of this paper is that the classical equality of functions or distributions splits into two concepts: a strong one still denoted by the symbol $=$, coherent with all operations (multiplications, differentiation), and a weak one denoted by the symbol $\approx$, coherent with the differentiation but incoherent with the multiplication. The weak equality, called from now on association, is a faithful generalization of the concept of a weak solution (of partial differential equations) in the sense of distribution theory. For instance, in this theory, the equation $u_t + uu_x = 0$ has no discontinuous solution, whereas $u_t + uu_x \approx 0$ has exactly the classical discontinuous weak solutions.

The main idea we used up to now in order to resolve the ambiguity in shock wave solutions of systems in nonconservative form can be summarized as follows: First, we view shock waves to have "width" (of the order of magnitude of a new mean free path; this is indeed known to be physically true); our mathematical tool expresses this fact, at the same time as the idealized fact that this width is "infinitely small". Then we assume the basic laws of continuum mechanics remain valid with this "infinitesimal" width of the shock; this is expressed in our theory by formulating them with the strong equality. Lastly, we express the fact that the constitutive equations do not necessarily hold fully in this infinitesimal width, although they hold on both sides of the shock; this is expressed by setting them with the weak equality. In many cases when one has only one constitutive equation, this is sufficient to resolve the ambiguities. When there are several constitutive equations, more physical information is needed to resolve the ambiguity.

In this method, the equations are not stated in the same sense: those stated with the strong equality hold, of course, in a much stronger sense than those stated with the association. The classical numerical methods stem from systems of conservation laws, in which there is no ambiguity in jump conditions. For the sake of simplicity all equations are treated at the same level of validity, which is natural since there is only one kind of equality in classical analysis. This is not adequate to express the above method, which is precisely based on the difference between our two kinds of equalities.

Indeed the solution is simple; it suffices to use a double scale: a large one for the equations stated with the association, and a smaller one for the equations stated with the strong equality. The shocks usually take place on a few (3 to 5) large meshes, which is not enough to express fully within the width of the shock those equations discretized by "large" mesh. Usually we divide a "large" mesh into 4 "small" meshes, and this is sufficient in order to express that the equations with strong equalities are fully valid within the shock. This method is not so much expensive in computational time since it suffices to use the double scaling only in a neighborhood of the discontinuities of solutions. The aim of this paper is to present this method on one dimensional problems, without taking care of the time of computation. It is fully justified by the fact that, in many cases, this is the only known finite difference method giving the solution.

2 Theoretical and numerical results in a system of two equations

Consider the system coupling the velocity u and the stress σ,

$$u_t + uu_x \approx \sigma_x, \tag{1}$$
$$\sigma_t + u\sigma_x \approx k^2 u_x. \tag{2}$$

It is a simplified model of elastoplastic dynamics in one dimension when the density is constant,

$$\rho_t + (\rho u)_x = 0, \tag{3}$$
$$(\rho u)_t + \left(\rho u^2\right)_x = 0, \tag{4}$$
$$\sigma_t + u\sigma_x = k^2 u_x, \tag{5}$$

with k^2 a positive constant representing the speed of propagation of the elastic waves.

We seek solutions which are constant except on the shock where they have a jump. This kind of solutions is represented by the formulas

$$u(x,t) = u_l + \Delta u \; H(x-ct), \tag{6}$$
$$\sigma(x,t) = \sigma_l + \Delta\sigma \; K(x-ct), \tag{7}$$

where H, K are Heaviside generalized functions, $\Delta u = u_r - u_l$ and $\Delta\sigma = \sigma_r - \sigma_l$. We denote by u_l and u_r, respectively, the values of u on the left and right sides of the discontinuity. One has ([1, 2, 8])

$$c - u_l = \frac{\Delta u}{2} - \frac{\Delta\sigma}{\Delta u}, \tag{8}$$
$$\left(A - \frac{1}{2}\right)\Delta u = k^2\frac{\Delta u}{\Delta\sigma} - \frac{\Delta\sigma}{\Delta u}, \tag{9}$$

where A is an arbitrary real number (depending in each jump) defined by the relation

$$HK' \approx A\delta, \tag{10}$$

and it may take different values depending on the choice of the functions H and K. Although the Riemann problem (1)–(2) admits an infinite number of jump conditions for shocks, we are interested in seeking its shock solution when the equation (1) or (2) is written in a strong equality.

In order to introduce our new numerical method, consider the following two systems:

$$\left\{\begin{array}{rcl} u_t + uu_x &=& \sigma_x, \\ \sigma_t + u\sigma_x &\approx& k^2 u_x, \end{array}\right. \tag{11}$$

$$\left\{\begin{array}{rcl} u_t + uu_x &\approx& \sigma_x, \\ \sigma_t + u\sigma_x &=& k^2 u_x. \end{array}\right. \tag{12}$$

To each of these systems there corresponds precisely one value of A that we compute in the following sections.

2.1 Computation of the jump conditions of (11)

In the particular case of (11), by setting $q = \sigma + u^2/2$, one can prove that (11) is equivalent to

$$\begin{cases} u_t + \left(u^2 - q\right)_x & = \; 0, \\ q_t + \left(\dfrac{u^3}{3} - k^2 u\right)_x & \approx \; 0, \end{cases} \tag{11'}$$

which is in the conservative form. In this case the new mathematical tool gives back at once nothing other than the classical Rankine-Hugoniot jump conditions. However this remark is not useful; so let us start from (11), (6) and (7). Plug (6) and (7) into the first equation of (11):

$$-c\Delta u H' + (u_l + \Delta u H)\Delta u H' = \Delta\sigma K'.$$

Using (8) and (9), one gets

$$K' = \left(1 - \frac{(\Delta u)^2}{2\Delta\sigma}\right) H' + \frac{(\Delta u)^2}{\Delta\sigma} H H'.$$

Thus

$$K = F(H) = H - \frac{(\Delta u)^2}{2\Delta\sigma} H(1 - H), \tag{13}$$

and

$$HK' = \left(1 - \frac{(\Delta u)^2}{2\Delta\sigma}\right) HH' + \frac{(\Delta u)^2}{\Delta\sigma} H^2 H'.$$

Since $H^n H' \approx \frac{1}{n+1}\delta$, so the value of the number A in (9) is given by:

$$A = \frac{1}{2} + \frac{1}{12}\frac{(\Delta u)^2}{\Delta\sigma}. \tag{14}$$

2.2 Computation of the jump conditions of (12)

Plug (6) and (7) into the second equation of (12):

$$-c\Delta u H' + (u_l + \Delta u H)\Delta u H' = \Delta\sigma K'.$$

Using (8) and (9), one obtains

$$K' = \frac{\frac{k^2}{\Delta\sigma} H'}{-\frac{1}{2} + \frac{\Delta\sigma}{(\Delta u)^2} + H}.$$

Integrate this equation to obtain

$$K = G(H) = \frac{k^2}{\Delta\sigma} \ln\left|\frac{-c + u_l + \Delta u H}{-c + u_l}\right|. \tag{15}$$

Since $K(\xi) = 1$ if $\xi > 0$, we have

$$1 = \frac{k^2}{\Delta\sigma} \ln \left| 1 + \frac{1}{-\frac{1}{2} + \frac{\Delta\sigma}{(\Delta u)^2}} \right|. \tag{16}$$

Now we describe our numerical schemes for solutions of (11) and (12). We start from some very simple and general discretization method consisting of computing means of some quantities to ensure stability. This method works very well. For many systems, it has been checked that it gives variables represented by the same Heaviside function, i.e. value 1/2 of the parameters like A. We set $\Delta t = rh$ where Δt and h are respectively the time and space steps. Consider the mean values

$$m_i^n = \frac{u_{i+1}^n + 2u_i^n + u_{i-1}^n}{4} \quad \text{and} \quad p_i^n = \frac{\sigma_{i+1}^n + 2\sigma_i^n + \sigma_{i-1}^n}{4}.$$

We introduce them in the scheme below to ensure its stability for $r > 0$ small enough. The scheme we used for the discretization of (1)–(2) is

$$\left\{ \begin{array}{lcl} u_i^{n+1} & = & m_i^n - rm_i^n \dfrac{u_{i+1}^n - u_{i-1}^n}{2} + r \dfrac{\sigma_{i+1}^n - \sigma_{i-1}^n}{2}, \\ \sigma_i^{n+1} & = & p_i^n - rm_i^n \dfrac{\sigma_{i+1}^n - \sigma_{i-1}^n}{2} + rk^2 \dfrac{u_{i+1}^n - u_{i-1}^n}{2}. \end{array} \right. \tag{17}$$

In this one-scale scheme (**ratio =1**), we use the same space step for u and σ. The numerical method used on each grid appears to be essentially central differencing with artificial dissipation added. Note that the averaged quantities m_i, p_i can be written as, e.g.,

$$m_i = u_i + \frac{1}{4}\left(u_{i+1} - 2u_i^n + u_{i-1}\right) \approx u + \frac{1}{4}h^2 u_{xx},$$

the latter term gives the artificial dissipation.

Now the new numerical methods presented here to solve the system (11) and (12) are based on the following idea: the equations holding in a stronger sense across the discontinuity should be discretized on a finer grid than the ones holding weakly. This is a very interesting approach, and is surprisingly effective at calculating solutions with the correct jump conditions in each case.

2.3 Proposed numerical scheme for solution of (11)

For the space discretization, the "large" meshes are divided into four equal size "small" meshes; the time discretization is only made of one scaling. The index i refers to the small (space) meshes discretization, the large scale being used for the discretization of the second equation in (11): the values σ_i^n are defined only when i is a multiple of four (**ratio =4**). For all $i \in \mathbb{Z}$ we set

$$u_i^{n+1} = m_i^n - rm_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2} + r\,(d\sigma)_i^n\,, \tag{11$_1$}$$

where $(d\sigma)_i^n$ is an approximation of the term σ_x. Since σ_i^n is defined only when i is a multiple of four, we adopt the following. First for the case $i \geq 0$;

$$\begin{aligned}
&\text{if} \quad i = 4k, && \text{then} \quad (d\sigma)_i^n = \frac{1}{8}\left(\sigma_{i+4}^n - \sigma_{i-4}^n\right), \\
&\text{if} \quad i = 4k+1, && \text{then} \quad (d\sigma)_i^n = \frac{3}{16}\left(\sigma_{4k+4}^n - \sigma_{4k}^n\right) + \frac{1}{16}\left(\sigma_{4k}^n - \sigma_{4k-4}^n\right), \\
&\text{if} \quad i = 4k+2, && \text{then} \quad (d\sigma)_i^n = \frac{1}{4}\left(\sigma_{4k+2}^n - \sigma_{4k-2}^n\right), \\
&\text{if} \quad i = 4k+3, && \text{then} \quad (d\sigma)_i^n = \frac{3}{16}\left(\sigma_{4k+4}^n - \sigma_{4k}^n\right) + \frac{1}{16}\left(\sigma_{4k+8}^n - \sigma_{4k+4}^n\right).
\end{aligned}$$

For the case $i < 0$, these definitions are extended by symmetry. For those integers i which are integer multiples of four, (11_1) is completed by

$$\sigma_i^{n+1} = \frac{\sigma_{i+4}^n + 2\sigma_i^n + \sigma_{i-4}^n}{4} - rm_i^n \frac{\sigma_{i+4}^n - \sigma_{i-4}^n}{8} + rk^2 \frac{u_{i+1}^n - u_{i-1}^n}{2}. \qquad (11_2)$$

2.4 Proposed numerical scheme for solution of (12)

In this case the value of u_i^n is defined when i is a multiple of four (**ratio =4**). Thus the scheme becomes: for all $i \in \mathbb{Z}$,

$$\sigma_i^{n+1} = p_i^n - r\,(Mu)_i^n \frac{\sigma_{i+1}^n - \sigma_{i-1}^n}{2} + rk^2\,(du)_i^n. \qquad (12_2)$$

Where $(Mu)_i^n$ and $(du)_i^n$ are defined similarly as before, due to the fact that u_i^n is defined only for i multiple of four: for $i \geq 0$,

$$\begin{aligned}
&\text{if} \quad i = 4k, && \text{then} \quad (du)_i^n = \frac{1}{8}\left(u_{i+4}^n - u_{i-4}^n\right), \\
& && \text{and} \quad (Mu)_i^n = \frac{u_{i+4}^n + 2u_i^n + u_{i-4}^n}{4}, \\
&\text{if} \quad i = 4k+1, && \text{then} \quad (du)_i^n = \frac{1}{16}\left(u_{4k+4}^n - u_{4k}^n\right) + \frac{3}{16}\left(u_{4k}^n - u_{4k-4}^n\right), \\
& && \text{and} \quad (Mu)_i^n = \frac{u_{4k+4}^n + 2u_{4k}^n + u_{4k-4}^n}{4} \\
&\text{if} \quad i = 4k+2, && \text{then} \quad (du)_i^n = \frac{1}{4}\left(u_{4k+4}^n - u_{4k}^n\right), \\
& && \text{and} \quad (Mu)_i^n = \frac{u_{4k+4}^n + u_{4k-4}^n}{2}, \\
&\text{if} \quad i = 4k+3, && \text{then} \quad (du)_i^n = \frac{3}{16}\left(u_{4k+4}^n - u_{4k}^n\right) + \frac{1}{16}\left(u_{4k+8}^n - u_{4k+4}^n\right), \\
& && \text{and} \quad (Mu)_i^n = \frac{u_{4k+8}^n + 2u_{4k+4}^n + u_{4k}^n}{4}.
\end{aligned}$$

For i multiple of 4, the first equation in (12) is discretized by (12_1)

$$u_i^{n+1} = \overline{u}_i^n - r\overline{u}_i^n \frac{u_{i+4}^n - u_{i-4}^n}{8} + r\frac{\sigma_{i+1}^n - \sigma_{i-1}^n}{2}, \qquad (12_1)$$

where $\overline{u}_i^n = \dfrac{u_{i+4}^n + 2u_i^n + u_{i-4}^n}{4}$.

2.5 Numerical results

In the numerical examples, the left hand side values of u_l are $1, 2, 3$ successively, and $\sigma_l = 0$; the right hand side values are always $u_r = 0$ and $\sigma_r = 0$. Three tests corresponding to the three values of u_l are given below.

In all cases, the solution consists of a pair of shock waves separated by constant values $\bar{u}$ and $\bar{\sigma}$ of u and σ respectively.

The theoretical values of $\bar{u}$ and $\bar{\sigma}$ corresponding to system (11) (respectively (12)) are computed from the algebraic formula (8), (9), and (14) (respectively (8), (9), and (16)) with an error $\leq 5 \cdot 10^{-4}$. The observed values of $\bar{u}$ and $\bar{\sigma}$ are given with an error $\leq 5 \cdot 10^{-3}$. We computed solutions with $r = 0.25$ for $u_l = 1, 2$ and $r = 0.2$ for $u_l = 3$. We fixed the simulation time T at 1.

System (11) : **ratio** 1

value of of u_l	theoretical value of $\bar{u}$	observed value of $\bar{u}$	theoretical value of $-\bar{\sigma}$	observed value of $-\bar{\sigma}$
1	0.5	0.5	0.4947	0.5
2	1.	1.	0.9575	1.
3	1.5	1.5	1.3521	1.5

System (12): **ratio** 1

value of of u_l	theoretical value of $\bar{u}$	observed value of $\bar{u}$	theoretical value of $-\bar{\sigma}$	observed value of $-\bar{\sigma}$
1	0.5	0.5	0.5052	0.5
2	1.	1.	1.0436	1.
3	1.5	1.5	1.6559	1.5

System (11): **ratio** 4

value of of u_l	theoretical value of $\bar{u}$	observed value of $\bar{u}$	theoretical value of $-\bar{\sigma}$	observed value of $-\bar{\sigma}$
1	0.5000	0.5003	0.4947	0.4942
2	1.0000	1.001	0.9575	0.9563
3	1.5000	1.505	1.3521	1.3513

System (12): **ratio** 4

value of of u_l	theoretical value of $\bar{u}$	observed value of $\bar{u}$	theoretical value of $-\bar{\sigma}$	observed value of $-\bar{\sigma}$
1	0.5000	0.50001	0.5052	0.5059
2	1.0000	0.998	1.0436	1.046
3	1.5000	1.498	1.6559	1.664

We remark that the intermediate step value for the velocity u and the stress σ are calculated with more accuracy with the double scale method with ratio 4. This particular ratio 4 is appropriate for these problems. With a stronger discontinuity a larger ratio would be needed.

Unfortunately, for each of the tests, the numerical solution presents a surplus of diffusion. In order to improve the quality and neatness of the numerical results we introduce a classical antidiffusion method. This modification does not play a role in the convergence of the scheme but it makes the shock neater. It is described as follows. For $w = u, \sigma$, we introduce a correction only in the two cases:

$$w_{i-2}^n > w_{i-1}^n > w_i^n > w_{i+1}^n, \tag{a}$$

or

$$w_{i-2}^n < w_{i-1}^n < w_i^n < w_{i+1}^n. \tag{b}$$

We set

$$\bar{c}_i^n = \pm \min\left(\frac{|w_i^n - w_{i-1}^n|}{4}, \frac{|w_{i-2}^n - w_{i-1}^n|}{2}, \frac{|w_{i+1}^n - w_i^n|}{2}\right),$$

with "$-$" in the case (b) and "$+$" in the case (a). When we are not in the case (a) and (b) we set $\bar{c}_i^n = 0$. In the scheme, w_i^n is replaced by $w_i^n - \bar{c}_i^n + \bar{c}_{i-1}^n$.

For ratio four, we introduce the following correction only in the two cases

$$w_{i-8}^n > w_{i-4}^n > w_i^n > w_{i+4}^n, \tag{c}$$

or

$$w_{i-8}^n < w_{i-4}^n < w_i^n < w_{i+4}^n. \tag{d}$$

We set

$$\bar{c}_i^n = \pm \min\left(\frac{|w_i^n - w_{i-4}^n|}{16}, \frac{|w_{i-8}^n - w_{i-4}^n|}{8}, \frac{|w_{i+4}^n - w_i^n|}{8}\right),$$

with "$-$" in the case (d) and "$+$" in the case (c). When we are not in the case (c) and (d) we set $\bar{c}_i^n = 0$. In the scheme, w_i^n is replaced by $w_i^n - \bar{c}_i^n + \bar{c}_{i-4}^n$.

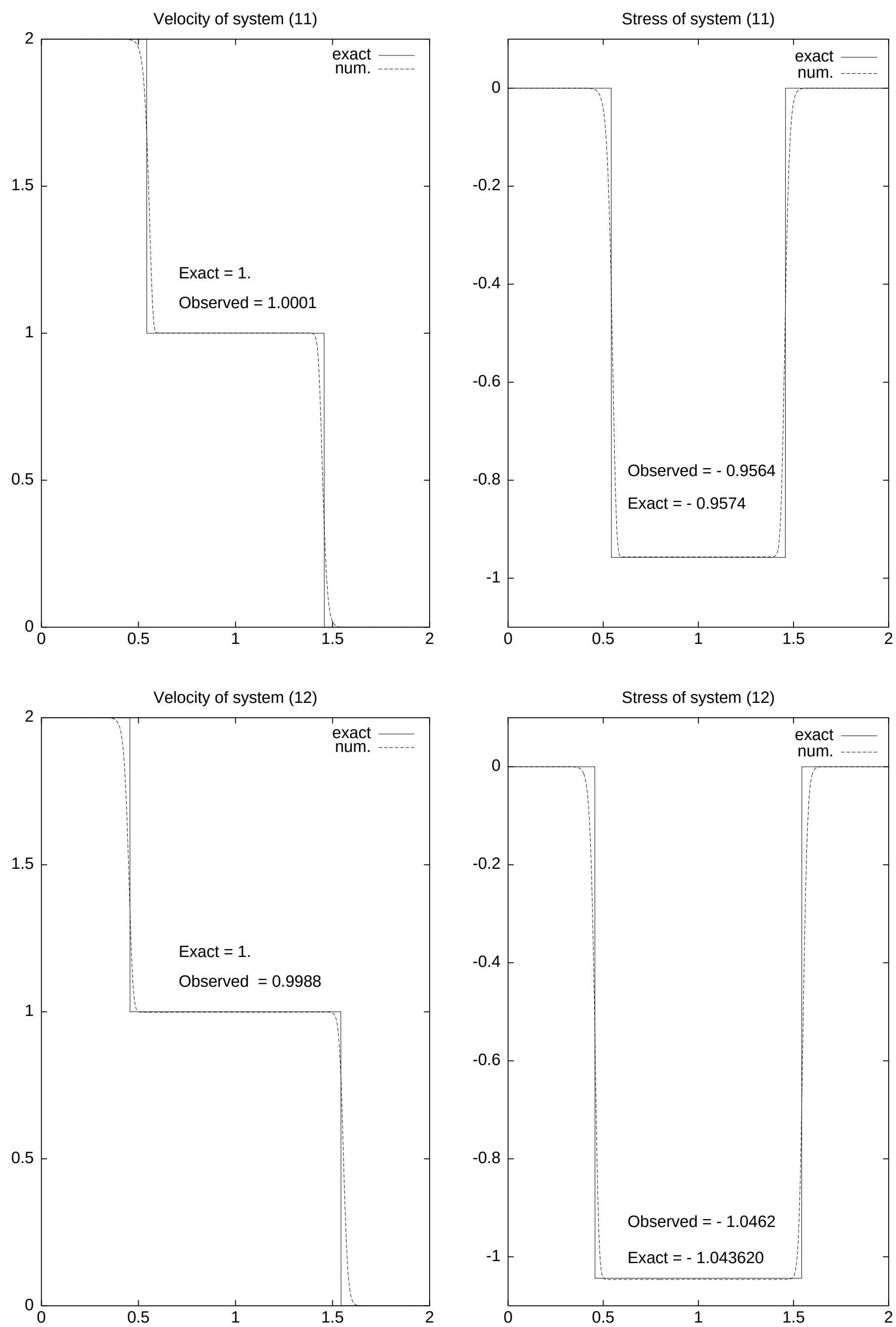

Figure 1: Comparison between the exact solutions of (11) and (12), and the numerical solutions obtained by double scale method for ratio 4

3 Theoretical and numerical results in systems of fluid dynamics

In this section, we consider the classical system of aerodynamics ([5, 3, 9])

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + p\right)_x = 0, \\ (\rho e)_t + (\rho e u + p u)_x = 0, \\ p \approx (\gamma - 1)\rho I, \end{cases} \tag{18}$$

where ρ, u, and p denote respectively, the density, the velocity, and the pressure. In this system, we state the laws of physics (the internal energy per unit mass $I = e - u^2/2$, e being the total energy per unit mass) with the strong equalities, and the constitutive equation with the association.

Then it can be transformed into various systems in the nonconservative form having the same piecewise continuous solutions. Two of these systems are

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x + \dfrac{1}{\rho} p_x = 0, \\ p_t + u p_x + \gamma p u_x \approx 0, \end{cases} \tag{19}$$

and

$$\begin{cases} v_t + u v_x - v u_x = 0, \\ u_t + u u_x + v p_x = 0, \\ p_t + u p_x + \gamma p u_x \approx 0, \end{cases} \tag{20}$$

where $v = 1/\rho$ is the specific volume. It is has been proved in [5, 3] that v, u, and p are represented by the same Heaviside generalized function; thus natural one-scale schemes work very well for (19). But ρ and $v = 1/\rho$ are not represented by the same Heaviside generalized function, so that the same method if applied to (20) would not lead to the correct solution. Both (19) and (20) have the same piecewise continuous solutions as (18). Thus it suffices to superpose the curves obtained from our new double scaling method with the classical solution. We observe agreement with the expected results.

We now present some numerical results. The computation is done for the initial data $(u_l, u_r) = (0, 0)$, $(p_l, p_r) = (1, 0.1)$, $(\rho_l, \rho_r) = (1, 1.25)$ with $\gamma = 1.4$. We have computed the solutions with a parameter $r = 0.2$, and $T = 0.2$.

The present test is the so-called shock tube test problem. Let us recall the initial conditions; air is confined in a tube (of infinite length). A membrane separates the tube in two zones of different pressures and densities. The membrane disappears at $t = 0$; this gives rise to a rarefaction wave traveling to the left, a contact discontinuity and a shock wave traveling to the right.

variable	$\bar{u}$	$(\bar{\rho_1}, \bar{\rho_2})$	$\bar{p}$
theoretical value	0.9274	$(0.4263, 0.2655)$	0.3031
observed value for ratio 1	0.918	$(0.429, 0.252)$	0.3080
observed value for ratio 4	0.9279	$(0.415, 0.264)$	0.3037

Table 1: Observed and theoretical values of test 1 of the intermediate step value for the density, pressure and velocity corresponding to system (20)

variable	$\bar{u}$	$(\bar{\rho_1}, \bar{\rho_2})$	$\bar{p}$
theoretical value	0.9274	$(0.4263, 0.2655)$	0.3031
observed value for ratio 1	0.907	$(0.435, 0.275)$	0.3127
observed value for ratio 4	0.9270	$(0.421, 0.265)$	0.3037

Table 2: Observed and theoretical values of test 1 of the intermediate step value for the density, pressure and velocity corresponding to system (19)

It appears from these results that the nonconservative form gives numerical results of good quality for ratio 4, it permits us to reach more precisely the intermediate step values, whereas the ratio 1 does not permit us to calculate an acceptable solution.

Moreover, in the curves obtained from the nonconservative method one observes the absence of oscillations in pressure and velocity, but for the density, this method gives a poor result on contact discontinuity (oscillations for instance) for ratio 4. This test shows that it is important to apply the double scale method only in a neighborhood of the discontinuity (shock wave) in the context of preserving the contact discontinuity.

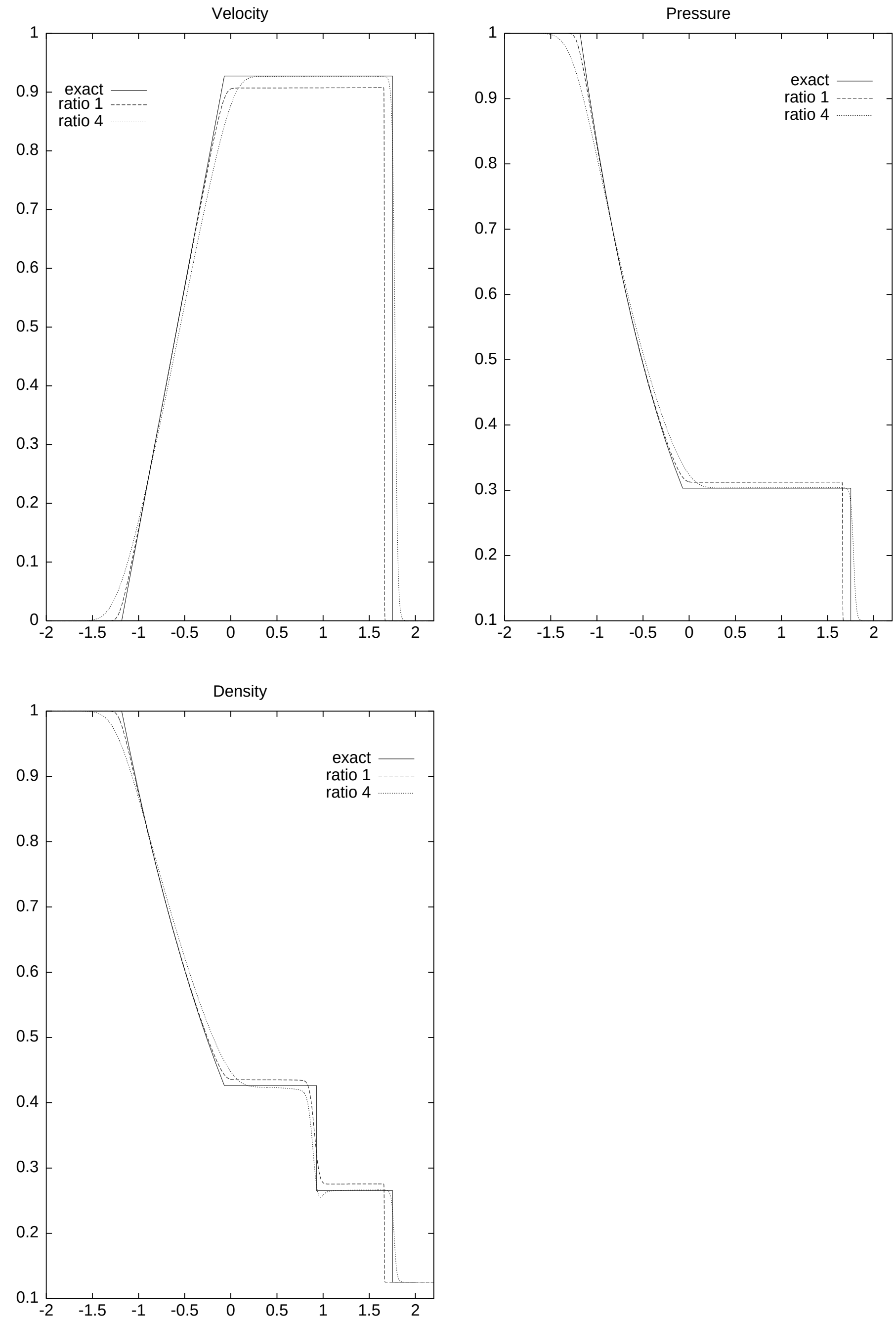

Figure 2: Comparison between the exact solutions of (19) and the numerical solutions obtained for ratio 1 and ratio 4

4 Conclusion

The nonlinear theory of generalized functions makes it possible to eliminate ambiguities associated with “multiplication of distributions” in systems of equations of physics. It provides new ideas to resolve the ambiguities and state more precisely the equations of physics.

This resolution of the ambiguities is done by algebraic calculations in the simple cases. In this paper, we have shown that the above theory suggests clearly new numerical methods lead to the theoretical algebraic results. We showed that the double scale method permits us to calculate acceptable solutions. These discontinuities are well resolved numerically and correct speeds are observed.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.1,53-73,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

Vibrations of Elastic Strings: Unilateral Problem

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Abstract

In a previous paper, [17] Part Two, was investigated an initial boundary value problem for the operator

$$Lu(x,t) = \frac{\partial^2 u}{\partial t^2} - \left[\frac{\tau_0}{m} + \frac{k}{m}\,\frac{\gamma(t)-\gamma_0}{\gamma_0} + \frac{k}{2m\gamma(t)}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right] \frac{\partial^2 u}{\partial x^2}\,,$$

which is a model for small vibrations of an elastic string with moving ends and variable tension. Without restrictions on the initial data we proved local solutions in t. The present paper is dedicated to study a unilateral problem for Lu with no restriction on the initial configuration u_0 and the initial velocity u_1 has a bounded gradient. We succeed to prove that the solution of the unilateral problem has a solution for all $t \in [0,T]$, $\ T$ a positive arbitrary number.

Keywords: Elastic strings, unilateral problem, moving ends, penalty method, nonlocal solutions.

Mathematics Subject Classification: 35L85, 35L20.

1 INTRODUCTION

In [17] it was deduced a model describing the small vertical vibrations of an elastic string in the case of moving ends and variable tension. In fact, it was deduced the mathematical model

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{m}\,\frac{\gamma(t)-\gamma_0}{\gamma_0} + \frac{k}{2m\gamma(t)}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.1}$$

Note that $u = u(x,t)$ is the deformation of the string; τ_0 the initial tension in the rest position $[\alpha_0, \beta_0]$; $[\alpha(t), \beta(t)]$ the deformations of $[\alpha_0, \beta_0]$ after the time $t > 0$, with $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$, $\gamma(t) = \beta(t) - \alpha(t)$, $\gamma_0 = \gamma(0)$, $0 < \alpha(t) < \alpha_0 < \beta_0 < \beta(t)$. By m we represent the mass of the string and $k = \sigma E$, with σ the area of the cross section of the string and E the Young's modulus of the material.

It is opportune to observe that when we have fixed ends, that is, $\alpha(t) = \alpha_0$, $\beta(t) = \beta_0$ for all $t \geq 0$, the model (1.1) reduces to

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{2m\gamma_0} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

called the Kirchhoff model, see [1], [6], [9], [10], [11], [12], [13], [16], [18], [20], [21], [22], [26].

If in (1.2) we suppose fixed ends and constant tension τ_0 , we ignore the non linear contribution

$$\sigma(t) = \frac{k}{2m\gamma_0} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx,$$

which appears from the variation of the tension, then we obtain, from (1.2), the well known D'Alembert model, [8],

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0}{m} \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.3}$$

In order to propose our problem we need some notation. Let

$$\widehat{Q} = \left\{ (x,t) \in \mathbb{R}^2;\ \alpha(t) < x < \beta(t),\ 0 < t < T \right\}, \tag{1.4}$$

with $0 < \alpha(t) < \alpha_0 < \beta_0 < \beta(t)$ for $t > 0$.

The lateral boundary of $\widehat{Q}$ is defined by

$$\overset{\wedge}{\sum} = \bigcup_{0<t<T} [\{\alpha(t), \beta(t)\} \times \{t\}]. \tag{1.5}$$

$$
\begin{aligned}
\text{Set} \qquad & \hat{a}(t) = \frac{\tau_0}{m} + \frac{k}{m}\,\frac{\gamma(t)-\gamma_0}{\gamma_0}\,; \quad \hat{b}(t) = \frac{k}{2m\gamma(t)} \ \text{ and} \\
& \widehat{M}(t,\lambda) = \hat{a}(t) + \hat{b}(t)\lambda.
\end{aligned}
\tag{1.6}
$$

Thus, we consider the nonlinear differential operator

$$
\widehat{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - \widehat{M}\left(t, \int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \frac{\partial^2 u}{\partial x^2}\,, \tag{1.7}
$$

defined for functions $u\colon \widehat{Q} \to \mathbb{R}$.

Let us consider the closed convex set K_t, contained in $H_0^1(\Omega_t)$, for $t \geq 0$, defined by

$$
K_t = \left\{ w \in H_0^1(\Omega_t); \left|\frac{\partial w}{\partial x}\right| \leq \frac{1}{\gamma(t)} \ \text{ a.e. in } \Omega_t \right\}. \tag{1.8}
$$

Note that $H_0^1(\Omega_t)$ is the Sobolev space on $\Omega_t = (\alpha(t), \beta(t))$, the sections of $\widehat{Q}$ at level t; see [15].

Remark 1.1. In previous work, cf. [17], was investigated an initial boundary value problem for $\widehat{L}u$ in $\widehat{Q}$, where we chose initial values $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$, $\Omega_0 = (\alpha_0, \beta_0)$. It was proved, cf. [17], that the initial boundary value problem for $\widehat{L}u$ in $\widehat{Q}$ with these initial values and zero on $\widehat{\Sigma}$ has only one local solution $u = u(x,t)$. It means that the solution $u = u(x,t)$ is defined for $(x,t) \in \widehat{Q}$, but for $0 < t < T_0$, T_0 a fixed number. However, to obtain solution defined for all $t > 0$, we need restrict u_0, u_1 to be inside a fixed ball, what is called "small initial data".

In the present work, we consider a unilateral problem or a variational inequality cf. [3] for the operator $\widehat{L}u$, to be defined in Section 2. We prove, that considering $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$ and $u_1 \in K_0 \subset H_0^1(\Omega_0)$, the unilateral problem for $\widehat{L}u$ in $\widehat{Q}$ has a unique solution $u = u(x,t)$, defined for all number $t \geq 0$. Note that u_0 is arbitrary in $H_0^1(\Omega_0) \cap H^2(\Omega_0)$. □

The methodology employed to study the unilateral problem of $\widehat{L}u$ in $\widehat{Q}$ consists in the transformation of the noncylindrical domain $\widehat{Q}$ into a cylinder Q and the operator $\widehat{L}u$ into an operator Lv defined for functions $v\colon Q \to \mathbb{R}$. Thus we obtain an equivalent cylindrical unilateral problem for Lv in Q and we are able to apply the penalty method idealized by Lions [14].

2 NOTATIONS AND RESULTS

We consider the following hypotheses:

(H1) $\alpha, \beta \in C^3([0,\infty);\mathbb{R})$, such that $0 < \alpha(t) < \alpha_0 < \beta_0 < \beta(t)$, for all $t > 0$, $\alpha'(t) < 0$, $\beta'(t) > 0$ and $\alpha'(0) = \beta'(0) = 0$, with f' the derivative of $f(t)$.

(H2) $|\alpha'(t) + y\gamma'(t)| \leq \sqrt{\dfrac{m_0}{2}}$, for all $t \geq 0$, $0 < y < 1$ with $0 < m_0 \leq \dfrac{\tau_0}{m}\cdot$

Theorem 2.1. *Suppose*

$$u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0) \quad \textit{and} \quad u_1 \in K_0 \subset H_0^1(\Omega_0).$$

There exists one and only one function $u\colon \widehat{Q} \to \mathbb{R}$, satisfying the conditions:

$$\begin{gathered} u \in L^\infty(0,T; H_0^1(\Omega_t) \cap H^2(\Omega_t));\ u' \in L^\infty(0,T; H^1(\Omega_t)); \\ u'' \in L^\infty(0,T; L^2(\Omega_t)), \end{gathered} \tag{2.1}$$

$$Du(t) \in K_t \ \textit{a.e. in}\ (0,T), \tag{2.2}$$

$$\int_0^T \int_{\alpha(t)}^{\beta(t)} \widehat{L}u(x,t)\big[w(x,t) - Du(x,t)\big]dxdt \geq 0, \tag{2.3}$$

for all $w \in L^1(0,T; H_0^1(\Omega_t))$, with $w(t) \in K_t$ a.e. in $(0,T)$.

$$u(x,0) = u_0(x),\ u'(x,0) = u_1(x) \ \textit{in}\ \Omega_0 = (\alpha_0, \beta_0). \tag{2.4}$$

The operator D is defined by

$$Du(x,t) = u'(x,t) + \left[\frac{\gamma'(t)}{\gamma(t)}\,(x - \alpha(t)) + \alpha'(t)\right]\frac{\partial u(x,t)}{\partial x}\cdot \tag{2.5}$$

□

To prove theorem 2.1 we transform it in an equivalent unilateral problem in a cylindrical domain.

In fact, if $(x,t) \in \widehat{Q}$, the point $(y,t) \in Q$, for $y = \dfrac{x-\alpha(t)}{\gamma(t)}$ and $Q = (0,1) \times (0,T)$. Thus the mapping $\tau_t x = y$, with $y = \dfrac{x-\alpha(t)}{\gamma(t)}$, transforms $(\alpha(t), \beta(t))$, $t \geq 0$, into $(0,1)$. The inverse is $\tau_t^{-1} y = x$, with $x = \gamma(t) y + \alpha(t)$. Note that τ_t and τ_t^{-1} are C^3, by (H1).

The next step is to obtain the operator $Lv(y,t)$ transformed from $\widehat{L}u(x,t)$ by τ_t. In fact, if we set $v(y,t) = u(x,t)$ with $y = \tau_t x$, we obtain

$$\begin{aligned} Lv(y,t) &= \frac{\partial^2 v}{\partial t^2} - \frac{1}{\gamma^2(t)} \left[-\frac{m_0}{2} + \hat{a}(t) + \frac{\hat{b}(t)}{\gamma(t)} \int_0^1 \left(\frac{\partial v}{\partial y} \right)^2 dy \right] \frac{\partial^2 v}{\partial y^2} - \\ &\quad - \frac{\partial}{\partial y} \left(a(y,t) \frac{\partial v}{\partial y} \right) + b(y,t) \frac{\partial^2 v}{\partial y \partial t} + c(y,t) \frac{\partial v}{\partial y}, \end{aligned} \tag{2.6}$$

where

$$a(y,t) = \left[\frac{m_0}{2\gamma^2(t)} - \left(\frac{\alpha'(t) + y\gamma'(t)}{\gamma(t)} \right)^2 \right], \tag{2.7}$$

$$b(y,t) = -2 \left[\frac{\alpha'(t) + y\gamma'(t)}{\gamma(t)} \right], \tag{2.8}$$

$$c(y,t) = - \left[\frac{\alpha''(t) + y\gamma''(t)}{\gamma(t)} \right]. \tag{2.9}$$

By (H2) we have $-\dfrac{m_0}{2} + \hat{a}(t) \geq \dfrac{m_0}{2}$, then the coefficient of $-\dfrac{\partial^2 v}{\partial y^2}$ is strictly positive. Also by (H2), $a(y,t) \geq 0$ for all $(y,t) \in Q$.

Observe also that $v'(y,t) = Du(x,t)$ and if $z(y,t) = w(x,t)$, $y = \tau_t x$, $\dfrac{\partial z}{\partial y} = \gamma \dfrac{\partial w}{\partial x}$ and K_t is transformed into the closed convex set

$$K = \left\{ z \in H_0^1(\Omega);\ \left| \frac{\partial z}{\partial y} \right| \leq 1 \text{ a.e. in } \Omega \right\} \tag{2.10}$$

with $\Omega = (0,1)$.

Prior to completing the proof of theorem 2.1, we state another result.

Theorem 2.2. *Suppose*

$$v_0 \in H_0^1(\Omega) \cap H^2(\Omega) \quad \textit{and} \quad v_1 \in K.$$

Then, there exists one and only one function $v\colon Q \to \mathbb{R}$, *such that*

$$\begin{gathered} v \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega));\ v' \in L^\infty(0,T;H_0^1(\Omega)) \cap L^4(0,T;W_0^{1,4}(\Omega)); \\ v'' \in L^\infty(0,T;L^2(\Omega)) \end{gathered} \tag{2.11}$$

$$v'(t) \in K \quad a.e.\ in\ (0,T) \tag{2.12}$$

$$\int_0^T \int_0^1 Lv(y,t)\big[z(y,t) - v'(y,t)\big]\gamma(t)\,dydt \geq 0, \tag{2.13}$$

for all $z \in L^4(0,T;W_0^{1,4}(\Omega))$, *with* $z(t) \in K$ *a.e. in* $(0,T)$

$$v(y,0) = v_0(y) \quad and \quad v'(y,0) = v_1(y) \quad in \quad \Omega = (0,1). \tag{2.14}$$

Remark 2.1. Note that, as we will prove in Section 3, T is any positive number.

By the inverse mapping τ_t^{-1} we prove that theorem 2.2 implies theorem 2.1. By this reason we need only to prove theorem 2.2.

To prove theorem 2.2 we transform, by penalty, the inequality (2.13) into a family of equations depending of a parameter $\varepsilon > 0$ and apply Galerkin's method.

First of all, let us define a penalty operator convenient to our problem, cf. Lions [14]. By $W_0^{1,4}(\Omega)$ we represent the Sobolev space whose topological dual is $W^{-1,4/3}(\Omega)$. The closed convex set K, defined in (2.10), is also contained in $W_0^{1,4}(\Omega)$. Represent by v^- the negative part of the function v defined by $v^-(y) = \max(-v(y),0)$. For $u, v \in W_0^{1,4}(\Omega)$, we have

$$\left[\left(1 - \left|\frac{\partial u}{\partial y}\right|^2\right)^- \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y}\right] \in L^1(\Omega).$$

For $u \in W_0^{1,4}(\Omega)$ consider the linear form

$$\langle P(u), v\rangle = \int_0^1 \left(1 - \left|\frac{\partial u}{\partial y}\right|^2\right)^- \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y}\,dy,$$

defined for $v \in W_0^{1,4}(\Omega)$, which is continuous, then it is an object of the dual $W^{-1,4/3}(\Omega)$. We obtain

$$P(u) = -\frac{\partial}{\partial y}\left[\left(1 - \left|\frac{\partial u}{\partial y}\right|^2\right)^- \frac{\partial u}{\partial y}\right] \tag{2.15}$$

in the sense of distributions on $\Omega = (0,1)$.

We prove, cf. Lions [14], that the operator $P\colon W_0^{1,4}(\Omega) \to W^{-1,4/3}(\Omega)$, is monotone, hemicontinuous, takes bounded sets of $W_0^{1,4}(\Omega)$ into bounded sets of $W^{-1,4/3}(\Omega)$ and its kernel is K. This operator is called the penalty operator relating to the closed convex set K. $\square$

We prove the following result from which we obtain the proof of theorem 2.2.

Theorem 2.3. *Suppose* $0 < \varepsilon < 1$, $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $v_1 \in K$. *There exists a unique function* $v_\varepsilon\colon Q \to \mathbb{R}$ *satisfying*

$$\begin{gathered} v_\varepsilon \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega));\ v_\varepsilon' \in L^\infty(0,T; H_0^1(\Omega)) \cap L^4(0,T; W_0^{1,4}(\Omega)); \\ v_\varepsilon'' \in L^\infty(0,T; L^2(\Omega)) \end{gathered} \tag{2.16}$$

$$\int_0^T (Lv_\varepsilon(t), w(t))\,dt + \frac{1}{\varepsilon}\int_0^T \langle P(v_\varepsilon'(t)), w(t)\rangle\,dt = 0, \tag{2.17}$$

for all $w \in L^4(0,T; W_0^{1,4}(\Omega))$.

$$v_\varepsilon(y,0) = v_0(y),\ v_\varepsilon'(y,0) = v_1(y) \quad in \quad \Omega = (0,1). \tag{2.18}$$

The proof of theorem 2.3 will be given in Section 3. For the moment let us prove that it implies the proof of theorem 2.2. Observe that in (2.17) we represent by $(\ ,\ )$ the scalar product in $L^2(\Omega)$ and $\langle\ ,\ \rangle$ the duality pairing between $W^{-1,4/3}(\Omega)$ and $W_0^{1,4}(\Omega)$.

In fact, set in (2.17) $w(t) = (z(t) - v_\varepsilon'(t))\gamma(t)$ with $z \in L^4(0,T; W_0^{1,4}(\Omega))$ such that $z(t) \in K$ a.e. in $(0,T)$. We have

$$\int_0^T (Lv_\varepsilon(t), z(t) - v_\varepsilon'(t))\gamma(t)dt + \frac{1}{\varepsilon}\int_0^T \langle P(v_\varepsilon'(t)), z(t) - v_\varepsilon'(t)\rangle\gamma(t)dt = 0. \tag{2.19}$$

By monotonicity of P and because $z(t) \in K$, we have $\langle P(v'_\varepsilon(t)) - P(z(t)), v'_\varepsilon(t) - z(t)\rangle \geq 0$, then it follows from (2.19) that

$$\int_0^T (Lv_\varepsilon(t), z(t) - v'_\varepsilon(t))\gamma(t)\, dt \geq 0 \tag{2.20}$$

for all $z \in L^4(0, T; W_0^{1,4}(\Omega))$ with $z(t) \in K$ a.e. in $(0, T)$. We prove in Section 3, that when $0 < \varepsilon < 1$ and if $\varepsilon \to 0$, (2.20) converges to

$$\int_0^T (Lv(t), z(t) - v'(t))\gamma(t) dt \geq 0$$

for all $z \in L^4(0, T; W^{1,4}(\Omega))$ with $z(t) \in K$ a.e. in $(0, T)$ and $v \colon Q \to \mathbb{R}$ satisfies the regularity, the unicity and the initial conditions of theorem 2.2.

3 PROOF OF THE THEOREM 2.3

We apply Galerkin's method with the Hilbertian basis of spectral objects $(w_\nu)_{\nu\in\mathbb{N}}$ and $(\lambda_\nu)_{\nu\in\mathbb{N}}$ for the operator $-\dfrac{\partial^2}{\partial y^2}$ in $H_0^1(\Omega)$, $\Omega = (0, 1)$, cf. Brezis [2]. We know that the eigenvectors $(w_\nu)_{\nu\in\mathbb{N}}$ are orthonormal and complete in $L^2(\Omega)$ and complete in $H_0^1(\Omega) \cap H^2(\Omega)$, $H_0^1(\Omega)$ and $W_0^{1,4}(\Omega)$. We represent by $V_N = [w_1, w_2, \ldots, w_N]$ the subspace of $H_0^1(\Omega)$ generated by the first N vectors w_ν. The approximate problem consists in determining $v_{\varepsilon N}(x, t) = \sum_{j=1}^{N} g_{jN}(t) w_j(x)$ in V_N, the solution of the system of ordinary differential equations

$$\left|\begin{array}{l} (Lv_{\varepsilon N}(t), w) + \dfrac{1}{\varepsilon}\langle P(v'_{\varepsilon N}(t)), w\rangle = 0 \quad \text{for all} \quad w \text{ in } V_N \\[2mm] v_{\varepsilon N}(0) = v_{0N} \to v_0 \quad \text{strong in} \quad H_0^1(\Omega) \cap H^2(\Omega) \\[2mm] v'_{\varepsilon N}(0) = v_{1N} \to v_1 \quad \text{strong in } H_0^1(\Omega), \text{ with } v_{1N} \in K. \end{array}\right. \tag{3.1}$$

The system (3.1) has a local solution $v_{\varepsilon N} = v_{\varepsilon N}(x, t)$, for $x \in \Omega$ and $0 \leq t < T_N$, cf. Coddington-Levinson [7]. The extension of $v_{\varepsilon N}$ from $[0, t_N)$ to $[0, T)$, for all number $T > 0$, is a consequence of an a priori estimate obtained in Estimate (i).

Remark 3.1. Since K is a closed convex set of $H_0^1(\Omega)$, there exists a projection operator $\pi_K \colon H_0^1(\Omega) \to K$, cf. Brezis [2]. We have $||\pi_K\, v_{1N} - \pi_K\, v_1|| \leq ||v_{1N} - v_1||$, which converges to zero. But $\pi_K\, v_1 = v_1$ because $v_1 \in K$. Then $\pi_K\, v_{1N} \in K$ approximates v_1 in $H_0^1(\Omega)$ norm. So we can consider the approximations of v_1 belonging to K. □

In order to have a better notation, we consider, in the computation, $v_{\varepsilon N} = v$, $\dfrac{\partial^2 v}{\partial y^2} = \Delta v$, $\dfrac{\partial v}{\partial y} = \nabla v$ and $\dfrac{\partial v}{\partial t} = v'$, $\dfrac{\partial^2 v}{\partial t^2} = v''$. By $|v(y,t)|$ we represent the absolute value of the real number $v(y,t)$ and $|v(t)|$, $||v(t)||$ the norms of $v = v(y,t)$ in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively, that is,

$$|v(t)|^2 = \int_\Omega |v(y,t)|^2\, dy \quad \text{and} \quad ||v(t)||^2 = \int_\Omega |\nabla v(y,t)|^2\, dy.$$

Estimate (i). Set $w = v'(t)$ in (3.1) and observe the definition of Lv given in (2.6). We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\,|v'(t)|^2 + \mu(t)\,\frac{1}{2}\frac{d}{dt}\,||v(t)||^2 + a(t, v(t), v'(t)) + \\ &\qquad + (b(t)\nabla v'(t), v'(t)) + (c(t)\nabla v(t), v'(t)) + \\ &\qquad + \frac{1}{\varepsilon}\,\langle P(v'(t)), v'(t)\rangle = 0. \end{aligned} \tag{3.2}$$

Observe that we employ the notation $a(t, v(t), w) = \displaystyle\int_\Omega a(y,t)\nabla v(y,t)\nabla w(y)\, dy$ and $(b(t)g(t), w) = \displaystyle\int_\Omega b(y,t)g(y,t)w(y)\, dy$. Note that $\mu(t) = \dfrac{1}{\gamma^2(t)}\left[-\dfrac{m_0}{2} + \hat{a}(t) + \dfrac{\hat{b}(t)}{\gamma(t)}\displaystyle\int_\Omega |\nabla v(y,t)|^2\, dy\right]$.

After computations we obtain for all $0 \le t < t_N$,

$$
\begin{aligned}
&\frac{d}{dt}\left\{ |v'(t)|^2 + \left(\frac{\hat{a}(t) - \dfrac{m_0}{2}}{\gamma^2(t)} \right) ||v(t)||^2 + \frac{\hat{b}(t)}{2\gamma^3(t)}\, ||v(t)||^4 + \right.\\
&\left. + \int_0^1 a(y,t)|\nabla v(y,t)|^2\, dy \right\} + 2\int_0^1 \frac{\gamma'(t)}{\gamma(t)}\, |v'(y,t)|^2\, dy + \\
&+ \frac{2}{\varepsilon}\, \langle P(v'(t), v'(t)\rangle \; = \\
&= \left[\frac{\hat{a}'(t)}{\gamma^2(t)} - \frac{2\left(\hat{a}(t) - \dfrac{m_0}{2}\right)\gamma'(t)}{\gamma^3(t)} \right] ||v(t)||^2 + \\
&+ \left[\frac{\hat{b}'(t)}{\gamma^3(t)} - \frac{3\hat{b}(t)\gamma'(t)}{\gamma^4(t)} \right] ||v(t)||^4 + \\
&+ \int_0^1 a'(y,t)|\nabla v(y,t)|^2\, dy + c(y,t)\left[||v(t)||^2 + |v'(t)|^2 \right].
\end{aligned}
\tag{3.3}
$$

Integrating (3.3) on $(0,t)$, $\;0 < t < t_N$, we obtain

$$
\begin{aligned}
&|v'(t)|^2 + \left(\frac{\hat{a}(t) - \dfrac{m_0}{2}}{\gamma^2(t)} \right) ||v(t)||^2 + \frac{\hat{b}(t)}{2\gamma^3(t)}\, ||v(t)||^4 + \\
&+ \int_0^1 a(y,t)|\nabla v(y,t)|^2\, dy + \frac{2}{\varepsilon}\int_0^t \langle P(v'(s), v'(s)\rangle\, ds \; \le \\
&\le |v_{1N}|^2 + \left(\frac{\hat{a}(0) - \dfrac{m_0}{2}}{\gamma_0^2} \right) ||v_{0N}||^2 + \frac{\hat{b}(0)}{2\gamma_0^3}\, ||v_{0N}||^4 + \\
&+ \frac{m_0}{2\gamma_0^2}\int_0^1 |\nabla v_{0N}(y)|^2\, dy + \int_0^t \left[\frac{\hat{a}'(s)}{\gamma^2(s)} - \frac{2\left(\hat{a}(s) - \dfrac{m_0}{2}\right)\gamma'(s)}{\gamma^3(s)} \right] ||v(s)||^2 ds + \\
&+ \int_0^t \left[\frac{\hat{b}'(s)}{\gamma^3(s)} - \frac{3\hat{b}(s)\gamma'(s)}{\gamma^4(s)} \right] ||v(s)||^4\, ds + \\
&+ \int_0^t \int_0^1 a'(y,s)|\nabla v(y,s)|^2\, dyds + \\
&+ \int_0^t |c(y,s)|\left[||v(s)||^2 + |v'(s)|^2 \right]\, ds.
\end{aligned}
\tag{3.4}
$$

Remark 3.2. In (3.4), by the convergences in (3.1), the sum of the terms evaluated in $t = 0$ is less than a positive constant C_2, independent of N and t_N. Also we have

- $\left(\dfrac{\hat{a}(t)-\dfrac{m_0}{2}}{\gamma^2(t)}\right) \geq \left(\dfrac{m_0}{2\gamma^2(t)}\right) > C_3$
- $\dfrac{\hat{b}(t)}{\gamma^3(t)} \geq \dfrac{k}{2m\gamma^3(t)} > C_4$

Note that C_3 and C_4 depend on $T>0$ but T is an arbitrary positive number, not depending of N and t_N.

- $\displaystyle\int_0^1 a(y,t)|\nabla v(y,t)|^2\,dy \geq 0$ by (H2) and (2.7).

□

From Remark 3.2 we modify (3.4), obtaining

$$\varphi(t)+\frac{1}{\varepsilon}\int_0^t \langle P(v'(s),v'(s)\rangle\,ds \leq C_5 + C_7\int_0^t \varphi(s)\,ds, \tag{3.5}$$

with $\varphi(t)=|v'(t)|^2+||v(t)||^2+||v(t)||^4$. Since the penalty term is positive, the Gronwall inequality implies $\varphi(t)\leq C_8$, that is, after the extension of the solution

$$|v'_{\varepsilon N}(t)|^2+||v_{\varepsilon N}(t)||^2+||v_{\varepsilon N}(t)||^4 < C_8\,, \tag{3.6}$$

for all $N\in\mathbb{N}$, $\varepsilon>0$ and $t\in[0,T]$, $T>0$.

From (3.5) and (3.6) it follows that

$$\int_0^T \langle P(v'_{\varepsilon N}(t)), v'_{\varepsilon N}(t)\rangle\,dt < C_9\,,$$

for all $N\in\mathbb{N}$, $0<\varepsilon<1$, and any fixed $T>0$.

By definition of P, this implies

$$\int_0^T\int_0^1 \left(|\nabla v'_{\varepsilon N}(y,t)|^2-1\right)|\nabla v'_{\varepsilon N}(y,t)|^2\,dydt < C_9 \tag{3.7}$$

for $|\nabla v'_{\varepsilon N}(y,t)|^2>1$. For $|\nabla v'_{\varepsilon N}(y,t)|^2\leq 1$ the duality is zero, because $P(v'_{\varepsilon N}(y,t))=0$.

From (3.7) and by Schwarz's inequality, we obtain an extra fundamental estimate

$$\int_0^T |\nabla v'_{\varepsilon N}(t)|\_L^4(\Omega)^4 \, dt < C_{10}\,,$$

for all $N \in \mathbb{N}$, $\;0 < \varepsilon < 1$, $\;T$ an arbitrary positive number.

Thus we have the estimate

$$|v'_{\varepsilon N}(t)|^2 + ||v_{\varepsilon N}(t)||^2 + \int_0^T |\nabla v'_{\varepsilon N}(t)|\_L^4(\Omega)^4 \, dt < C_{11}. \tag{3.8}$$

Estimate (ii). Set $w = -\Delta v'(t)$ in (3.1). We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\,||v'(t)||^2 + \mu(t)\,\frac{1}{2}\frac{d}{dt}\,|\Delta v(t)|^2 + \\ &+ a(t, v(t), -\Delta v'(t)) + (b(t)\nabla v'(t), -\Delta v'(t)) + \\ &+ (c(t)\nabla v(t), -\Delta v'(t)) + \frac{1}{\varepsilon}\,\langle P((v'(t)), -\Delta v'(t)\rangle = 0. \end{aligned} \tag{3.9}$$

By definition of $P(v'(t))$, see (2.15), we obtain $P(v'(t)) = 0$ when $|\nabla v'(y,t)|^2 \leq 1$, that is $v'(t) \in K$. It follows that

$$\frac{1}{\varepsilon}\,\langle P(v'(t)), -\Delta v'(t)\rangle = \frac{1}{\varepsilon}\int_{|\nabla v'(y,t)|>1} \left[3(\nabla v'(y,t))^2 - 1\right] [\Delta v'(y,t)]^2 \, dy \geq 0.$$

By a similar argument as we did to obtain Estimate (i), we transform (3.9) to the following inequality:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{||v'(t)||^2+\mu(t)|\Delta v(t)|^2+a(t,\nabla v(t),\nabla v(t)\right\}+\\
&+\left[\frac{\beta'(t)}{\gamma(t)}\,(\nabla v'(1,t))^2-\frac{\alpha'(t)}{\gamma(t)}\,(\nabla v'(0,t))^2\right]\ \le\\
&\le 2\left[\frac{\beta'(t)\gamma'(t)}{\gamma^2(t)}\,\nabla v(1,t)\nabla v'(1,t)-\frac{\alpha'(t)\gamma'(t)}{\gamma^2(t)}\,\nabla v(0,t)\nabla v'(0,t)\right]+\\
&+\left[-\frac{\beta''(t)}{\gamma(t)}\,\nabla v(1,t)\nabla v'(1,t)+\frac{\alpha''(t)}{\gamma(t)}\,\nabla v(0,t)\nabla v'(0,t)\right]+\\
&+\frac{1}{2}\left[\frac{\hat{a}'(t)}{\gamma^2(t)}-\frac{2\left(\hat{a}(t)-\dfrac{m_0}{2}\right)\gamma'(t)}{\gamma^3(t)}\right]\,|\Delta v(t)|^2+\\
&+\frac{1}{2}\left[\frac{\hat{b}'(t)}{\gamma^3(t)}-\frac{3\hat{b}(t)\gamma'(t)}{\gamma^4(t)}\right]\,||v(t)||^2\,|\Delta v(t)|^2+\\
&+\frac{\hat{b}(t)}{\gamma^3(t)}\,(\nabla v(t),\nabla v'(t))|\Delta v(t)|^2-2\left[\frac{\gamma'(t)}{\gamma(t)}\right]^2(\nabla v(t),\nabla v'(t))-\\
&-2\left[\frac{\alpha'(t)+y\gamma'(t)}{\gamma(t)}\right]\,\frac{\gamma'(t)}{\gamma(t)}\,(\Delta v(t),\nabla v'(t)),+\\
&+\frac{1}{2}\int_0^1 a'(y,t)|\Delta v(y,t)|^2\,dy+\int_0^1\frac{\gamma'(t)}{\gamma(t)}\,|\nabla v'(y,t)|^2\,dy+\\
&+\int_0^1\frac{\gamma''(t)}{\gamma(t)}\,\nabla v(y,t)\nabla v'(y,t)\,dy+\\
&+\int_0^1\left[\frac{\alpha''+y\gamma''(t)}{\gamma(t)}\right]\Delta v(y,t)\nabla v'(y,t)\,dy.
\end{aligned}
\tag{3.10}
$$

Now, by hypothesis (H1), (H2) and Estimate (i), we modify the right hand side of (3.10) obtaining

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{||v'(t)||^2+\mu(t)|\Delta v(t)|^2+a(t,\nabla v(t),\nabla v(t))\right\}+\\
&+\left[\frac{\beta'(t)}{\gamma(t)}\,(\nabla v'(1,t))^2-\frac{\alpha'(t)}{\gamma(t)}\,(\nabla v'(0,t))^2\right]\ \le\\
&\le 2\left[\frac{\beta'(t)\gamma'(t)}{\gamma^2(t)}\,\nabla v(1,t)\nabla v'(1,t)-\frac{\alpha'(t)\gamma'(t)}{\gamma^2(t)}\,\nabla v(0,t)\nabla v'(0,t)\right]+\\
&+\left[-\frac{\beta''(t)}{\gamma(t)}\,\nabla v(1,t)\nabla v'(1,t)+\frac{\alpha''(t)}{\gamma(t)}\,\nabla v(0,t)\nabla v'(0,t)\right]+\\
&+K_1+K_2\left[|\nabla v'(t)|^2+(1+|\nabla v'(t)|_{L^4(\Omega)})|\Delta v(t)|^2\right].
\end{aligned}
\tag{3.11}
$$

By an argument similar to that employed in [17] Part Two, we transform (3.11) and obtain

$$\begin{aligned} &\frac{d}{dt}\,\left\{||v'(t)||^2 + \mu(t)|\Delta v(t)|^2 + a(t, \nabla v(t), \nabla v(t))\right\} \leq \\ &\leq 2K_1 + K_3\left[||v'(t)||^2 + (1 + |\nabla v'(t)|_{L^4(\Omega)})|\Delta v(t)|^2\right]. \end{aligned} \tag{3.12}$$

Here we are in the fundamental point in our proof. By Estimate (i) we have

$$\int_0^T |\nabla v'_{\varepsilon N}(t)|^4_{L^4(0,1)}\,dt < C_{11}\,.$$

We have, by Hölder's inequality with $p = 4$, $p' = \dfrac{4}{3}$, and the above estimate, that

$$\int_0^T \int_0^1 |\nabla v'_{\varepsilon N}(y,t)|\,dydt$$

is bounded. Note, also, by the Schwarz inequality and the above estimate, we obtain

$$\int_0^T \int_0^1 |\nabla v'_{\varepsilon N}(y,t)|^2\,dydt$$

is bounded. Thus, since $a(t, \nabla v(t), \nabla v(t)) \geq 0$, $\mu(t) \geq \dfrac{m_0}{2\gamma^2(T)}$, we obtain from (3.12)

$$|\Delta v(t)|^2 \leq K_4 + K_3 \int_0^t (1 + |\nabla v'(s)|_{L^4(0,1)})|\Delta v(s)|^2\,ds$$

for all $0 < t < T$. We are in the case of a Gronwall inequality of the type

$$\varphi(t) \leq C + \int_0^t \theta(s)\varphi(s)\,ds,$$

with $\theta \in L^1(0,T)$. It implies that $|\Delta v(t)|^2$ is bounded in $[0,T]$ for all number $T > 0$.

Thus we obtain the estimate

$$||v'_{\varepsilon N}(t)||^2 + |\Delta v_{\varepsilon N}(t)|^2 < C_{12}\,, \tag{3.13}$$

for all $N \in \mathbb{N}$, $0 < \varepsilon < 1$, $t \in [0, T]$, $T > 0$ an arbitrary number.

Estimate (iii). We estimate $v''_{\varepsilon N}$ in the norm $L^2(\Omega)$ for $0 < t < T$. First we need estimate $v''_{\varepsilon N}$ at $t = 0$. From (3.1), for $t = 0$ we obtain

$$(v''_{\varepsilon N}(0), w) = \mu(0)(\Delta v_{0N}, w) - \frac{m_0}{2\gamma_0^2}(\Delta v_{0N}, w) - (c(0)\nabla v_{0N}, w). \tag{3.14}$$

If we set $w = v''_{\varepsilon N}(0)$ in (3.14), observing the convergences in (3.1), we obtain

$$|v''_{\varepsilon N}(0)| < C_{12}\,, \text{ for all } N \in \mathbb{N},\ \ 0 < \varepsilon < 1. \tag{3.15}$$

To estimate $v''_{\varepsilon N}$ it is not simple because the penalty term in (3.1) is not derivable. However it is monotone and this helps substantially to estimate $v''_{\varepsilon N}$. We employ an argument of Lions [15], Browder [5]. See also Brezis [3] and Vieira & Rabello [25], for the same difficulty.

We define the operator

$$\delta_h\, v(y,t) = \frac{1}{h}\,[v(y, t+h) - v(y,t)],$$

for $0 < y < 1$, $h > 0$ and $0 < t < T - h$.

From the approximate equation (3.1) we obtain

$$(Lv(t+h) - Lv(t), w) + \frac{1}{\varepsilon}\,\langle P(v'(t+h)) - P(v'(t)), w\rangle = 0.$$

Dividing both sides by $h > 0$, we obtain

$$(\delta_h\, Lv(t), w) + \frac{1}{\varepsilon}\,\langle \delta_h\, P(v'(t)), w\rangle = 0. \tag{3.16}$$

For $w = \delta_h\, v'$, we obtain, by monotonicity,

$$\frac{1}{\varepsilon}\,\langle \delta_h\, P(v'(t)), \delta_h\, v'(t)\rangle \geq 0.$$

Thus we have

$$(\delta_h\, Lv(t), \delta_h\, v'(t)) \leq 0, \quad \text{for all} \quad 0 < t < T - h. \tag{3.17}$$

After computations similar to the one done in Estimate (i), if we set $\varphi(t) = |\delta_h\, v'_{\varepsilon N}(t)|^2 + ||\delta_h\, v_{\varepsilon N}(t)||^2$, we obtain

$$\varphi(t) \leq K_5(1 + \varphi(0)) + K_6 \int_0^t \varphi(s)\, ds. \tag{3.18}$$

We prove that as $h \to 0$, we have

$$|\delta_h\, v'_{\varepsilon N}(0)|^2 \to |v''_{\varepsilon N}(0)|^2 \text{ and } ||\delta_h\, v_{\varepsilon N}(0)||^2 \to ||v'_{\varepsilon N}(0)||^2.$$

Since $|v''_{\varepsilon N}(0)|^2 < C_{13}$ and $||v'_{\varepsilon N}(0)||^2 = ||v_{1N}||^2$ is also bounded, see (3.1), we obtain, from (3.18)

$$|\delta_h\, v'_{\varepsilon N}(t)|^2 \leq (K_7 + K_8\, r(h))e^{K_9 T},$$

with $r(h) \to 0$ when $h \to 0$, for $0 < \varepsilon < 1$, $T > 0$ is an arbitrary number.

Taking the limit when $h \to 0$ in the last inequality, we obtain

$$|v''_{\varepsilon N}(t)|^2 < C_{14}\,, \quad \text{for all} \quad N \in \mathbb{N},\ 0 < \varepsilon < 1,\ t \in [0, T],\ T > 0 \text{ arbitrary}. \tag{3.19}$$

From the estimates, uniform in N and $0 < \varepsilon < 1$, we obtain a subsequence $(v_{\varepsilon N})_{N \in \mathbb{N}}$, for ε fixed, such that

$$\left|\begin{array}{l}
v_{\varepsilon N} \rightharpoonup v_\varepsilon \quad \text{weak star in} \quad L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \\
v'_{\varepsilon N} \rightharpoonup v'_\varepsilon \quad \text{weak star in} \quad L^\infty(0, T; H_0^1(\Omega)) \\
v'_{\varepsilon N} \rightharpoonup v'_\varepsilon \quad \text{weakly in} \quad L^4(0, T; W_0^{1,4}(\Omega)) \\
v''_{\varepsilon N} \rightharpoonup v''_\varepsilon \quad \text{weak star in} \quad L^\infty(0, T; L^\infty(0, T; L^2(\Omega)) \\
P(v'_{\varepsilon N}) \rightharpoonup \chi_\varepsilon \quad \text{weakly in} \quad L^{4/3}(0, T; W^{-1, \frac{4}{3}}(\Omega)).
\end{array}\right. \tag{3.20}$$

Note that the last convergence is because the penalty operator takes bounded sets of $L^4(0, T; W_0^{1,4}(\Omega))$ into bounded sets of the dual $L^{4/3}(0, T; W^{-1, \frac{4}{3}}(\Omega))$. To pass to the limit in the approximate equation we have a problem in the nonlinear term $\mu(t)\Delta v_{\varepsilon N}$. We have the first convergence in (3.20) which gives $\Delta v_{\varepsilon N} \rightharpoonup \Delta v_\varepsilon$ weak star in $L^\infty(0,T;L^2(\Omega))$ but we need some strong convergence for $\mu(t)$. We have $v_{\varepsilon N}$ bounded in $L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$ and $v'_{\varepsilon N}$ bounded in $L^2(0,T;H_0^1(\Omega))$. Since $H_0^1(\Omega)\cap H^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ with the first embedding compact, there exists a subsequence, still represented by $(v_{\varepsilon N})$, such that

$$v_{\varepsilon N} \to v_\varepsilon \quad \text{strongly in} \quad L^2(0,T;H_0^1(\Omega)). \tag{3.21}$$

This is an application of the compactness argument of Aubin-Lions, cf. [14], [23], [24].

By the estimates (ii), (iii) and the same argument of compactness, we obtain a subsequence $(v_{\varepsilon N})$ such that

$$v'_{\varepsilon N} \to v'_\varepsilon \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.22}$$

By means of the convergences (3.20) and (3.21) we can pass to the limit in (3.1) when $N\to\infty$ and obtain

$$(Lv_\varepsilon(t),w(t)) + \frac{1}{\varepsilon}\,\langle \chi_\varepsilon(t),w(t)\rangle = 0 \tag{3.23}$$

for all $w\in L^4(0,T;W_0^{1,4}(\Omega))$.

Equation (3.23) says that

$$Lv_\varepsilon + \frac{1}{\varepsilon}\,\chi_\varepsilon = 0 \quad \text{in} \quad L^{4/3}(0,T;W^{-1,\frac{4}{3}}(\Omega)). \tag{3.24}$$

The next step is to prove that $\chi_\varepsilon(t) = P(v'_\varepsilon(t))$. This is a consequence of monotonicity of P and (3.24). In fact, for $z\in L^4(0,T;W_0^{1,4}(\Omega))$, we have

$$\int_0^T \langle P(v'_{\varepsilon N}(t) - P(z(t)), v'_{\varepsilon N}(t) - z(t)\rangle\, dt \geq 0.$$

Then

$$\begin{aligned}\overline{\lim_{N\to\infty}} \int_0^T \langle P(v'_{\varepsilon N}(t)), v'_{\varepsilon N}(t)\rangle\, dt - \int_0^T \langle \chi_\varepsilon(t), z(t)\rangle\, dt -\\ - \int_0^T \langle P(z(t), v'_\varepsilon(t) - z(t)\rangle\, dt \geq 0.\end{aligned} \tag{3.25}$$

From the approximate equation (3.1) and since $v'_{\varepsilon N} \to v'_\varepsilon$ strongly in $L^2(0,T;L^2(\Omega))$, we obtain

$$\overline{\lim_{N\to\infty}} \int_0^T \langle P(v'_{\varepsilon N}(t), v'_{\varepsilon N}(t)\rangle \, dt = -\lim_{N\to\infty} \varepsilon \int_0^T (Lv_{\varepsilon N}(t), v'_{\varepsilon N}(t))dt \; =$$
$$= -\varepsilon \int_0^T (Lv_\varepsilon(t), v'_\varepsilon(t))dt = \int_0^T \langle \chi_\varepsilon(t), v'_\varepsilon(t)\rangle \, dt, \text{ by (3.24).}$$

Substituting in (3.25) we get

$$\int_0^T \langle \chi_\varepsilon(t) - P(z(t)), v'_\varepsilon(t) - z(t)\rangle \, dt \geq 0.$$

This implies $\chi_\varepsilon(t) = P(v'_\varepsilon(t))$. It is sufficient to set $z = v'_\varepsilon - \lambda w$, $\lambda > 0$, w arbitrary in $L^4(0,T;W_0^{1.4}(\Omega))$ and let $\lambda \to 0$. Note that $v'_\varepsilon \in L^4(0,T;W_0^{1,4}(\Omega))$.

Thus we have, in fact,

$$Lv_\varepsilon + \frac{1}{\varepsilon} P(v'_\varepsilon) = 0 \quad \text{in} \quad L^{4/3}(0,T;W^{-1,4/3}(\Omega)),$$

that is

$$\int_0^T (Lv_\varepsilon(t), w(t)) \, dt + \frac{1}{\varepsilon} \int_0^T \langle P(v'_\varepsilon(t)), w(t)\rangle \, dt = 0$$

for all $w \in L^4(0,T;W_0^{1,4}(\Omega))$.

The function v_ε satisfies all the conditions of theorem 2.3, which is now proved. □

From the convergences (3.20) and Banach-Steinhaus theorem, it follows from (3.20), (3.21) and (3.22) that there exists a subnet $(v_\varepsilon)_{0<\varepsilon<1}$, such that it converges to v as $\varepsilon \to 0$, in the sense of (3.20), (3.21) and (3.22). This function satisfies (2.11). Thus for $z \in L^4(0,T;W_0^{1,4}(\Omega))$ with $z(t) \in K$ a.e. in $(0,T)$, we obtain

$$\int_0^T (Lv_\varepsilon(t), z(t) - v'_\varepsilon(t))\gamma(t)dt \geq 0.$$

When $\varepsilon \to 0$, we deduce that v is a solution of

$$\int_0^T (Lv(t), z(t) - v(t))\gamma(t)\, dt \geq 0.$$

We have uniqueness and v satisfies the initial conditions. To prove that $v'(t) \in K$ a.e. in $(0,T)$, observe that $P(v'_\varepsilon) \to 0$ in $L^{4/3}(0,T; W_0^{-1,4/3}(\Omega))$ when $\varepsilon \to 0$. This strong convergence happens because, for all $w \in L^4(0,T; W_0^{1,4}(\Omega))$,

$$\left| \int_0^T \langle P(v'_\varepsilon(t)), w(t)\rangle\, dt \right| = \left| -\varepsilon \int_0^T (L(v_\varepsilon(t)), w(t))\, dt \right| \leq \varepsilon\, M\, ||w||_{L^4\left(0,T; W_0^{1,4}(\Omega)\right)}$$

with M a constant not depending on ε.

Thus, since P is monotone, we have, for all $w \in L^4(0,T; W_0^{1,4}(\Omega))$,

$$\int_0^T \langle P(w(t)), w(t) - v'(t)\rangle\, dt = \lim_{\varepsilon \to 0} \int_0^T \langle P(w(t)) - P(v'_\varepsilon(t)), w(t) - v'_{\varepsilon N}(t)\rangle dt \geq 0\,.$$

As P is hemicontinuous, we choose $w = v' + \lambda\xi$, with $\lambda > 0$ and ξ arbitrary in $L^4(0,T; W_0^{1,4}(\Omega))$, in the above inequality. So, it follows that

$$\int_0^T \langle P(v'(t)), \xi(t)\rangle\, dt = 0,$$

for all $\xi \in L^4(0,T; W_0^{1,4}(\Omega))$, what implies that $P(v'(t)) = 0$ a.e. in $(0,T)$, that is, $v'(t) \in K$ a.e. in $(0,T)$. □

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.1,75-87,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

On Solution Sensitivity of Generalized Relaxed Cocoercive Implicit Quasivariational Inclusions with A-monotone Mappings[1]

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Abstract. In this paper, we introduce and study a new class of parametric generalized relaxed cocoercive implicit quasivariational inclusions with A-monotone mappings. By using the parametric implicit resolvent operator technique for A-monotone, we analyze solution sensitivity for this kind of generalized relaxed cocoercive inclusions in Hilbert spaces. Our results generalize sensitivity analysis results on strongly monotone quasivariational inclusions and nonlinear implicit quasivariational inclusions. Furthermore, relaxed cocoercivity is illustrated by some examples.

Key words and phrases: Sensitive analysis, relaxed cocoercive implicit quasivariational inclusions, relaxed maximal monotone mapping, A-monotone mapping, implicit resolvent operator.

2000 Mathematics Subject classification: 49J40, 47H05

1 INTRODUCTION

In order to enable us to study the behavior and sensitivity analysis of solution sets of many important nonlinear problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional structural, transportation, elasticity, and various applied sciences in a general and unified framework, it is well known that many authors had studied sensitivity analysis of solutions for variational inequalities and inclusions in quite different methods. For example, by using the project

[1]This work was supported by the Korea Research Foundation Grant (KRF-2004-041-C00033) and the Educational Science Foundation of Sichuan, Sichuan of China (2004C018).

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technique, Ding and Luo [3], Ding et al. [5], Moudafi [14] dealt with the sensitivity analysis of solutions for variational inequalities and nonlinear project equations in Hilbert spaces. By using the implicit function approach, Jittorntrum [10], Kyparisis [11], Robinson [16] studied the sensitivity analysis of solutions for variational inequalities under suitable second order and regularity assumptions. Recently, Agarwal et al. [1], Dong et al. [6], analyzed solution sensitivity analysis for variational inequalities and variational inclusions by using resolvent operator technique.

On the other hand, Verma [18, 19] introduced the concept of A-monotone mappings, which generalizes the well-known general class of maximal monotone mappings, and originates way back from an earlier work of the Verma [17]. The author also studied some properties of A-monotone mappings and defined resolvent operators associated with A-monotone mappings. Further, Verma [21] analyzed solution sensitivity for a relaxed cocoercive quasivariational inclusions based on the generalized resolvent operator technique, which generalizes the results on the sensitivity analysis for strongly monotone quasivariational inclusions as well as for relaxed cocoercive quasivariational inclusions [1, 3, 14, 20] and others since the class of relaxed cocoercive mappings is more general than the existing classes of mapping in literature. Some examples of relaxed cocoercive mappings are also included. For more details, we recommend [1-11, 13, 14, 16-22].

Inspired and motivated by the works of [1, 2, 4, 9, 21], in this paper, we introduce and study a new class of parametric generalized relaxed cocoercive implicit quasivariational inclusions with A-monotone mappings. By using the parametric implicit resolvent operator technique for A-monotone, we analyze solution sensitivity for this kind of generalized relaxed cocoercive inclusions in Hilbert spaces. Our results generalize sensitivity analysis results on strongly monotone quasivariational inclusions and nonlinear implicit quasivariational inclusions. Furthermore, relaxed cocoercivity is illustrated by some examples.

2 PRELIMINARIES

Throughout this paper, we suppose that $\mathbb{H}$ is a real Hilbert space with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, $2^{\mathbb{H}}$ denotes the family of all the nonempty subsets of $\mathbb{H}$, $C(\mathbb{H})$ denotes the family of all the nonempty compact subsets of $\mathbb{H}$ and $\mathcal{L}$ is a nonempty open subset of $\mathbb{H}$ in which the parameter λ take values.

The notion of the cocoercivity is applied in several directions, especially to solving variational inequality problems using the auxiliary problem principle and projection methods [18, 20], while the notion of the relaxed cocoercivity is more general than the strong monotonicity as well as cocoercivity. Several classes of relaxed cocoercive variational inequalities have been studied in [18-20].

Definition 2.1. A mapping $T : \mathbb{H} \times \mathbb{H} \times \mathcal{L} \to \mathbb{H}$ is said to be

(i) m-relaxed monotone in the first argument if there exists a positive constant m such that
$$\langle T(x, u, \lambda) - T(y, u, \lambda), x - y\rangle \geq -m\|x - y\|^2$$

for all $(x, y, u, \lambda) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathcal{L}$;

(ii) s-cocoercive in the first argument if there exists a constant $s > 0$ such that

$$\langle T(x,u,\lambda) - T(y,u,\lambda), x - y\rangle \geq s\|T(x,u,\lambda) - T(y,u,\lambda)\|^2$$

for all $(x, y, u, \lambda) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathcal{L}$;

(iii) γ-relaxed cocoercive with respect to A in the first argument if there exists a positive constant γ such that

$$\langle T(x,u,\lambda) - T(y,u,\lambda), A(x) - A(y)\rangle \geq -\gamma\|T(x,u,\lambda) - T(y,u,\lambda)\|^2$$

for all $(x, y, u, \lambda) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathcal{L}$;

(iv) (ϵ, α)-relaxed cocoercive with respect to A in the first argument if there exist positive constants ϵ and α such that

$$\langle T(x,u,\lambda) - T(y,u,\lambda), A(x) - A(y)\rangle \geq -\alpha\|T(x,u,\lambda) - T(y,u,\lambda)\|^2 + \epsilon\|x - y\|^2$$

for all $(x, y, u, \lambda) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathcal{L}$.

In a similar way, we can define (relaxed) cocoercivity of the mapping $T(\cdot,\cdot,\cdot)$ in the second argument.

Example 2.1. Consider a nonexpansive mapping $T : \mathbb{H} \to \mathbb{H}$. If we set $F = I - T$, where I is the identity mapping, then F is $(\frac{1}{2})$-cocoercive.

Proof. For any two elements $x, y \in \mathbb{H}$, we have

$$\begin{aligned}
\|F(x) - F(y)\|^2 &= \|(I - T)(x) - (I - T)(y)\|^2 \\
&= \langle (I - T)(x) - (I - T)(y), (I - T)(x) - (I - T)(y)\rangle \\
&\leq 2[\|x - y\|^2 - \langle x - y, T(x) - T(y)\rangle] \\
&= 2\langle x - y, F(x) - F(y)\rangle,
\end{aligned}$$

that is, F is $(\frac{1}{2})$-cocoercive.

Example 2.2. Consider a projection $P : \mathbb{H} \to C$, where C is a nonempty closed convex subset of $\mathbb{H}$. Then P is 1-cocoercive since P is nonexpansive.

Proof. For any $x, y \in \mathbb{H}$, we have

$$\begin{aligned}
\|P(x) - P(y)\|^2 &= \langle P(x) - P(y), P(x) - P(y)\rangle \\
&\leq \langle x - y, P(x) - P(y)\rangle,
\end{aligned}$$

that is, P is 1-cocoercive.

Example 2.3. Consider an r-strongly monotone (and hence r-expanding) mapping $T : \mathbb{H} \to \mathbb{H}$. Then T is $(r + r^2, 1)$-relaxed cocoercive with respect to I.

Proof. For any two elements $x, y \in \mathbb{H}$, we have

$$\|T(x) - T(y)\| \geq r\|x - y\|,$$
$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2,$$

and so

$$\|T(x) - T(y)\|^2 + \langle T(x) - T(y), x - y\rangle \geq (r + r^2)\|x - y\|^2,$$

i.e., for all $x, y \in \mathbb{H}$, we get

$$\langle T(x) - T(y), x - y\rangle \geq (-1)\|T(x) - T(y)\|^2 + (r + r^2)\|x - y\|^2.$$

Therefore, T is $(r + r^2, 1)$-relaxed cocoercive with respect to I.

Remark 2.1. Clearly, every m-cocoercive mapping is m-relaxed cocoercive, while each r-strongly monotone mapping is $(r + r^2, 1)$-relaxed cocoercive with respect to I.

Definition 2.2. A mapping $T : \mathbb{H} \times \mathbb{H} \times \mathcal{L} \to \mathbb{H}$ is said to be μ-Lipschitz continuous in the first argument if there exists a constant $\mu > 0$ such that

$$\|T(x, u, \lambda) - T(y, u, \lambda)\| \leq \mu\|x - y\|$$

for all $(x, y, u, \lambda) \in \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathcal{L}$. In a similar way, we can define Lipschitz continuity of the mapping $T(\cdot, \cdot, \cdot)$ in the second and third argument.

Definition 2.3. Let $F : \mathbb{H} \times \mathcal{L} \to 2^{\mathbb{H}}$ be a multivalued mapping. Then F is said to be τ-$\hat{\mathbf{H}}$-Lipschitz continuous in the first argument if there exists a constant $\tau > 0$ such that

$$\hat{\mathbf{H}}(F(x, \lambda), F(y, \lambda)) \leq \tau\|x - y\|$$

for all $x, y \in \mathbb{H}$ and $\lambda \in \mathcal{L}$, where $\hat{\mathbf{H}} : 2^{\mathbb{H}} \times 2^{\mathbb{H}} \to (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff metric, i.e.,

$$\hat{\mathbf{H}}(A, B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x - y\|, \sup_{x\in B}\inf_{y\in A}\|x - y\|\}$$

for all $A, B \in 2^{\mathbb{H}}$.

In a similar way, we can define $\hat{\mathbf{H}}$-Lipschitz continuity of the mapping $F(\cdot, \cdot)$ in the second argument.

Lemma 2.1. ([12]) Let $(\mathbb{X}, d)$ be a complete metric space and $T_1, T_2 : \mathbb{X} \to C(\mathbb{X})$ be two set-valued contractive mappings with same contractive constant $t \in (0, 1)$, i.e.,

$$\hat{\mathbf{H}}(T_i(x), T_i(y)) \leq td(x, y)$$

for all $x, y \in \mathbb{X}$ and $i = 1, 2$. Then we have

$$\hat{\mathbf{H}}(F(T_1), F(T_2)) \leq \frac{1}{1 - t}\sup_{x\in\mathbb{X}}\hat{\mathbf{H}}(T_1(x), T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are fixed-point sets of T_1 and T_2, respectively.

3 A-MONOTONICITY

Recently, Verma [18, 19] introduced and studied a new class of mappings A-monotone mappings, which have a wide range of applications. The class of A-monotone mappings generalizes the well-known class of maximal monotone mappings. The notion of the A-monotonicity is illustrated by some examples.

Definition 3.1. ([18]) Let $A : \mathbb{H} \to \mathbb{H}$ be a nonlinear mapping on a Hilbert space $\mathbb{H}$ and let $M : \mathbb{H} \to 2^{\mathbb{H}}$ be a multivalued mapping on $\mathbb{H}$. The mapping M is said to be A-monotone if M is m-relaxed monotone and $R(A + \rho M) = \mathbb{H}$ holds for $\rho > 0$.

Note that this is equivalent to stating that M is A-monotone with constant m if

(i) M is m-relaxed monotone,

(ii) $A + \rho M$ is maximal monotone.

Remark 3.1. Obviously, if $m = 0$, that is, M is 0-relaxed monotone, then the A-monotone mappings reduces to an H-monotone operators (see, for example, [2]). Therefore, the class of A-monotone mappings provides a unifying frameworks for classes of maximal monotone operators and H-monotone operators. For details about these operators, we refer the reader to [2, 22] and the references therein.

Example 3.1. ([19]) Let $\mathbb{H}$ be a reflexive Banach space with $\mathbb{H}^*$ its dual space and $A : \mathbb{H} \to \mathbb{H}^*$ be r-strongly monotone. Let $f : \mathbb{H} \to R$ be locally Lipschitz such that ∂f is m-relaxed monotone. Then ∂f is A-monotone, which is equivalent to stating that $A + \partial f$ is pseudomonotone (and in fact, maximal monotone).

Proposition 3.1. Let $A : \mathbb{H} \to \mathbb{H}$ be an r-strongly monotone single-valued mapping and $M : \mathbb{H} \to 2^{\mathbb{H}}$ be an A-monotone mapping with constant m on a real Hilbert space $\mathbb{H}$. Then M is maximal monotone.

Proof. Given that M is m-relaxed monotone, it suffices to show:

$$\langle u - v, x - y\rangle \geq -m\|x - y\|^2$$

if $(y, v) \in graph(M)$ implies $u \in M(x)$. Assume that $(x_0, u_0) \notin graph(M)$ such that

$$\langle u_0 - v, x_0 - y\rangle \geq (-m)\|x_0 - y\|^2 \tag{3.1}$$

for all $(y, v) \in graph(M)$. Since M is A-monotone, $R(A + \rho M) = \mathbb{H}$ for all $\rho > 0$. This implies that there exists an element $(x_1, u_1) \in graph(M)$ such that

$$A(x_1) + \rho u_1 = A(x_0) + \rho u_0. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\rho\langle u_0 - u_1, x_0 - x_1\rangle = -\langle A(x_0) - A(x_1), x_0 - x_1\rangle \geq -m\rho\|x_0 - x_1\|^2.$$

Since A is r-strongly monotone, it implies $x_0 = x_1$ for $\rho < \frac{r}{m}$. As a result, we have $u_0 = u_1$, that is, $(x_0, u_0) \in graph(M)$, a contradiction. Hence, M is maximal monotone.

The next property is helpful in shaping up the generalized resolvent operator, which is crucial to the main results on sensitivity analysis on hand.

Proposition 3.2. Let $A : \mathbb{H} \to \mathbb{H}$ be an r-strongly monotone mapping and let $M : \mathbb{H} \to 2^{\mathbb{H}}$ be an A-monotone mapping with constant m. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Proof. If, for a given $x \in \mathbb{H}$, $u, v \in (A+\rho M)^{-1}(x)$, then we have $-A(u)+x \in \rho M(u)$ and $-A(v) + x \in \rho M(v)$. Since M is m-relaxed monotone, it implies that

$$\begin{aligned}\langle -A(u) + x - (-A(v) + x), u - v\rangle &= \langle A(v) - A(u), u - v\rangle \\ &\geq -m\|u - v\|^2.\end{aligned}$$

Since A is r-strongly monotone, it implies $u = v$ for $m < r$. Therefore, $(A + \rho M)^{-1}$ is single-valued.

This leads to the generalized definition of the resolvent operator:

Definition 3.2. ([18]) Let $A : \mathbb{H} \to \mathbb{H}$ be an r-strongly monotone mapping and $M : \mathbb{H} \to 2^{\mathbb{H}}$ be an A-monotone mapping with constant m. Then the generalized resolvent operator $J^M_{\rho,A} : \mathbb{H} \to \mathbb{H}$ is defined by

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Lemma 3.1. ([18, 19]) Let $A : \mathbb{H} \to \mathbb{H}$ be $r - \mathit{strongly}$ monotone and $M : \mathbb{H} \to 2^{\mathbb{H}}$ be A-monotone with constant m. Then M is maximal monotone and the A-resolvent operator $J^M_{\rho,A} : \mathbb{H} \to \mathbb{H}$ associated with M and defined by

$$J^M_{\rho,A}(x) = (A + \rho M)^{-1}(x)$$

for all $x \in \mathbb{H}$ is $\frac{1}{r-\rho m}$-Lipschitz continuous for $0 < \rho < \frac{r}{m}$, i.e.,

$$\|J^M_{\rho,A}(x) - J^M_{\rho,A}(y)\| \leq \frac{1}{r - \rho m}\|x - y\|$$

for all $x, y \in \mathbb{H}$.

4 THE MAIN RESULTS

Let $N : \mathbb{H}\times\mathbb{H}\times\mathcal{L} \to \mathbb{H}$, $T : \mathbb{H}\times\mathcal{L} \to 2^{\mathbb{H}}$ and $g : \mathbb{H}\times\mathcal{L} \to \mathbb{H}$ be three nonlinear mapping and $M : \mathbb{H}\times\mathbb{H}\times\mathcal{L} \to 2^{\mathbb{H}}$ be a nonlinear mapping such that for each given $(y, \lambda) \in \mathbb{H}\times\mathcal{L}$, $M(\cdot, y, \lambda) : \mathbb{H} \to 2^{\mathbb{H}}$ be a A-monotone mapping with $g(\mathbb{H}, \lambda) \cap \mathrm{dom}M(\cdot, y, \lambda) \neq \emptyset$.

We will consider the following parametric generalized relaxed cocoercive implicit quasivariational inclusion problem:

For each fixed $\lambda \in \mathcal{L}$, find $x(\lambda) \in \mathbb{H}$ such that $u(\lambda) \in T(x(\lambda), \lambda)$ and

$$0 \in N(u(\lambda), x(\lambda), \lambda) + M(g(x(\lambda), \lambda), x(\lambda), \lambda). \tag{4.1}$$

Example 4.1. If $g = I$, the identity mapping and $T : \mathbb{H} \times \mathcal{L} \to \mathbb{H}$ is a single-valued mapping, then a special case of the problem (4.1) is: determine element $x(\lambda) \in \mathbb{H}$ such that

$$0 \in N(T(x(\lambda), \lambda), x(\lambda), \lambda) + M(x(\lambda), x(\lambda), \lambda). \tag{4.2}$$

Further, if $T = I$, then the problem (4.2) is equivalent to finding $x(\lambda) \in \mathbb{H}$ such that

$$0 \in N(x(\lambda), x(\lambda), \lambda) + M(x(\lambda), x(\lambda), \lambda), \tag{4.3}$$

which is studied by Verma [21] when $x(\lambda) = x$ for all $\lambda \in \mathcal{L}$ in (4.3).

Remark 4.1. For appropriate and suitable choices of N, T, g and M, it is easy to see that the problem (4.1) includes a number of (parametric) quasi-variational inclusions, (parametric) generalized quasi-variational inclusions, (parametric) quasi-variational inequalities, (parametric) implicit quasi-variational inequalities studied by many authors as special cases, see, for example, [1-11, 13, 17-22] and the references therein.

Now, for each fixed $\lambda \in \mathcal{L}$, the solution set $S(\lambda)$ of the problem (4.1) is denoted as

$$\begin{aligned} S(\lambda) = \quad & \{x(\lambda) \in \mathbb{H} : \text{ there exists } u(\lambda) \in T(x(\lambda), \lambda) \text{ such that} \\ & \quad 0 \in N(u(\lambda), x(\lambda), \lambda) + M(g(x(\lambda), \lambda), x(\lambda), \lambda)\}. \end{aligned}$$

In this paper, our main aim is to study the behavior of the solution set $S(\lambda)$, and the the conditions on these mappings T, N, M, g under which the function $S(\lambda)$ is continuous or Lipschitz continuous with respect to the parameter $\lambda \in \mathcal{L}$.

Next, we first transfer the problem (4.1) into a problem of finding the parametric fixed point of the associated resolvent operator.

Lemma 4.1. For each fixed $\lambda \in \mathcal{L}$, an element $x(\lambda) \in S(\lambda)$ is a solution to (4.1) if and only if there is $x(\lambda) \in \mathbb{H}$ and $u(\lambda) \in T(x(\lambda), \lambda)$ such that

$$g(x(\lambda), \lambda) = J_{\rho,A}^{M(\cdot, x(\lambda), \lambda)}(A(g(x(\lambda), \lambda)) - \rho N(u(\lambda), x(\lambda), \lambda)), \tag{4.4}$$

where $J_{\rho,A}^{M(\cdot,x(\lambda),\lambda)} = (A + \rho M(\cdot, x(\lambda), \lambda))^{-1}$ is the corresponding resolvent operator in first argument and of an A-monotone mapping $M(\cdot, \cdot, \cdot)$, A is an r-strongly monotone mapping and $\rho > 0$.

Proof. For each fixed $\lambda \in \mathcal{L}$, by the definition of the resolvent operator $J_{\rho,A}^{M(\cdot,x(\lambda),\lambda)}$ of $M(\cdot, x(\lambda), \lambda)$, we know that there exist $x(\lambda) \in \mathbb{H}$ and $u(\lambda) \in T(x(\lambda), \lambda)$ such that (4.4) holds if and only if

$$A(g(x(\lambda), \lambda)) - \rho N(u(\lambda), x(\lambda), \lambda) \in A(g(x(\lambda), \lambda)) + \rho M(g(x(\lambda), \lambda), x(\lambda), \lambda),$$

i.e.,

$$0 \in N(u(\lambda), x(\lambda), \lambda) + M(g(x(\lambda), \lambda), x(\lambda), \lambda).$$

It follows from the definition of $S(\lambda)$, we obtain that $x(\lambda) \in S(\lambda)$ is a solution of the problem (4.1) if and only if there exist $x(\lambda) \in \mathbb{H}$ and $u(\lambda) \in T(x(\lambda), \lambda)$ such that (4.4) holds. This completes the proof.

Remark 4.2. The equality (4.4) can be written as

$$x(\lambda) = x(\lambda) - g(x(\lambda), \lambda) + J_{\rho,A}^{M(\cdot,x(\lambda),\lambda)}(A(g(x(\lambda), \lambda)) - \rho N(u(\lambda), x(\lambda), \lambda)). \tag{4.5}$$

Theorem 4.1. Let $A : \mathbb{H} \to \mathbb{H}$ be r-strongly monotone and s-Lipschitz continuous, $T : \mathbb{H} \times \mathcal{L} \to C(\mathbb{H})$ be τ-$\hat{\mathbf{H}}$-Lipschitz continuous in the first variable, $g : \mathbb{H} \times \mathcal{L} \to \mathbb{H}$ is δ-strongly monotone and σ-Lipschitz continuous in the first variable, and $M : \mathbb{H} \times \mathbb{H} \times \mathcal{L} \to 2^{\mathbb{H}}$ be A-monotone with constant m in the first variable. Let $N : \mathbb{H} \times \mathbb{H} \times \mathcal{L} \to \mathbb{H}$ be (γ, α)-relaxed cocoercive with respect to g_1 and μ-Lipschitz continuous in the second variable, and let N be β-Lipschitz continuous in the first variable, respectively, where $g_1 : \mathbb{H} \times \mathcal{L} \to \mathbb{H}$ is defined by $g_1(x) = A \circ g(x, \lambda) = A(g(x, \lambda))$ for all $(x, \lambda) \in \mathbb{H} \times \mathcal{L}$. If

$$\|J_{\rho,A}^{M(\cdot,u,\lambda)}(w) - J_{\rho,A}^{M(\cdot,v,\lambda)}(w)\| \leq \eta \|u - v\| \tag{4.6}$$

for all $(u, v, \lambda) \in \mathbb{H} \times \mathbb{H} \times \mathcal{L}$ and there exists a constant $\rho > 0$ such that

$$\begin{cases} k = \eta + \sqrt{1 - 2\delta + \sigma^2} < 1, \quad s\sigma > r(1 - k), \\ h = \beta\tau + m(1 - k) < \mu, \\ \rho < \min\{\frac{r}{m}, \frac{r(1-k)}{h}\}, \\ |\rho - \frac{\gamma - \alpha\mu^2 - rh(1-k)}{\mu^2 - h^2}| < \frac{\sqrt{[r(1-h+hk) - \alpha\mu^2]^2 - (\mu^2 - h^2)[s^2\sigma^2 - r^2(1-k)^2]}}{\mu^2 - h^2}, \\ r(1 - h + hk) > \alpha\mu^2 + \sqrt{(\mu^2 - h^2)[s^2\sigma^2 - r^2(1 - k)^2]}, \end{cases} \tag{4.7}$$

then, for each $\lambda \in \mathcal{L}$, the following results hold:

(1) the solution set $S(\lambda)$ of the problem (4.1) is nonempty;

(2) $S(\lambda)$ is a closed subset in $\mathbb{H}$.

Proof. In the sequel, from (4.5), we first define a multivalued mapping $G : \mathbb{H} \times \mathcal{L} \to 2^{\mathbb{H}}$ by

$$G(x, \lambda) \quad = \quad \bigcup_{u \in T(x,\lambda)} [x - g(x, \lambda) + J_{\rho,A}^{M(\cdot,x,\lambda)}(A(g(x, \lambda)) - \rho N(u, x, \lambda))]$$

for all $(x, \lambda) \in \mathbb{H} \times \mathcal{L}$. For any $(x, \lambda) \in \mathbb{H} \times \mathcal{L}$, since $T(x, \lambda) \in C(\mathbb{H})$, g, A, N and $J_{\rho,A}^{M(\cdot,x,\lambda)}$ are continuous, we have $G(x, \lambda) \in C(\mathbb{H})$.

Now, for each fixed $\lambda \in \mathcal{L}$, we prove that $G(x, \lambda)$ is a multivalued contractive mapping. In fact, for any $(x, \lambda), (\hat{x}, \lambda) \in \mathbb{H} \times \mathcal{L}$ and any $a \in G(x, \lambda)$, there exists $u \in T(x, \lambda)$ such that

$$a = x - g(x, \lambda) + J_{\rho,A}^{M(\cdot,x,\lambda)}(A(g(x, \lambda)) - \rho N(u, x, \lambda)).$$

Note that $T(\hat{x}, \lambda) \in C(\mathbb{H})$, it follows from Nadler's result [15] that there exists $\hat{u} \in T(\hat{x}, \lambda)$ such that

$$\|u - \hat{u}\| \leq \hat{\mathbf{H}}(T(x, \lambda), T(\hat{x}, \lambda)). \tag{4.8}$$

Setting

$$b = \hat{x} - g(\hat{x}, \lambda) + J_{\rho,A}^{M(\cdot,\hat{x},\lambda)}(A(g(\hat{x}, \lambda)) - \rho N(\hat{u}, \hat{x}, \lambda)),$$

then we have $b \in G(\hat{x}, \lambda)$. It follows (4.6) and Lemma 3.1 that

$$\begin{aligned}
&\|a-b\| \\
&= \|x - g(x,\lambda) + J_{\rho,A}^{M(\cdot,x,\lambda)}(A(g(x,\lambda)) - \rho N(u,x,\lambda)) \\
&\quad -\{\hat{x} - g(\hat{x},\lambda) + J_{\rho,A}^{M(\cdot,\hat{x},\lambda)}(A(g(\hat{x},\lambda)) - \rho N(\hat{u},\hat{x},\lambda))\}\| \\
&\le \|x - \hat{x} - [g(x,\lambda) - g(\hat{x},\lambda)]\| \\
&\quad +\|J_{\rho,A}^{M(\cdot,x,\lambda)}(A(g(x,\lambda)) - \rho N(u,x,\lambda)) - J_{\rho,A}^{M(\cdot,\hat{x},\lambda)}(A(g(x,\lambda)) - \rho N(u,x,\lambda))\| \\
&\quad +\|J_{\rho,A}^{M(\cdot,\hat{x},\lambda)}(A(g(x,\lambda)) - \rho N(u,x,\lambda)) - J_{\rho,A}^{M(\cdot,\hat{x},\lambda)}(A(g(\hat{x},\lambda)) - \rho N(\hat{u},\hat{x},\lambda))\| \\
&\le \|x - \hat{x} - [g(x,\lambda) - g(\hat{x},\lambda)]\| + \eta\|x-\hat{x}\| \\
&\quad +\frac{1}{r-\rho m}\|A(g(x,\lambda)) - \rho N(u,x,\lambda) - (A(g(\hat{x},\lambda)) - \rho N(\hat{u},\hat{x},\lambda))\| \\
&\le \|x - \hat{x} - [g(x,\lambda) - g(\hat{x},\lambda)]\| + \eta\|x-\hat{x}\| + \frac{\rho}{r-\rho m}\|N(u,\hat{x},\lambda) - N(\hat{u},\hat{x},\lambda))\| \\
&\quad +\frac{1}{r-\rho m}\|A(g(x,\lambda)) - A(g(\hat{x},\lambda)) - \rho[N(u,x,\lambda) - N(u,\hat{x},\lambda)]\|
\end{aligned}$$

The δ-strongly monotonicity and σ-Lipschitz continuity of g in the first argument, the τ-$\hat{\mathbf{H}}$-Lipschitz continuity of T in the first argument, the (γ, α)-relaxed cocoercivity with respect to g_1 and μ-Lipschitz continuity of N in the second argument, the β-Lipschitz continuity of N in the first argument and the s-Lipschitz continuity of A, and the inequality (4.8) imply that

$$\begin{aligned}
&\|x - \hat{x} - [g(x,\lambda) - g(\hat{x},\lambda)]\| \\
&\le \sqrt{1-2\delta+\sigma^2}\|x-\hat{x}\|, \\
&\|N(u,\hat{x},\lambda) - N(\hat{u},\hat{x},\lambda))\| \\
&\le \beta\|u-\hat{u}\| \le \beta\hat{\mathbf{H}}(T(x,\lambda), T(\hat{x},\lambda)) \le \beta\tau\|x-\hat{x}\|, \\
&\|A(g(x,\lambda)) - A(g(\hat{x},\lambda)) - \rho[N(u,x,\lambda) - N(u,\hat{x},\lambda)]\|^2 \\
&\le \|A(g(x,\lambda)) - A(g(\hat{x},\lambda))\|^2 - 2\rho\langle N(u,x,\lambda) - N(u,\hat{x},\lambda), A(g(x,\lambda)) - A(g(\hat{x},\lambda))\rangle \\
&\quad +\rho^2\|N(u,x,\lambda) - N(u,\hat{x},\lambda)\|^2 \\
&\le \|A(g(x,\lambda)) - A(g(\hat{x},\lambda))\|^2 - 2\rho[-\alpha\|N(u,x,\lambda) - N(u,\hat{x},\lambda)\|^2 + \gamma\|x-\hat{x}\|^2] \\
&\quad +\rho^2\|N(u,x,\lambda) - N(u,\hat{x},\lambda)\|^2 \\
&\le (s^2\sigma^2 - 2\rho\gamma + \rho^2\mu^2 + 2\rho\alpha\mu^2)\|x-\hat{x}\|^2.
\end{aligned}$$

In light of above arguments, we infer

$$\|a-b\| \le \theta\|x-\hat{x}\|, \tag{4.9}$$

where

$$\theta = \eta + \sqrt{1-2\delta+\sigma^2} + \frac{\rho\beta\tau + \sqrt{s^2\sigma^2 - 2\rho\gamma + \rho^2\mu^2 + 2\rho\alpha\mu^2}}{r-\rho m}.$$

It follows from condition (4.7) that $\theta < 1$. Hence, from (4.9), we get

$$d(a, G(\hat{x},\lambda)) = \inf_{b\in G(\hat{x},\lambda)} \|a-b\| \le \theta\|x-\hat{x}\|.$$

Since $a \in G(x,\lambda)$ is arbitrary, we obtain

$$\sup_{a\in G(x,\lambda)} d(a, G(\hat{x},\lambda)) \le \theta\|x-\hat{x}\|.$$

By using same argument, we can prove

$$\sup_{b\in G(\hat{x},\lambda)} d(G(x,\lambda), b) \le \theta\|x-\hat{x}\|.$$

It follows from the definition of the Hausdorff metric $\hat{\mathbf{H}}$ on $C(\mathbb{H})$ that

$$\hat{\mathbf{H}}(G(x,\lambda), G(\hat{x},\lambda)) \le \theta\|x-\hat{x}\|$$

for all $(x,\hat{x},\lambda) \in \mathbb{H}\times\mathbb{H}\times\mathcal{L}$, i.e., $G(x,\lambda)$ is a multivalued contractive mapping, which is uniform with respect to $\lambda \in \mathcal{L}$. By a fixed point theorem of Nadler [15], for each $\lambda \in \mathcal{L}$, $G(x,\lambda)$ has a fixed point $x(\lambda) \in \mathbb{H}$, i.e., $x(\lambda) \in G(x(\lambda),\lambda)$. By the definition of G, we know that there exists $u(\lambda) \in T(x(\lambda),\lambda)$ such that (4.5) holds. Therefore, it follows from Lemma 4.1 that $x(\lambda) \in S(\lambda)$ is a solution of the problem (4.1) and so $S(\lambda) \neq \emptyset$ for all $\lambda \in \mathcal{L}$.

Next, we prove the conclusion (2). For each $\lambda \in \mathcal{L}$, let $\{x_n\} \subset S(\lambda)$ and $x_n \to x_0$ as $n \to \infty$. Then we have $x_n \in G(x_n,\lambda)$ for all $n = 1, 2, \cdots$. By the proof of conclusion (1), we have

$$\hat{\mathbf{H}}(G(x_n,\lambda), G(x_0,\lambda)) \le \theta\|x_n - x_0\|$$

for all $\lambda \in \mathcal{L}$. It follows that

$$\begin{aligned} d(x_0, G(x_0,\lambda)) &\le \|x_0 - x_n\| + d(x_n, G(x_n,\lambda)) + \hat{\mathbf{H}}(G(x_n,\lambda), G(x_0,\lambda)) \\ &\le (1+\theta)\|x_n - x_0\|. \end{aligned}$$

Hence we have $x_0 \in G(x_0,\lambda)$ and $x_0 \in S(\lambda)$. Therefore, $S(\lambda)$ is a nonempty closed subset of $\mathbb{H}$. This completes the proof.

Theorem 4.2. Under the hypotheses of Theorem 4.1, further, assume that

(i) for any $x \in \mathbb{H}$, $\lambda \to T(x,\lambda)$ is l_T-$\hat{\mathbf{H}}$-Lipschitz continuous (or continuous);

(ii) for any $u, v, z, \omega \in \mathbb{H}$, $\lambda \to N(u,v,\lambda)$, $\lambda \to g(u,\lambda)$ and $\lambda \to J_{\rho,A}^{M(\cdot,v,\lambda)}(w)$ both are Lipschitz continuous (or continuous) with Lipschitz constants l_N, l_g and l_J, respectively.

Then the solution set $S(\lambda)$ of the problem (4.1) is a Lipschitz continuous (or continuous) from $\mathcal{L}$ to $\mathbb{H}$.

Proof. From the hypotheses of the theorem and Theorem 4.1, for any $\lambda, \bar{\lambda} \in \mathcal{L}$, we know that $S(\lambda)$ and $S(\bar{\lambda})$ are both nonempty closed subset. By the proof of Theorem 4.1, $G(x,\lambda)$ and $G(x,\bar{\lambda})$ are both multivalued contractive mappings with the same contraction constant $\theta \in (0,1)$. It follows from Lemma 2.1 that

$$\hat{\mathbf{H}}(S(\lambda), S(\bar{\lambda})) \le \frac{1}{1-\theta}\sup_{x\in\mathbb{H}} \hat{\mathbf{H}}(G(x,\lambda), G(x,\bar{\lambda})). \tag{4.10}$$

Setting any $a \in G(x,\lambda)$, there exists $u(\lambda) \in T(x,\lambda)$ such that

$$a = x - g(x,\lambda) + J_{\rho,A}^{M(\cdot,x,\lambda)}(A(g(x,\lambda)) - \rho N(u(\lambda),x,\lambda)).$$

Since $T(x,\lambda), T(x,\bar{\lambda}) \in C(\mathbb{H})$, it follows from Nadler's result [15] that there exists $u(\bar{\lambda}) \in T(x,\bar{\lambda})$ such that

$$\|u(\lambda) - u(\bar{\lambda}))\| \le \hat{\mathbf{H}}(T(x,\lambda), T(x,\bar{\lambda})).$$

Let

$$b = x - g(x,\bar{\lambda}) + J_{\rho,A}^{M(\cdot,x,\bar{\lambda})}(A(g(x,\bar{\lambda})) - \rho N(u(\bar{\lambda}),x,\bar{\lambda})),$$

then $b \in G(x,\bar{\lambda})$. It follows the assumptions of $g, J_{\rho,A}^{M(\cdot,\cdot,\cdot)}, N, A$ and T that

$$\begin{aligned}
&\|a-b\| \\
&= \|x - g(x,\lambda) + J_{\rho,A}^{M(\cdot,x,\lambda)}(A(g(x,\lambda)) - \rho N(u(\lambda),x,\lambda)) \\
&\quad -\{x - g(x,\bar{\lambda}) + J_{\rho,A}^{M(\cdot,x,\bar{\lambda})}(A(g(x,\bar{\lambda})) - \rho N(u(\bar{\lambda}),x,\bar{\lambda})\}\| \\
&\le \|g(x,\lambda) - g(x,\bar{\lambda})\| \\
&\quad +\|J_{\rho,A}^{M(\cdot,x,\lambda)}(A(g(x,\lambda)) - \rho N(u(\lambda),x,\lambda)) - J_{\rho,A}^{M(\cdot,x,\bar{\lambda})}(A(g(x,\lambda)) - \rho N(u(\lambda),x,\lambda))\| \\
&\quad +\|J_{\rho,A}^{M(\cdot,x,\bar{\lambda})}(A(g(x,\lambda)) - \rho N(u(\lambda),x,\lambda)) - J_{\rho,A}^{M(\cdot,x,\bar{\lambda})}(A(g(x,\bar{\lambda})) - \rho N(u(\bar{\lambda}),x,\bar{\lambda}))\| \\
&\le l_g\|\lambda-\bar{\lambda}\| + l_J\|\lambda-\bar{\lambda}\| \\
&\quad +\frac{1}{r-\rho m}\|A(g(x,\lambda)) - \rho N(u(\lambda),x,\lambda) - (A(g(x,\bar{\lambda})) - \rho N(u(\bar{\lambda}),x,\bar{\lambda}))\| \\
&\le (l_g+l_J)\|\lambda-\bar{\lambda}\| + \frac{\rho}{r-\rho m}\|N(u(\lambda),x,\lambda) - N(u(\bar{\lambda}),x,\lambda)\| \\
&\quad +\frac{\rho}{r-\rho m}\|N(u(\bar{\lambda}),x,\lambda) - N(u(\bar{\lambda}),x,\bar{\lambda})\| + \frac{1}{r-\rho m}\|A(g(x,\lambda)) - A(g(x,\bar{\lambda}))\| \\
&\le (l_g+l_J)\|\lambda-\bar{\lambda}\| + \frac{\rho\beta}{r-\rho m}\|u(\lambda) - u(\bar{\lambda})\| \\
&\quad +\frac{\rho l_N}{r-\rho m}\|\lambda-\bar{\lambda}\| + \frac{s}{r-\rho m}\|g(x,\lambda) - g(x,\bar{\lambda})\| \\
&\le (l_g + l_J + \frac{\rho l_N}{r-\rho m})\|\lambda-\bar{\lambda}\| + \frac{\rho\beta}{r-\rho m}\hat{\mathbf{H}}(T(x,\lambda), T(x,\bar{\lambda})) + \frac{s l_g}{r-\rho m}\|\lambda-\bar{\lambda}\| \\
&\le \Gamma\|\lambda-\bar{\lambda}\|,
\end{aligned}$$

where

$$\Gamma = l_g + l_J + \frac{\rho l_N + \rho\beta l_T + s l_g}{r-\rho m}.$$

Hence we obtain

$$\sup_{a\in G(x,\lambda)} d(a, G(x,\bar{\lambda})) \le \Gamma\|\lambda-\bar{\lambda}\|.$$

By using a similar argument as above, we get

$$\sup_{b\in G(x,\bar{\lambda})} d(G(x,\lambda), b) \le \Gamma\|\lambda-\bar{\lambda}\|.$$

It follows that

$$\hat{\mathbf{H}}(G(x,\lambda), G(x,\bar{\lambda})) \leq \Gamma\|\lambda - \bar{\lambda}\|$$

for all $(x, \lambda, \bar{\lambda}) \in \mathbb{H} \times \mathcal{L} \times \mathcal{L}$. Thus (4.10) implies

$$\hat{\mathbf{H}}(S(\lambda), S(\bar{\lambda})) \leq \frac{\Gamma}{1-\theta}\|\lambda - \bar{\lambda}\|.$$

This proves that $S(\lambda)$ is Lipschitz continuous in $\lambda \in \mathcal{L}$. If, each mapping in conditions (i) and (ii) is assumed to be continuous in $\lambda \in \mathcal{L}$, then, by similar argument as above, we can show that $S(\lambda)$ is continuous in $\lambda \in \mathcal{L}$. This completes the proof.

Remark 4.3. In Theorems 4.1 and 4.2, if $N : \mathbb{H}\times\mathbb{H}\times\mathcal{L} \to \mathbb{H}$ is α-strongly monotone in the second variable, i.e., when $\gamma = 0$ in (4.7), then we can obtain the corresponding results. Theorems 4.1 and 4.2 improve and generale the known results in [1, 3, 7, 8, 14, 20, 21].

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Volume 8,Number 2 April 2006
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

A quarterly international publication of Eudoxus Press, LLC

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Analysis of Support Vector Machine Classification

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Abstract. This paper studies support vector machine classification algorithms. We analyze the 1-norm soft margin classifier. The consistency is considered in two forms. When the regularization error decays to zero, the Bayes-risk consistency is proved and learning rates are derived by means of techniques of uniform convergence. The main difficulty we overcome here is to bound the offset. For the consistency with hypothesis space, we present a counterexample.

Key words: Support vector machine classification, misclassification error, Bayes-risk consistency, consistency with hypothesis space, Mercer kernel, regularization error.

1 Introduction

Support vector machines (SVMs) form an important part of learning theory. They are very efficient for many applications in science and engineering, especially for classification problems (pattern recognition).

Motivated by classification algorithms for separating data of Fisher [11], Rosenblatt [18], and Vapnik [24], the **support vector machines** were introduced by Boser, Guyon and Vapnik [4] with polynomials kernels, and by Cortes and Vapnik [6] with general kernels. Since then there has been a rich study of SVM: applications to various practical problems; many variances of the original model; and some theoretical investigation. Some convergence analysis has been done recently [23, 29]. In this paper we investigate the original model, SVM 1-norm soft margin classifier, probably the most important SVM classification algorithm.

Let (X, d) be a compact metric space and $Y = \{1, -1\}$. A binary **classifier** $f : X \to \{1, -1\}$ is a function from X to Y which divides the input space X into two classes.

Let ρ be a probability distribution on $Z := X \times Y$ and $(\mathcal{X}, \mathcal{Y})$ be the corresponding random variable. Then the **misclassification error** for a classifier $f : X \to Y$ is defined to be the probability of the event $f(\mathcal{X}) \neq \mathcal{Y}$:

$$\mathcal{R}(f) = \text{Prob}\{f(\mathcal{X}) \neq \mathcal{Y}\} = \int_X P(\mathcal{Y} \neq f(x)|x) d\rho_X(x). \tag{1}$$

Here ρ_X is the marginal distribution of ρ on X and $\rho(\cdot|x) = P(\cdot|x)$ is the conditional probability measure given $\mathcal{X} = x$.

If we define the regression function of ρ as

$$f_\rho(x) = \int_Y y d\rho(y|x) = P(\mathcal{Y} = 1|x) - P(\mathcal{Y} = -1|x), \qquad x \in X, \tag{2}$$

then one can see (e.g. [9]) that the best classifier, called the **Bayes rule**, is given by $f_c := \text{sgn}(f_\rho)$, the sign of the regression function. Here for a function $f : X \to \mathbb{R}$, the sign function is defined as $\text{sgn}(f)(x) = 1$ if $f(x) \geq 0$, $\text{sgn}(f)(x) = -1$ if $f(x) < 0$. That means,

$$f_c(x) = \text{sgn}(f_\rho)(x) = \begin{cases} 1, & \text{if } P(\mathcal{Y} = 1|x) \geq P(\mathcal{Y} = -1|x), \\ -1, & \text{if } P(\mathcal{Y} = 1|x) < P(\mathcal{Y} = -1|x). \end{cases} \tag{3}$$

As ρ is unknown, the best classifier f_c cannot be found directly. What we have in hand is a set of random samples $\mathbf{z} = (z_i)_{i=1}^m = (x_i, y_i)_{i=1}^m$. Throughout this paper, as usual [25, 7], we assume that $\{z_i\}_{i=1}^m$ are independent and identically distributed drawers according to a Borel probability distribution ρ. A classification algorithm is a map from the set of samples to a set of classifiers $\mathcal{H}$:

$$\mathcal{A} : \bigcup_{i=1}^{\infty} Z^m \longrightarrow \mathcal{H},$$

which produces for every $\mathbf{z}$ a classifier $\mathcal{A}(\mathbf{z})$. The set $\mathcal{H}$ is called the **hypothesis space**.

Definition 1. A classification algorithm $\mathcal{A}$ is said to be **Bayes-risk consistent** (with ρ) if $\mathcal{R}(\mathcal{A}(\mathbf{z}))$ converges to $\mathcal{R}(f_c)$ in probability, i.e., for every $\varepsilon > 0$,

$$\lim_{m \to \infty} \text{Prob}\left\{\mathbf{z} \in Z^m : \mathcal{R}(\mathcal{A}(\mathbf{z})) - \mathcal{R}(f_c) > \varepsilon\right\} = 0.$$

It is said to be **consistent with hypothesis space** $\mathcal{H}$ (and ρ) if $\mathcal{R}(\mathcal{A}(\mathbf{z}))$ converges to $\inf_{f \in \mathcal{H}} \mathcal{R}(f)$ in probability, i.e., for every $\varepsilon > 0$,

$$\lim_{m \to \infty} \text{Prob}\left\{\mathbf{z} \in Z^m : \mathcal{R}(\mathcal{A}(\mathbf{z})) - \inf_{f \in \mathcal{H}} \mathcal{R}(f) > \varepsilon\right\} = 0.$$

It is easy to see that these two concepts coincide if the Bayes rule can be well approximated by the hypothesis space $\mathcal{H}$ in the sense that

$$\inf_{f\in\mathcal{H}} \mathcal{R}(f) = \mathcal{R}(f_c). \tag{4}$$

When (4) does not hold, the Bayes-risk consistency cannot hold no matter which algorithm is used, since $\mathcal{A}(\mathbf{z}) \in \mathcal{H}$. But the consistency with hypothesis space may still be true. So consistency with hypothesis space concerns the algorithm only, but Bayes-risk consistency also concerns the approximation power of the hypothesis space.

The 1-norm soft margin SVM is a classification algorithm depending on a reproducing kernel Hilbert space associated with a Mercer kernel.

Let $K : X \times X \to \mathbb{R}$ be continuous, symmetric and positive semidefinite, *i.e.*, for any finite set of distinct points $\{x_1, \cdots, x_\ell\} \subset X$, the matrix $(K(x_i, x_j))_{i,j=1}^{\ell}$ is positive semidefinite. Such a function is called a **Mercer kernel**.

The **Reproducing Kernel Hilbert Space** (RKHS) $\mathcal{H}_K$ associated with the kernel K is defined (see [1]) to be the completion of the linear span of the set of functions $\{K_x := K(x,\cdot) : x \in X\}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_K} = \langle\cdot,\cdot\rangle_K$ satisfying

$$\left\|\sum_{i=1}^m c_i K_{x_i}\right\|_K^2 = \left\langle \sum_{i=1}^m c_i K_{x_i}, \sum_{i=1}^m c_i K_{x_i} \right\rangle_K = \sum_{i,j=1}^m c_i K(x_i, x_j) c_j.$$

The reproducing property is given by

$$< K_x, g >_K = g(x), \qquad \forall x \in X, g \in \mathcal{H}_K. \tag{5}$$

Denote $C(X)$ as the space of continuous functions on X with the norm $\|\cdot\|_\infty$. Then (5) leads to

$$\|g\|_\infty \le \kappa \|g\|_K, \qquad \forall g \in \mathcal{H}_K, \tag{6}$$

where $\kappa = \sup_{x\in X} \sqrt{K(x,x)}$. This means $\mathcal{H}_K$ can be embedded into $C(X)$.

Define $\overline{\mathcal{H}}_K = \mathcal{H}_K + \mathbb{R}$. For a function $f(x) = f_1(x) + b$ with $f_1 \in \mathcal{H}_K$ and $b \in \mathbb{R}$, we denote $f^* = f_1$ and $b_f = b \in \mathbb{R}$. The constant term b is called the **offset**.

The **SVM 1-norm soft margin classifier** associated with the kernel K is defined as $\operatorname{sgn}(f_{\mathbf{z}})$, where $f_{\mathbf{z}}$ is a minimizer of the following optimization problem:

$$\begin{aligned} f_{\mathbf{z}} = \arg\min_{f\in\overline{\mathcal{H}}_K} & \left\{ \frac{1}{2}\|f^*\|_K^2 + \frac{C}{m}\sum_{i=1}^m \xi_i, \right\} \\ \text{subject to} \quad & y_i f(x_i) \ge 1 - \xi_i, \xi_i \ge 0, \text{ for } i = 1, \ldots, m. \end{aligned} \tag{7}$$

Here $C = C_m > 0$ is a trade-off parameter which may depend on m. The original model used the linear kernel $K(x, y) = x \cdot y$ for $X \subset \mathbb{R}^n$.

The performance of the 1-norm SVM on strictly separable distributions has been well understood in the literature. We say that the distribution ρ is strictly separable by $\overline{\mathcal{H}}_K$ with margin $\gamma > 0$ if there is a function $f_\gamma \in \overline{\mathcal{H}}_K$ such that $y f_\gamma(x) \geq \gamma$ almost surely. Margin-based analysis shows the learning rate $\mathcal{R}\big(\mathrm{sgn}(f_{\mathbf{z}})\big) - \mathcal{R}(f_c)$ for separable distributions and $C = \infty$ is $O(\frac{1}{m})$, see [25, 19, 7].

For nonseparable distributions, some data dependent upper bounds have also been given, mainly based on the VC (or V_γ) theory. These bounds are posteriori. Even they cannot be used to verify the Bayes-risk consistency.

When K is a universal kernel in the sense that $\mathcal{H}_K$ is dense in $C(X)$, the Bayes-risk consistency for all distributions was confirmed in [23, 29]. But this result on the consistency does not cover the most important case of polynomial kernels.

Observe that in (7), ξ_i can be found:

$$\xi_i = \max\{0, 1 - y_i f(x_i)\} = (1 - y_i f(x_i))_+,$$

where $(t)_+ := \max\{0, t\}$. Thus, if we define the loss function V as $V(y, f(x)) = (1 - y f(x))_+$, then the scheme (7) can be written as

$$f_{\mathbf{z}} = \arg\min_{f \in \overline{\mathcal{H}}_K} \Big\{ \mathcal{E}_{\mathbf{z}}(f) + \frac{1}{2C} \|f^*\|_K^2 \Big\}, \tag{8}$$

where

$$\mathcal{E}_{\mathbf{z}}(f) = \frac{1}{m} \sum_{i=1}^{m} V(y_i, f(x_i))$$

is the **empirical error** associated with the loss V. This is a regularization scheme [10].

Define the **generalization error**

$$\mathcal{E}(f) = \int_Z V(y, f(x))\, d\rho(x, y) = E(V(y, f(x)).$$

Then f_c is a minimizer of $\mathcal{E}(f)$ [27]. The empirical risk minimization (ERM) technique for the uniform convergence tells us that $\mathcal{E}(f_{\mathbf{z}}) \to \inf_{f \in \overline{\mathcal{H}}_K} \mathcal{E}(f)$ as $m, C \to \infty$. But we are interested in the excess misclassification error $\mathcal{R}(f_{\mathbf{z}}) - \inf_{f \in \overline{\mathcal{H}}_K} \mathcal{R}(f)$ for classification algorithms. A bridge between $\mathcal{R}(f)$ and $\mathcal{E}(f)$ was established by Zhang [29]: for any $f : X \to \mathbb{R}$,

$$\mathcal{R}(f) - \mathcal{R}(f_c) \leq \mathcal{E}(f) - \mathcal{E}(f_c). \tag{9}$$

Thus, when $\inf_{f\in\overline{\mathcal{H}}_K}\mathcal{E}(f)=\mathcal{E}(f_c)$, the consistency and error analysis for (7) can be given, as done in [29] by a leave-one-out analysis. But no offset term is involved in Zhang's analysis.

In this paper, we shall further investigate the SVM 1-norm soft margin classifier. First, we shall do the error analysis in the regularization framework in Section 2. It is different from the known methods, and can handle the case when $\inf_{f\in\overline{\mathcal{H}}_K}\mathcal{E}(f)\neq\mathcal{E}(f_c)$. In our analysis, we will overcome the difficulty caused by the offset in Section 3, which will be essential to determine the hypothesis space for the ERM analysis in Section 4. Also, our analysis will give a strategy to choose the trade-off parameter so that the convergence in probability or the almost sure convergence holds. Secondly, we study the consistency with hypothesis space which has not been studied in the literature. A counterexample for the divergence will be presented in Section 5.

2 Error Analysis

In this section study the convergence in the regularization framework.

Let

$$f_{K,C}=\arg\min_{f\in\overline{\mathcal{H}}_K}\Big\{\mathcal{E}(f)+\frac{1}{2C}\|f^*\|_K^2\Big\}. \tag{10}$$

We have the following proposition.

Proposition 1. *For every $C>0$, there holds*

$$\mathcal{R}(f_{\mathbf{z}})-\mathcal{R}(f_c)\leq\mathcal{S}(m,C)+\mathcal{D}(C),$$

where

$$\mathcal{S}(m,C):=\Big\{\mathcal{E}(f_{\mathbf{z}})-\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}})\Big\}+\Big\{\mathcal{E}_{\mathbf{z}}\big(f_{K,C}\big)-\mathcal{E}\big(f_{K,C}\big)\Big\}. \tag{11}$$

and

$$\mathcal{D}(C):=\inf_{f\in\overline{\mathcal{H}}_K}\Big\{\mathcal{E}(f)-\mathcal{E}(f_c)+\frac{1}{2C}\|f^*\|_K^2\Big\}.$$

Proof. Write

$$\begin{aligned}\mathcal{E}(f_{\mathbf{z}})-\mathcal{E}(f_c)=&\Big\{\mathcal{E}(f_{\mathbf{z}})-\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}})\Big\}+\Big\{\Big(\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}})+\frac{1}{2C}\|f_{\mathbf{z}}^*\|_K^2\Big)\\&-\Big(\mathcal{E}_{\mathbf{z}}(f_{K,C})+\frac{1}{2C}\|f_{K,C}^*\|_K^2\Big)\Big\}+\Big\{\mathcal{E}_{\mathbf{z}}(f_{K,C})-\mathcal{E}(f_{K,C})\Big\}\\&+\Big\{\mathcal{E}(f_{K,C})-\mathcal{E}(f_c)+\frac{1}{2C}\|f_{K,C}^*\|_K^2\Big\}-\frac{1}{2C}\|f_{\mathbf{z}}^*\|_K^2.\end{aligned}$$

By the definition of $f_{\mathbf{z}}$, the second term is ≤ 0. By the definition of $f_{K,C}$ we see the fourth term is just $\mathcal{D}(C)$. Hence $\mathcal{E}(f_{\mathbf{z}})-\mathcal{E}(f_c)$ can be bounded by $\mathcal{S}(m,C)+\mathcal{D}(C)$. This together with (9) finishes the proof. $\square$

The first term $\mathcal{S}(m,C)$ is called the **sample error** and the second term $\mathcal{D}(C)$ is called the **regularization error** [21].

To bound the sample error, since $f_{\mathbf{z}}$ runs over a set of functions as $\mathbf{z}$ changes, we will use the concentration inequalities concerning the uniform convergence. This technique has been well understood in learning theory, e.g. [24, 2, 9, 8]. To use this technique, we need the concept of covering numbers to measure the capacity of the hypothesis space.

Definition 2. For a compact set $\mathcal{F}$ in a metric space and $\varepsilon > 0$, the **covering number** $\mathcal{N}(\mathcal{F},\varepsilon)$ is defined to be the minimal integer $\ell \in \mathbb{N}$ such that there exist ℓ balls with radius ε covering $\mathcal{F}$.

Note that various covering numbers measured by the empirical metric are also used in the literature, see e.g. [2]. For their comparisons, see [2, 16].

Let $\mathcal{B}_R = \{f \in \mathcal{H}_K : \|f\|_K \leq R\}$ be the ball of $\mathcal{H}_K$ with radius $R > 0$ centered at 0. Denote

$$\mathcal{N}(\varepsilon) = \mathcal{N}(\mathcal{B}_1, \varepsilon), \qquad \varepsilon > 0.$$

The covering number $\mathcal{N}(\varepsilon)$ has been extensively studied, see e.g. [3, 28, 30, 31].

Proposition 2. *For every $C > 0$ and $\varepsilon > 0$, there holds*

$$\mathrm{Prob}\{\mathcal{S}(m,C) > \varepsilon\} \leq \exp\left\{-\frac{3m\varepsilon^2}{256B}\right\} + \left(\frac{32B}{\varepsilon}+1\right)\mathcal{N}\left(\frac{\varepsilon}{32\sqrt{2C}}\right)\exp\left\{-\frac{3m\varepsilon^2}{2^{14}B}\right\}$$

where $B = B_C := 1 + \kappa\sqrt{2C}$.

The proof of Proposition 2 will be given in Section 4.

By Proposition 1 and Proposition 2 we immediately obtain that for $0 < \varepsilon < 1$,

$$\mathrm{Prob}\left\{\mathcal{R}(\mathrm{sgn}(f_{\mathbf{z}}))-\mathcal{R}(f_c) > \varepsilon+\mathcal{D}(C)\right\} \leq \frac{34B}{\varepsilon}\mathcal{N}\left(\frac{\varepsilon}{32\sqrt{2C}}\right)\exp\left\{-\frac{3m\varepsilon^2}{2^{14}B}\right\}. \quad (12)$$

If the regularization error $\mathcal{D}(C)$ decays to 0 as $C \to \infty$, then the consistency holds by choosing the trade-off parameter properly.

Corollary 1. *Assume* $\lim_{C\to\infty} \mathcal{D}(C) = 0$. *Choose the parameter* $C = C_m$ *to satisfy*

$$C_m \to \infty \qquad \textit{and} \qquad \frac{\sqrt{C_m}}{m} \log\Big(\mathcal{N}\Big(\frac{1}{\sqrt{C_m}}\Big)\Big) \to 0 \tag{13}$$

as $m \to \infty$, *then*

$$\lim_{m\to\infty} \mathrm{Prob}\Big\{\mathcal{R}(\mathrm{sgn}(f_{\mathbf{z}})) - \mathcal{R}(f_c) > \varepsilon\Big\} = 0.$$

If, in addition, $C_m \le m^\alpha$ *for some* $\alpha < 2$, *then the almost sure convergence holds.*

Proof. The first assertion follows directly from (12).

To show the almost sure convergence, we apply the Borel-Cantelli Theorem (see e.g. [25]), because the right hand side of (12) decays exponentially fast when $C \le m^\alpha$ with $\alpha < 2$. □

Corollary 1 gives a strategy of choosing the trade-off parameter for the consistency. To get better learning rates, the parameter needs to trade-off the sample error and the regularization error.

Let us derive the error bound and see how to choose the constant C correspondingly. For every $0 < \delta < 1$, set

$$\left(\frac{16\sqrt{2}B}{\varepsilon} + 1\right) \mathcal{N}\left(\frac{\varepsilon}{16\sqrt{2}R}\right) \exp\left\{-\frac{m\varepsilon^2}{2048}\right\} = \delta. \tag{14}$$

This equation has a unique solution $\varepsilon(\delta, m, C)$ since the left hand side is strictly decreasing as a function of $\varepsilon \in [0, +\infty)$. Once the information of the covering number is available (which can be obtained from the kernel K), the sample error bound $\varepsilon(\delta, m, C)$ can be explicitly estimated. Thus, with confidence at least $1 - \delta$ the excess misclassification error can be bounded as

$$\mathcal{R}(f_{\mathbf{z}}) - \mathcal{R}(f_c) \le \varepsilon(\delta, m, C) + \mathcal{D}(C). \tag{15}$$

We need to bound the the regularization error. Since V is Lipschitz:

$$|V(y, f(x)) - V(y, g(x))| \le |f(x) - g(x)|,$$

we have the following proposition.

Proposition 3. *For every* $C > 0$, *there holds*

$$\mathcal{D}(C) \le \inf_{f\in\overline{\mathcal{H}}_K} \Big\{\|f - f_c\|_{L^1_{\rho_X}} + \frac{1}{2C}\|f^*\|_K^2\Big\}.$$

Proposition 3 tells that the regularization error can be estimated by the approximation in a weighted L^1 space. A direct corollary is for those distributions ρ such that f_c lies in the closure of $\overline{\mathcal{H}}_K$ in $L^1_{\rho_X}$. For such a distribution, the K-functional in Proposition 3, and hence the generalization error, tend to 0 as $C \to \infty$. This together with Corollary 1 gives the consistency of the 1-norm soft margin SVM for these distributions. In particular, if K is a universal kernel, for any Borel probability measure ρ, the consistency holds, since $\mathcal{H}_K$ is dense in $C(X)$ and hence also dense in $L^1_{\rho_X}$.

Define

$$I_1(g,R) = \inf_{f\in\overline{\mathcal{H}}_K, \|f^*\|_K\le R}\big\{\|g-f\|_{L^1_{\rho_X}}\big\}.$$

Then there holds

$$\inf_{f\in\overline{\mathcal{H}}_K}\left\{\|f-f_c\|_{L^1_{\rho_X}} + \frac{1}{2C}\|f^*\|_K^2\right\} \le \inf_{R>0}\left\{I_1(f_c,R)+\frac{R^2}{2C}\right\}. \tag{16}$$

The functional $I_1(g,R)$ is closely related to the approximation error studied by Smale and Zhou in [20] (see also [15] for related discussion):

$$I_2(g,R) = \inf_{f\in\mathcal{H}_K, \|f\|_K\le R}\big\{\|g-f\|_{L^2_{\rho_X}}\big\}.$$

In fact, as $\|f\|_{L^1_{\rho_X}} \le \|f\|_{L^2_{\rho_X}}$, with the choice $b_f = 0$ we obtain

$$I_1(f_c,R) \le I_2(f_c,R), \qquad \forall R>0. \tag{17}$$

The following example shows how to get learning rates from the above analysis.

Example 1. *Let $X=[0,1]^n$, $\sigma>0, 0<s, n/2$ and K be the Gaussian kernel*

$$K(x,y) = \exp\Big\{-\frac{|x-y|^2}{\sigma^2}\Big\}.$$

Assume $\frac{d\rho_X(x)}{dx}\le C_0$ for almost every $x\in X$. If f_c is the restriction of some function $\widetilde{f}_c \in H^s(\mathbb{R}^n)$ onto X, then with probability at least $1-\delta$ there holds

$$\mathcal{E}(f_{\mathbf{z}}) - \mathcal{E}(f_c) \le \mathcal{O}\left(C^{1/4}\frac{(\log m)^{n+1}}{m^{1/2}}\right) + \mathcal{O}\Big((\log C)^{-s/4}\Big).$$

This yields the learning rate $\mathcal{O}((\log m)^{-s/4})$ by choosing $C=m^\alpha$ with $0<\alpha<2$.

Proof. First we estimate the sample error. A result in [30] tells that the covering number can be bounded as

$$\log \mathcal{N}(\varepsilon) \le c\left(\log\frac{1}{\varepsilon}\right)^{n+1}. \tag{18}$$

By solving (14), with confidence at least $1-\delta$ we have

$$\mathcal{S}(m, C) \leq \varepsilon(\delta, m, C) = \mathcal{O}\left(C^{1/2}\frac{(\log m)^{n+1}}{m^{1/2}}\right).$$

Second, we estimate $\mathcal{D}(C)$. By the approximation error estimate given for $d\rho_X = dx$ in [20] (see also [32]) we see that

$$I_2(f_c, R) \leq C_0 C_s \big(\log R\big)^{-s/4}, \qquad \forall R > C_s,$$

where C_s is a constant depending on s, σ, n and $\|\widetilde{f}_c\|_{H^s} + \|\widetilde{f}_c\|_{L^2}$. This in connection with (17) implies that $I_1(f_c, R)$ has the same order. Choose R to be $\sqrt{C}\big(\log C\big)^{-s/4-1}$. Then by (16) and Proposition 3 we obtain

$$\mathcal{D}(C) \leq \mathcal{O}\left(\big(\log C\big)^{-s/4}\right).$$

Combing the estimates for the sample error and generalization error, our statement follows. $\square$

Further improvements of our analysis for the Bayes-risk consistency are possible. Better bounds for the regularization error estimates may be obtained by refining the approximation in $L^1_{\rho_X}$. The sample error bound can also be improved if some priori knowledge is known, since we only consider the worst case in our analysis.

3 Bounding the Offset

Regularization schemes without offset are much easier to analyze, see [5, 29]. When the offset is involved, the analysis becomes more difficult. This difficulty can be seen from the stability analysis [5]. The 1-norm soft margin SVM without offset is uniformly stable, as shown in [5]. However, the 1-norm soft margin SVM with offset is not uniformly stable. To see this, we choose $x_0 \in X$ and samples $\mathbf{z} = \{(x_0, y_i)\}_{i=1}^{2n+1}$ with $y_i = 1$ for $i = 1, \ldots, n+1$, and $y_i = -1$ for $i = n+2, \ldots, 2n+1$. Take $\mathbf{z}'$ to be the same as $\mathbf{z}$ except that (x_0, y_{n+1}) is replaced by $(x_0, -1)$. As x_i's are identical, one can see from the definition (8) that $f^*_{\mathbf{z}} = 0$ since $\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) = \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}(x_0))$. It follows that $f_{\mathbf{z}} = 1$ while $f_{\mathbf{z}'} = -1$. Thus, $|f_{\mathbf{z}} - f_{\mathbf{z}'}| = 2$ which does not converge to zero as n tends to infinity. Thus we cannot use the stability analysis to illustrate the statistical performance of the 1-norm soft margin SVM.

In order to bound the sample error (11), we use the uniform convergence. This means we allow $f_{\mathbf{z}}$ to run over a hypothesis space for which the capacity can be measured. To this end, we need bound the offset involved in $f_{\mathbf{z}}$ and $f_{K,C}$. The difficulty of bounding the offset has been realized in the literature (e.g. [23]). We shall overcome this difficulty by means of the special feature of the loss function V. By $x \in (X, \rho_X)$ we mean that x lies in the support of the measure ρ_X on X.

Lemma 1. *For any $C > 0$, $m \in \mathbb{N}$ and $\mathbf{z} \in Z^m$, there is a minimizer of (8) satisfying*

$$\min_{1 \le i \le m} |f_{\mathbf{z}}(x_i)| \le 1 \tag{19}$$

and a minimizer of (10) satisfying

$$\inf_{x \in (X, \rho_X)} \left| f_{K,C}(x) \right| \le 1. \tag{20}$$

Proof. Suppose a minimizer of (8) $f_{\mathbf{z}}$ satisfies

$$r := \min_{1 \le i \le m} |f_{\mathbf{z}}(x_i)| = f_{\mathbf{z}}(x_{i_0}) > 1.$$

Then for each i, either $y_i f_{\mathbf{z}}(x_i) \ge r > 1$ or $y_i f_{\mathbf{z}}(x_i) \le -r < -1$. For $\varepsilon \in \{1, -1\}$, set

$$I_\varepsilon := \{i \in \{1, \ldots, m\} : \, y_i = \varepsilon, y_i f_{\mathbf{z}}(x_i) \le -r\}.$$

Denote $\# I_\varepsilon$ the number of elements in the set I_ε.

If $\# I_1 = \# I_{-1}$ (possibly zero), then the function $\tilde{f}_{\mathbf{z}} := f_{\mathbf{z}} - d$ with $d = (r - 1)\mathrm{sgn} f_{\mathbf{z}}(x_{i_0})$ satisfies $|\tilde{f}_{\mathbf{z}}(x_{i_0})| = 1$ and $|\tilde{f}_{\mathbf{z}}(x_i)| \ge 1$ for each i. Hence

$$\mathcal{E}_{\mathbf{z}}(\tilde{f}_{\mathbf{z}}) = \sum_{i \in I_1 \cup I_{-1}} \big(1 - y_i f_{\mathbf{z}}(x_i) + y_i d\big) = \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) + \sum_{i \in I_1} d - \sum_{i \in I_{-1}} d = \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}).$$

This means $\tilde{f}_{\mathbf{z}}$ is a minimizer of (8) satisfying (19).

If for some $\varepsilon \in \{1, -1\}$, $\# I_\varepsilon > \# I_{-\varepsilon}$, then we see that $y_i \varepsilon = 1$ for $i \in I_\varepsilon$ and $y_i \varepsilon = -1$ for $i \in I_{-\varepsilon}$. Hence the function $\tilde{f}_{\mathbf{z}} := f_{\mathbf{z}} + \varepsilon(r-1)$ satisfies

$$\begin{aligned}
\mathcal{E}_{\mathbf{z}}(\tilde{f}_{\mathbf{z}}) &= \sum_{i \in I_1 \cup I_{-1}} \big(1 - y_i f_{\mathbf{z}}(x_i) - y_i \varepsilon (r-1)\big) \\
&= \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) - (r-1)\#(I_\varepsilon) + (r-1)\#(I_{-\varepsilon}) < \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}).
\end{aligned}$$

This is a contradiction to the definition of $f_{\mathbf{z}}$.

Therefore, (19) is always true for a minimizer of (8).

In the same way, suppose $r := \inf_{x\in(X,\rho_X)} |f_{K,C}(x)| > 1$ for a minimizer $f_{K,C}$ of (10). Consider the sets

$$I_\varepsilon := \{x \in (X,\rho_X) : P(\mathcal{Y}=\varepsilon|x) > 0, \, \varepsilon f_{K,C}(x) \le -r\}, \qquad \varepsilon = 1, -1.$$

Then $I_1 \cap I_{-1} = \emptyset$ and for $\varepsilon \in \{1,-1\}$,

$$\begin{aligned}\mathcal{E}(f_{K,C}+\varepsilon(r-1)) &= \int_{I_1} \big(1 - f_{K,C}(x) - \varepsilon(r-1)\big) P(\mathcal{Y}=1|x)d\rho_X \\ &\quad + \int_{I_{-1}} \big(1 + f_{K,C}(x) + \varepsilon(r-1)\big) P(\mathcal{Y}=-1|x)d\rho_X \\ &= \mathcal{E}(f_{K,C}) - \varepsilon(r-1)\left\{\int_{I_1} P(\mathcal{Y}=1|x)d\rho_X - \int_{I_{-1}} P(\mathcal{Y}=-1|x)d\rho_X\right\}.\end{aligned}$$

If

$$\int_{I_1} P(\mathcal{Y}=1|x)d\rho_X = \int_{I_{-1}} P(\mathcal{Y}=-1|x)d\rho_X,$$

we can define

$$\tilde{f}_{K,C} = \begin{cases} f_{K,C} + r - 1, & \text{when } \sup\{f_{K,C}(x) : x \in (X,\rho_X), f_{K,C}(x) < 0\} = -r, \\ f_{K,C} - r + 1, & \text{when } \inf\{f_{K,C}(x) : x \in (X,\rho_X), f_{K,C}(x) > 0\} = r. \end{cases}$$

Then $\mathcal{E}(\tilde{f}_{K,C}) = \mathcal{E}(f_{K,C})$ and hence $\tilde{f}_{K,C}$ is a minimizer of (10) satisfying (20).

If for some $\varepsilon \in \{1,-1\}$,

$$\int_{I_\varepsilon} P(\mathcal{Y}=\varepsilon|x)d\rho_X > \int_{I_{-\varepsilon}} P(\mathcal{Y}=-\varepsilon|x)d\rho_X,$$

then

$$\varepsilon\left\{\int_{I_1} P(\mathcal{Y}=1|x)d\rho_X - \int_{I_{-1}} P(\mathcal{Y}=-1|x)d\rho_X\right\} > 0.$$

Set $\tilde{f}_{K,C} = f_{K,C} + \varepsilon(r-1)$. We find that $\mathcal{E}(\tilde{f}_{K,C}) < \mathcal{E}(f_{K,C})$ which is a contradiction to the definition of $f_{K,C}$.

Thus, (20) can always be realized by a minimizer satisfying (10). $\square$

In what follows we shall always choose $f_{\mathbf{z}}$ and $f_{K,C}$ to satisfy (19) and (20), respectively. Also, denote $b_{f_{K,C}}$ simply as $b_{K,C}$.

Lemma 2. *For any $C > 0, m \in \mathbb{N}$ and $\mathbf{z} \in Z^m$, there hold*

(1) $\|f_{\mathbf{z}}^*\|_K \le \sqrt{2C}, \quad |b_{\mathbf{z}}| \le 1 + \kappa\sqrt{2C},$ *and* $\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) \le 1.$

(2) $\|f_{K,C}\|_\infty \le 1 + 2\kappa\sqrt{2C}$ *and* $\mathcal{E}(f_{K,C}) \le 1$.

Proof. By the definition (8) and the choice $f = 0 + 0$, we see that

$$\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) + \frac{1}{2C}\|f^*_{\mathbf{z}}\|^2_K \le \frac{1}{m}\sum_{i=1}^m V(y_i, 0) + 0 = 1.$$

This gives $\|f^*_{\mathbf{z}}\|_K \le \sqrt{2C}$ and $\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) \le 1$. The former in connection with (6) and (19) leads to

$$|b_{\mathbf{z}}| \le \min_{1\le i\le m} |f_{\mathbf{z}}(x_i)| + \|f^*_{\mathbf{z}}\|_\infty \le 1 + \kappa\sqrt{2C}.$$

This proves Part (1).

By (10) and the choice $f = 0 + 0$, we see that

$$\mathcal{E}(f_{K,C}) + \frac{1}{2C}\|f^*_{K,C}\|^2_K \le \int_Z V(y, 0)d\rho + 0 = 1.$$

This gives $\|f^*_{K,C}\|_K \le \sqrt{2C}$ and $\mathcal{E}(f_{K,C}) \le 1$. The former in connection with (6) and (20) leads to

$$|b_{K,C}| \le \inf_{x\in(X,\rho_X)} |f_{K,C}(x)| + \|f^*_{K,C}\|_\infty \le 1 + \kappa\sqrt{2C}.$$

Hence

$$\|f_{K,C}\|_\infty \le \|f^*_{K,C}\|_\infty + |b_{K,C}| \le 1 + 2\kappa\sqrt{2C}$$

and Part (2) follows. □

4 Estimating the Sample Error

In this section we prove Proposition 2. For this purpose, we shall establish some probability inequalities. These inequalities are modified versions of Bernstein inequality and motivated by sample error estimates for the square loss [3, 12, 8].

Recall the Bernstein inequality: Suppose a random variable ξ has mean $\mu = E\xi$ and variance $\sigma^2 = \sigma^2(\xi)$ and satisfies $|\xi - \mu| \le M$. Let $\mathbf{z} = (z_i)_{i=1}^m$ be independent samples. Then

$$\text{Prob}\left\{\left|\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i)\right| > \varepsilon\right\} \le 2\exp\left\{-\frac{m\varepsilon^2}{2(\sigma^2 + \frac{1}{3}M\varepsilon)}\right\}.$$

The one-sided Bernstein inequality has no leading factor 2.

Lemma 3. *Suppose a random variable ξ satisfies $0 \le \xi \le M$ and $\mu = E\xi$. Then for every $\varepsilon > 0$ and $0 < \alpha \le 1$, there holds*

$$\mathrm{Prob}\left\{\frac{\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\mu + \varepsilon} > \alpha\right\} \le \exp\left\{-\frac{3m\alpha^2\varepsilon}{8M}\right\}.$$

Proof. As ξ satisfies $|\xi - \mu| \le M$, the one-sided Bernstein inequality tells that

$$\mathrm{Prob}\left\{\frac{\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\mu + \varepsilon} > \alpha\right\} \le \exp\left\{-\frac{m\alpha^2(\mu+\varepsilon)^2\varepsilon}{2\left(\sigma^2 + \frac{1}{3}M\alpha(\mu+\varepsilon)\right)}\right\}.$$

Here $\sigma^2 \le E(\xi^2) \le ME(\xi) = M\mu$ since $0 \le \xi \le M$. Then we find that

$$\sigma^2 + \frac{1}{3}M\alpha(\mu+\varepsilon) \le \frac{4}{3}M(\mu+\varepsilon) \le \frac{4M(\mu+\varepsilon)^2}{3\varepsilon}.$$

This yields the desired inequality. □

In the same way, we have

$$\mathrm{Prob}\left\{\frac{\frac{1}{m}\sum_{i=1}^m \xi(z_i) - \mu}{\mu + \varepsilon} > \alpha\right\} \le \exp\left\{-\frac{3m\alpha^2\varepsilon}{8M}\right\}. \tag{21}$$

Lemma 4. *Under the assumptions of Lemma 3, for every $\varepsilon > 0$ and $0 < \alpha \le 1$, there holds*

$$\mathrm{Prob}\left\{\frac{\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\frac{1}{m}\sum_{i=1}^m \xi(z_i) + \varepsilon} > \alpha\right\} \le \exp\left\{-\frac{3m\alpha^2\varepsilon}{32M}\right\}.$$

Proof. By Lemma 3, it suffices to show that

$$\frac{\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\mu + \varepsilon} \le \frac{\alpha}{2} \Longrightarrow \frac{\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\frac{1}{m}\sum_{i=1}^m \xi(z_i) + \varepsilon} \le \alpha. \tag{22}$$

The left hand side of (22) implies

$$\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i) \le \frac{\mu}{2} + \frac{1}{2}\varepsilon.$$

This gives

$$\mu \le 2\left(\frac{1}{m}\sum_{i=1}^m \xi(z_i)\right) + \varepsilon.$$

Then we get

$$\mu - \frac{1}{m}\sum_{i=1}^m \xi(z_i) \le \frac{1}{m}\sum_{i=1}^m \xi(z_i) + \varepsilon$$

and

$$\frac{\mu+\varepsilon}{\frac{1}{m}\sum_{i=1}^m \xi(z_i)+\varepsilon} = \frac{\mu-\frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\frac{1}{m}\sum_{i=1}^m \xi(z_i)+\varepsilon} + 1 \le 2.$$

Thus

$$\frac{\mu-\frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\frac{1}{m}\sum_{i=1}^m \xi(z_i)+\varepsilon} = \frac{\mu-\frac{1}{m}\sum_{i=1}^m \xi(z_i)}{\mu+\varepsilon} \cdot \frac{\mu+\varepsilon}{\frac{1}{m}\sum_{i=1}^m \xi(z_i)+\varepsilon} \le \alpha.$$

This proves (22) and hence finishes the proof. □

By the Lipschitz property of the loss function V, we find that

$$|\mathcal{E}_{\mathbf{z}}(f)-\mathcal{E}_{\mathbf{z}}(g)| \le \|f-g\|_\infty, \quad |\mathcal{E}(f)-\mathcal{E}(g)| \le \|f-g\|_\infty. \tag{23}$$

Now we can prove a result concerning the uniform convergence.

Lemma 5. *Let $\mathcal{F}$ be a subset of $C(X)$ such that $\|f\|_\infty \le M$ for every $f \in \mathcal{F}$. Then for every $\varepsilon > 0$ and $0 < \alpha \le 1$, we have*

$$\operatorname{Prob}\left\{\sup_{f\in\mathcal{F}} \frac{\mathcal{E}(f)-\mathcal{E}_{\mathbf{z}}(f)}{\mathcal{E}_{\mathbf{z}}(f)+\varepsilon} \ge 4\alpha\right\} \le \mathcal{N}(\mathcal{F},\alpha\varepsilon)\exp\left\{-\frac{m\alpha^2\varepsilon}{32(1+M)}\right\}.$$

Proof. Let $\{f_j\}_{j=1}^N \subset \mathcal{F}$ with $N = \mathcal{N}(\mathcal{F},\alpha\varepsilon)$ such that $\mathcal{F}$ is covered by balls centered at f_j with radius $\alpha\varepsilon$. Note that for every $f \in \mathcal{F}$ the random variable $\xi = V(y, f(x))$ satisfies $0 \le \xi \le 1+\|f\|_\infty \le 1+M$. Then for each j, Lemma 4 tells

$$\operatorname{Prob}\left\{\frac{\mathcal{E}(f_j)-\mathcal{E}_{\mathbf{z}}(f_j)}{\mathcal{E}_{\mathbf{z}}(f_j)+\varepsilon} \ge \alpha\right\} \le \exp\left\{-\frac{3m\alpha^2\varepsilon}{32(1+M)}\right\}.$$

For each $f \in \mathcal{F}$, there is some j such that $\|f-f_j\|_\infty \le \alpha\varepsilon$. This in connection with (23) tells us that $|\mathcal{E}_{\mathbf{z}}(f)-\mathcal{E}_{\mathbf{z}}(f_j)|$ and $|\mathcal{E}(f)-\mathcal{E}(f_j)|$ are both bounded by $\alpha\varepsilon$. Hence

$$\frac{|\mathcal{E}_{\mathbf{z}}(f)-\mathcal{E}_{\mathbf{z}}(f_j)|}{\mathcal{E}_{\mathbf{z}}(f)+\varepsilon} \le \alpha \qquad \text{and} \qquad \frac{|\mathcal{E}(f)-\mathcal{E}(f_j)|}{\mathcal{E}_{\mathbf{z}}(f)+\varepsilon} \le \alpha.$$

The former implies that $\mathcal{E}_{\mathbf{z}}(f_j)+\varepsilon \le 2\big[\mathcal{E}_{\mathbf{z}}(f)+\varepsilon\big]$. Therefore,

$$\operatorname{Prob}\left\{\sup_{f\in\mathcal{F}} \frac{\mathcal{E}(f)-\mathcal{E}_{\mathbf{z}}(f)}{\mathcal{E}_{\mathbf{z}}(f)+\varepsilon} \ge 4\alpha\right\} \le \sum_{j=1}^N \operatorname{Prob}\left\{\frac{\mathcal{E}(f_j)-\mathcal{E}_{\mathbf{z}}(f_j)}{\mathcal{E}_{\mathbf{z}}(f_j)+\varepsilon} \ge \alpha\right\}$$

which is bounded by $N\exp\big\{-\frac{m\alpha^2\varepsilon}{32(1+M)}\big\}$. □

By Lemma 2 (1), $f_{\mathbf{z}}$ always lies in the set

$$\mathcal{F}_{R,B} := \{f : f = f^* + b_f \in \mathcal{B}_R + [-B,B]\} \tag{24}$$

with $R = \sqrt{2C}$ and $B = 1+\kappa\sqrt{2C}$. In order to apply Lemma 5, we need the covering numbers of the function set $\mathcal{F}_{R,B}$.

Lemma 6. *Let $\mathcal{F}_{R,B}$ be given by (24) with $R > 0$ and $B > 0$. For any $\varepsilon > 0$ there holds*

$$\mathcal{N}(\mathcal{F}_{R,B}, \varepsilon) \leq \left(\frac{2B}{\varepsilon} + 1\right) \mathcal{N}\left(\frac{\varepsilon}{2R}\right).$$

Proof. It is easy to see that $\mathcal{N}(\mathcal{F}_{R,B}, \varepsilon)$ is bounded by $(\frac{2B}{\varepsilon} + 1)\mathcal{N}\left(\mathcal{B}_R, \frac{\varepsilon}{2}\right)$ since

$$\|(f^* + b_f) - (g^* + b_g)\|_\infty \leq \|f^* - g^*\|_\infty + |b_f - b_g|.$$

But an $\frac{\varepsilon}{2R}$-covering of $\mathcal{B}_1$ is the same as an $\frac{\varepsilon}{2}$-covering of $\mathcal{B}_R$, our conclusion follows. $\square$

Proof of Proposition 2. Set $R = \sqrt{2C}$ and $B = 1 + \kappa\sqrt{2C}$.

From Lemma 2 we know the random variable $\xi = V(y, f_{K,C}(x))$ satisfies $0 \leq \xi \leq 2B$. By the fact $\mathcal{E}(f_{K,C}) \leq 1$ we obtain

$$\text{Prob}\left\{\mathcal{E}_{\mathbf{z}}(f_{K,C}) - \mathcal{E}(f_{K,C}) > \frac{\varepsilon}{2}\right\} \leq \text{Prob}\left\{\frac{\mathcal{E}_{\mathbf{z}}(f_{K,C}) - \mathcal{E}(f_{K,C}}{\mathcal{E}(f_{K,C}) + 1} > \frac{\varepsilon}{4}\right\}.$$

Applying (21), we find that the right hand side above is bounded by $\exp\left\{-\frac{3m\varepsilon^2}{256B}\right\}$.

Let $\mathcal{F}_{R,B}$ be given by (24). Then $f_{\mathbf{z}} \in \mathcal{F}_{R,B}$. This together with the fact $\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) \leq 1$ leads to

$$\begin{aligned} \text{Prob}\left\{\mathcal{E}(f_{\mathbf{z}}) - \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) > \tfrac{\varepsilon}{2}\right\} &\leq \text{Prob}\left\{\frac{\mathcal{E}(f_{\mathbf{z}}) - \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}})}{\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) + 1} > \frac{\varepsilon}{4}\right\} \\ &\leq \text{Prob}\left\{\sup_{f \in \mathcal{F}_{R,B}} \frac{\mathcal{E}(f) - \mathcal{E}_{\mathbf{z}}(f)}{\mathcal{E}_{\mathbf{z}}(f) + 1} > \frac{\varepsilon}{4}\right\}. \end{aligned}$$

According to Lemma 5, it can be bounded by

$$\mathcal{N}\left(\mathcal{F}_{R,B}, \frac{\varepsilon}{16}\right) \exp\left\{-\frac{3m\varepsilon^2}{2^{14}B}\right\}.$$

Bounding the covering number $\mathcal{N}\left(\mathcal{F}_{R,B}, \frac{\varepsilon}{16}\right)$ by Lemma 6 completes the proof . $\square$

5 Consistency with Hypothesis Space may Fail

The above analysis shows that the Bayes-risk Consistency holds if $\mathcal{D}(C) \to 0$ as $C \to \infty$.

In this section we consider the case when $\mathcal{D}(C) \not\to 0$. In this case we would not expect the Bayes-risk consistency in general. But the consistency with hypothesis space becomes a natural question. This kind of consistency is meaningful, since in practice, one may not need a classifier to approximate the Bayes rule very well. All we need is that the misclassification error is endurably small. Thus, we may face the following question:

Suppose a priori knowledge ensures that $\overline{\mathcal{H}}_K$ contains a classifier with endurably small misclassification error which does not approximate the Bayes rule. Does SVM produce a sufficiently good classifier?

Unfortunately, this is not true in general, as shown by the following example.

Example 2. *Let $X = [-1, 1]$, $K(x, y) = x \cdot y$, and ρ be the probability measure supported at four points defined as*

$$P(-1, 1) = P(1, -1) = \frac{1}{8}, \quad P(-\frac{1}{24}, -1) = P(\frac{1}{24}, 1) = \frac{3}{8}.$$

Define $f_{\mathbf{z}}$ by (7). Then for $m \in \mathbb{N}$ and $C > 0$, with confidence at least $1 - 8\exp\{\frac{-3m}{64(1+45/(2m))}\}$, there holds

$$\mathcal{R}\big(sgn(f_{\mathbf{z}})\big) \geq \inf_{f \in \overline{\mathcal{H}}_K} \mathcal{R}(sgn(f)) + \frac{1}{8}.$$

Proof . Notice that $\mathcal{H}_K = \{ax : a \in \mathbb{R}\}, \|ax\|_K = |a|$, and $\overline{\mathcal{H}}_K = \{ax + b : a, b \in \mathbb{R}\}$. For $j = 1, \ldots, 4$, denote $\mathbf{z}^{(j)} = (x^{(j)}, y^{(j)})$ where $\mathbf{z}^{(1)} = (-1, 1), \mathbf{z}^{(2)} = (-\frac{1}{24}, -1), \mathbf{z}^{(3)} = (\frac{1}{24}, 1)$, and $\mathbf{z}^{(4)} = (1, -1)$.

The definition of the misclassification error $\mathcal{R}$ tells us that

$$\mathcal{R}(f) = \sum_{j=1}^{4} \rho_X(x^{(j)}) \chi_{\{\mathrm{sgn}(f(x^{(j)})) \neq y^{(j)}\}}.$$

If $f \in \overline{\mathcal{H}}_K$ satisfies $\mathrm{sgn}(f(x^{(j)})) \neq y^{(j)}$ for $j = 2$ or 3, then $\mathcal{R}(f) \geq 3/8 > 1/4$. It follows that

$$\inf_{f \in \overline{\mathcal{H}}_K} \mathcal{R}(f) = \inf_{f \in \mathcal{H}_K} \mathcal{R}(f) = \frac{1}{4}$$

and a best classifier in $\mathcal{H}_K$ can be taken as $f_K(x) = x$.

Now we consider the misclassification error of $f_{\mathbf{z}}$.

Let $\mathbf{z}$ consist of m_j copies of $\mathbf{z}^{(j)}$, $j = 1, \ldots, 4$. We claim that

$$\Big|\frac{m_j}{m} - \rho_X(x^{(j)})\Big| < \frac{1}{32}, \ \forall j \Longrightarrow f_{\mathbf{z}}(x) = a_{\mathbf{z}}x + b_{\mathbf{z}} \text{ with } a_{\mathbf{z}} \leq 0 \text{ or } -\frac{b_{\mathbf{z}}}{a_{\mathbf{z}}} \notin [-\frac{1}{24}, \frac{1}{24}].$$

Suppose to the contrary that $f_{\mathbf{z}}(x) = ax + b$ with $a > 0$ and $-b/a \in [-\frac{1}{24}, \frac{1}{24}]$. Denote $x_0 := -b/a$. Then $f_{\mathbf{z}}(x) = a(x - x_0)$. Set $f_{\mathbf{z}}^-(x) = -a(x - x_0) \in \overline{\mathcal{H}}_K$. Observe that $\rho_X(x^{(j)}) - 1/32 < \frac{m_j}{m} < \rho_X(x^{(j)}) + 1/32$ for each j. Also, $\|f_{\mathbf{z}}^*\|_K = \|(f_{\mathbf{z}}^-)^*\|_K = |a|$.

If $a > 5$, then

$$\begin{aligned}\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) &\geq \frac{1}{m}\left\{m_1\left(1 - y^{(1)} f_{\mathbf{z}}(x^{(1)})\right)_+ + m_4\left(1 - y^{(4)} f_{\mathbf{z}}(x^{(4)})\right)_+\right\} \\ &= \frac{m_1}{m}(1 + a(1 + x_0)) + \frac{m_4}{m}(1 + a(1 - x_0)) \geq \frac{3}{16}(1 + a).\end{aligned}$$

For $j = 1, 4$, we have $y^{(j)} f_{\mathbf{z}}^-(x^{(j)}) = -a y^{(j)}(x^{(j)} - x_0) = a|x^{(j)} + y^{(j)} x_0| > 1$. Hence

$$\begin{aligned}\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}^-) &\leq \frac{1}{m}\left\{m_2\left(1 - y^{(2)} f_{\mathbf{z}}^-(x^{(2)})\right)_+ + m_3\left(1 - y^{(3)} f_{\mathbf{z}}^-(x^{(3)})\right)_+\right\} \\ &= \frac{m_2}{m}(1 - a(-\frac{1}{24} - x_0)) + \frac{m_3}{m}(1 + a(\frac{1}{24} - x_0)) \leq \frac{13}{16}(1 + \frac{1}{24}a).\end{aligned}$$

Since $a > 5$, we have

$$\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) + \frac{1}{2C}\|f_{\mathbf{z}}^*\|_K^2 > \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}^-) + \frac{1}{2C}\|(f_{\mathbf{z}}^-)^*\|_K^2$$

which is a contradiction to the definition (7) of $f_{\mathbf{z}}$.

If $a \leq 5$, then for $j = 2, 3$, we have $|y^{(j)} f_{\mathbf{z}}(x^{(j)})| = a|x^{(j)} - x_0| \leq 5/12 < 1$. For $j = 1, 4$, we also have $-y^{(j)} f_{\mathbf{z}}(x^{(j)}) = -y^{(j)} a(x^{(j)} - x_0) > 0$. Hence

$$\begin{aligned}\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) = &\frac{m_1}{m}(1 - a(x^{(1)} - x_0)) + \frac{m_2}{m}(1 + a(x^{(2)} - x_0)) \\ &+ \frac{m_3}{m}(1 - a(x^{(3)} - x_0)) + \frac{m_4}{m}(1 + a(x^{(4)} - x_0)) \\ &\geq 1 + \frac{3}{16}a - \frac{13}{16}\frac{a}{24} > 1 + \frac{4}{32}a.\end{aligned}$$

For $f_{\mathbf{z}}^-$ we have

$$\begin{aligned}\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}^-) &\leq \frac{m_1}{m} + \frac{m_2}{m}(1 - a(-\frac{1}{12} - x_0)) + \frac{m_3}{m}(1 + a(\frac{1}{12} - x_0)) + \frac{m_4}{m} \\ &\leq 1 + \frac{13}{32}\frac{a}{6} < 1 + \frac{3}{32}a.\end{aligned}$$

Therefore, we also have a contradiction to the definition (7) of $f_{\mathbf{z}}$:

$$\mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}) + \frac{1}{2C}\|f_{\mathbf{z}}^*\|_K^2 > \mathcal{E}_{\mathbf{z}}(f_{\mathbf{z}}^-) + \frac{1}{2C}\|(f_{\mathbf{z}}^-)^*\|_K^2.$$

Thus our claim has been verified, and we must have $a_{\mathbf{z}} \le 0$ or $-b_{\mathbf{z}}/a_{\mathbf{z}} \notin [-\frac{1}{24}, \frac{1}{24}]$. In this case, we see that $\mathrm{sgn}(f_{\mathbf{z}}(x^{(j)})) \ne y^{(j)}$ for $j = 2$ or 3, hence $\mathcal{R}(f_{\mathbf{z}}) \ge 3/8$.

What we need to check finally is the probability of the event

$$\bigcap_{j=1}^{4} \left\{ \left| \frac{m_j}{m} - \rho_X(x^{(j)}) \right| < \frac{1}{32} \right\}.$$

For each fixed j, the random variable $\xi = \chi_{\mathbf{z}=\mathbf{z}^{(j)}}$ is a binomial distribution with mean $\mu = \rho_X(x^{(j)})$ and variance $\sigma^2 = \rho_X(x^{(j)})(1 - \rho_X(x^{(j)}))/m$. By the Bernstein inequality, we have

$$\begin{aligned} \mathrm{Prob}\left\{ \left| \frac{m_j}{m} - \mu \right| \ge \frac{1}{32} \right\} &\le 2\exp\left\{ -\frac{m(\frac{1}{32})^2}{2(\mu(1-\mu)/m + 1/(3\cdot 32))} \right\} \\ &\le 2\exp\left\{ -\frac{3m}{64(1 + 45/(2m))} \right\}. \end{aligned}$$

Thus, the desired confidence is at least

$$\mathrm{Prob}\left\{ \left| \frac{m_j}{m} - \mu \right| < \frac{1}{32} \quad \forall j \right\} \ge 1 - 8\exp\left\{ -\frac{3m}{64(1 + 45/(2m))} \right\}.$$

The statements of Example 2 have been verified. $\square$

In Example 2, the geometric structure of the underlying distribution is very singular. There is a subset of X which, with respect to the optimal classifier over the space, results in small misclassification error but large generalization error for it is distributed far from the decision boundary. Generally speaking, when this phenomenon happens, the sign function of the minimizer of $\mathcal{E}(f)$ over $\overline{\mathcal{H}}_K$ may not coincide with the optimal classifier and the convergence $\mathcal{R}(f_{\mathbf{z}}) \to \inf_{f \in \overline{\mathcal{H}}_K} \mathcal{R}(f)$ fails.

In practice, the SVM is still very efficient due to two reasons. Firstly, the geometric structure of the underlying distribution is usually regular (i.e., those points which are hard to classify are usually close to the decision boundary). Secondly, one may vary the kernels (equivalent to using larger hypothesis spaces) to reduce the gap between the misclassification error and the generalization error. For instance, in Example 2, $\mathrm{sgn}(f_{\mathbf{z}})$ will approximate the Bayes rule f_c very well if one uses the kernel $K(x, y) = (1 + x \cdot y)^3$.

The consistency with hypothesis space needs further study. It is interesting (in mathematics) and useful (for applications) to have some positive results.

Acknowledgement. The work described in this paper was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.2,121-132,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

Piecewise constant wavelets defined on closed surfaces

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Abstract

In a previous paper we constructed piecewise constant wavelets on spherical triangulations, orthogonal with respect to a given inner product. In this paper we generalize this construction to closed surfaces, finding conditions which have to be satisfied by a closed surface to assure the Riesz stability of the wavelets.

Key words: spherical wavelets, Haar wavelets, triangulation.

MSC 2000: 42C40, 41A63, 41A15.

1 Introduction

Consider the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ with center O and the surface $\mathcal{S} \subseteq \mathbb{R}^3$ defined by the function $\sigma : \mathbb{S}^2 \to \mathbb{R}^3$,

$$\sigma(\eta) = \rho(\eta)\eta, \tag{1.1}$$

for all $\eta \in \mathbb{S}^2$, where $\rho : \mathbb{S}^2 \to \mathbb{R}_+$ is a continuous function. We intend to use the piecewise constant locally supported wavelets defined on $\mathbb{S}^2$, presented in [3], constructing piecewise constant locally supported wavelets defined on $\mathcal{S}$. Actually, we try to find conditions which have to be satisfied by the function ρ to ensure the Riesz stability of these wavelets.

The paper is structured as follows. In Section 2 we recall the construction of wavelets defined on $\mathbb{S}^2$, construction which was realized in [3]. In Section 3 we show how this construction can be extended to closed surfaces. In Section 4 we introduce some inner products and prove some norm equivalencies in $L^2(\mathcal{S})$. They are used in Section 6 for studying the properties of our wavelets (orthogonality, Riesz stability). We show that, under some assumptions on the function ρ, our wavelets are Riesz stable in $L^2(\mathcal{S})$. Finally, we present some types of closed surfaces $\mathcal{S}$ where our wavelets are Riesz stable in $L^2(\mathcal{S})$.

2 Piecewise constant wavelets defined on spherical triangulations

The construction of locally supported piecewise constant wavelets defined on $\mathbb{S}^2$ was realized in [3]. Let Π be a convex polyhedron having triangular faces[1] and the vertices situated on the sphere $\mathbb{S}^2$. Also we have to suppose that no face contains the origin O and O is situated inside the polyhedron. We denote by $\mathcal{T}^0 = \{T_1, T_2, \ldots, T_n\}$ the set of the faces of Π and by Ω the surface (the "cover") of Π. Then we consider the radial projection onto $\mathbb{S}^2$, $p : \Omega \to \mathbb{S}^2$,

$$p(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z), \ (x,y,z) \in \Omega. \tag{2.1}$$

Being given Ω, we can say that $\mathcal{T} = \mathcal{T}^0$ is a triangulation of Ω. Next we consider its uniform refinement $\mathcal{T}^1$. For a given triangle $[M_1M_2M_3]$ in $\mathcal{T}^0$, let A_1, A_2, A_3 denote the midpoints of the edges M_2M_3, M_3M_1 and M_1M_2, respectively. Then we consider the set

$$\mathcal{T}^1 = \bigcup_{[M_1M_2M_3]\in\mathcal{T}^0} \{[M_1A_2A_3], [A_1M_2A_3], [A_1A_2M_3], [A_1A_2A_3]\},$$

which is also a triangulation of Ω. Continuing in the same manner the refinement process we can obtain a triangulation $\mathcal{T}^j$ of Ω, for $j \in \mathbb{N}$. The projection of $\mathcal{T}^j$ onto the

[1] The polyhedron could also have faces which are not triangles. In that case we triangulate each of these faces and consider it as having triangular faces, with some of the faces coplanar.

sphere will be $\mathcal{U}^j = \{p(T^j), \ T^j \in \mathcal{T}^j\}$, which is a triangulation of $\mathbb{S}^2$. The number of triangles in $\mathcal{U}^j$ will be $|\mathcal{U}^j| = n \cdot 4^j$.

Let $\langle \cdot, \cdot \rangle_\Omega$ be the following inner product, based on the initial coarsest triangulation $\mathcal{T}^0$:

$$\langle f, g \rangle_\Omega = \sum_{T \in \mathcal{T}^0} \frac{1}{a(T)} \int_T f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}, \ \text{ for } f, g \in C(T) \ \forall T \in \mathcal{T}^0.$$

Here $a(T)$ denotes the area of the triangle T. Also, we consider the induced norm

$$\|f\|_\Omega = \langle f, f \rangle_\Omega^{1/2}.$$

For the L^2-integrable functions F and G defined on $\mathbb{S}^2$, the following inner product associated to the given polyhedron Π was defined in [4]:

$$\langle F, G \rangle_{*,\mathbb{S}^2} = \langle F \circ p, G \circ p \rangle_\Omega . \tag{2.2}$$

There it was proved that, in the space $L^2(\mathbb{S}^2)$, the norm $\|\cdot\|_{*,\mathbb{S}^2}$ induced by this inner product is equivalent to the usual norm $\|\cdot\|_{L^2(\mathbb{S}^2)}$ of $L^2(\mathbb{S}^2)$. Denoting

$$d_T = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

for each triangle T having the vertices $B_i(x_i, y_i, z_i)$, $i = 1, 2, 3$, we proved that

$$m \|F\|^2_{L^2(\mathbb{S}^2)} \le \|F\|^2_{*,\mathbb{S}^2} \le M \|F\|^2_{L^2(\mathbb{S}^2)}, \tag{2.3}$$

with $m = \frac{1}{4} \min\limits_{T \in \mathcal{T}^0} \frac{d_T^2}{a(T)^3}$, $M = 2 \max\limits_{T \in \mathcal{T}^0} \frac{1}{|d_T|}$. If we use the relation $|d_T| = 2a(T)\operatorname{dist}(O, T)$, with $\operatorname{dist}(O, T)$ representing the distance from the origin to the plane of the triangle T, then the values m and M become

$$\begin{aligned} m &= \min_{T \in \mathcal{T}^0} \frac{\operatorname{dist}^2(O, T)}{a(T)}, \\ M &= \max_{T \in \mathcal{T}^0} \frac{1}{a(T)\operatorname{dist}(O, T)}. \end{aligned}$$

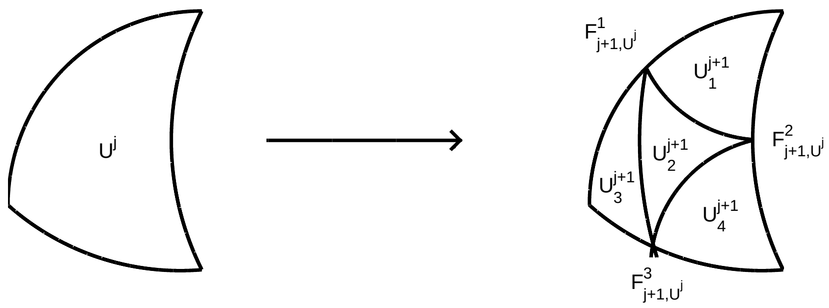

Figure 1: The triangle U^j and its refined triangles $U_k^{j+1},\ k = 1, 2, 3, 4.$

Then we constructed a multiresolution on $\mathbb{S}^2$ consisting of piecewise constant functions on the triangles of $\mathcal{U}^j = \left\{U_1^j, U_2^j, \ldots, U_{n\cdot 4^j}^j\right\}$, $j \in \mathbb{N}$.

By definition, a multiresolution of $L^2(\mathbb{S}^2)$ is a sequence of subspaces $\{V^j : j \geq 0\}$ of $L^2(\mathbb{S}^2)$ which satisfies the following properties:

1. $V^j \subseteq V^{j+1}$ for all $j \in \mathbb{N}$,
2. $\operatorname{clos}_{L^2(\mathbb{S}^2)} \bigcup\limits_{j=0}^{\infty} V^j = L^2(\mathbb{S}^2)$,
3. There are index sets $\mathcal{K}_j \subseteq \mathcal{K}_{j+1}$ such that for every level j there exists a Riesz basis $\left\{\varphi_t^j,\ t \in \mathcal{K}_j\right\}$ of the space V^j. This means that there exist constants $0 < c \leq C < \infty$, independent of the level j, such that

$$c2^{-j} \left\| \left\{c_t^j\right\}_{t\in\mathcal{K}_j} \right\|_{l_2(\mathcal{K}_j)} \leq \left\| \sum_{t\in\mathcal{K}_j} c_t^j \varphi_t^j \right\|_{L^2(\mathbb{S}^2)} \leq C2^{-j} \left\| \left\{c_t^j\right\}_{t\in\mathcal{K}_j} \right\|_{l_2(\mathcal{K}_j)},$$

where $\left\| \left\{c_t^j\right\}_{t\in\mathcal{K}_j} \right\|_{l_2(\mathcal{K}_j)} = \left(\sum_{t\in\mathcal{K}_j} (c_t^j)^2\right)^{1/2}$.

For a fixed $j \in \mathbb{N}$, to each triangle $U_k^j \in \mathcal{U}^j$, $k = 1, 2, \ldots, n\cdot 4^j$, we associate the function $\varphi_{U_k^j} : \mathbb{S}^2 \to \mathbb{R}$,

$$\varphi_{U_k^j}(\eta) = \begin{cases} 1, & \text{inside the triangle } U_k^j, \\ 1/2, & \text{on the edges of } U_k^j, \\ 0, & \text{elsewhere.} \end{cases}$$

Then we constructed the spaces of functions $V^j = \operatorname{span}\left\{\varphi_{U_k^j},\ k = 1, 2, \ldots, n \cdot 4^j\right\}$, consisting of piecewise constant functions on the triangles of $\mathcal{U}^j$. If $U_k^{j+1} = p\left(T_k^{j+1}\right),\ k =$

$1, 2, 3, 4$ are the refined triangles obtained from U^j as in Figure 1, we have

$$\varphi_{U^j} = \varphi_{U_1^{j+1}} + \varphi_{U_2^{j+1}} + \varphi_{U_3^{j+1}} + \varphi_{U_4^{j+1}},$$

which holds[2] in $L^2(\mathbb{S}^2)$. Thus, $V^j \subseteq V^{j+1}$ for all $j \in \mathbb{N}$. With respect to the scalar product $\langle \cdot, \cdot \rangle_{*,\mathbb{S}^2}$, the spaces V^j and V^{j+1} become Hilbert spaces, with the corresponding norm $\|\cdot\|_{*,\mathbb{S}^2} = \langle \cdot, \cdot \rangle_{*,\mathbb{S}^2}^{1/2}$.

Next we defined the space W^j as the orthogonal complement, with respect to the scalar product $\langle \cdot, \cdot \rangle_{*,\mathbb{S}^2}$, of the coarse space V^j in the fine space V^{j+1} :

$$V^{j+1} = V^j \bigoplus W^j.$$

The spaces W^j are called *the wavelet spaces.* The dimension of W^j is $\dim W^j = \dim V^{j+1} - \dim V^j = 3n \cdot 4^j$. In [3] we proved that we have two classes of orthonormal [3] wavelet bases. They include the "nearly orthogonal" wavelets obtained by Bonneau in [1] and Nielson, Jung and also those constructed by Sung in [2]. With the notations of Figure 1, the wavelets have the expressions

$$\begin{aligned}
{}^1\Psi_{F_{j+1}^1, U^j} &= \alpha_1 \varphi_{U_1^{j+1}} + \alpha_2 \varphi_{U_3^{j+1}} + \frac{1}{2}\varphi_{U_2^{j+1}} - \left(\frac{1}{2} + \alpha_1 + \alpha_2\right)\varphi_{U_4^{j+1}},\\
{}^1\Psi_{F_{j+1}^2, U^j} &= \alpha_1 \varphi_{U_4^{j+1}} + \alpha_2 \varphi_{U_1^{j+1}} + \frac{1}{2}\varphi_{U_2^{j+1}} - \left(\frac{1}{2} + \alpha_1 + \alpha_2\right)\varphi_{U_3^{j+1}},\\
{}^1\Psi_{F_{j+1}^3, U^j} &= \alpha_1 \varphi_{U_3^{j+1}} + \alpha_2 \varphi_{U_4^{j+1}} + \frac{1}{2}\varphi_{U_2^{j+1}} - \left(\frac{1}{2} + \alpha_1 + \alpha_2\right)\varphi_{U_1^{j+1}},
\end{aligned}$$

with $\alpha_1,\ \alpha_2$ such that $4\left(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2\right) + 2\left(\alpha_1 + \alpha_2\right) - 1 = 0$ and

$$\begin{aligned}
{}^2\Psi_{F_{j+1}^1, U^j} &= \alpha_1 \varphi_{U_1^{j+1}} + \alpha_2 \varphi_{U_3^{j+1}} - \frac{1}{2}\varphi_{U_2^{j+1}} + \left(\frac{1}{2} - \alpha_1 - \alpha_2\right)\varphi_{U_4^{j+1}},\\
{}^2\Psi_{F_{j+1}^2, U^j} &= \alpha_1 \varphi_{U_4^{j+1}} + \alpha_2 \varphi_{U_1^{j+1}} - \frac{1}{2}\varphi_{U_2^{j+1}} + \left(\frac{1}{2} - \alpha_1 - \alpha_2\right)\varphi_{U_3^{j+1}},\\
{}^2\Psi_{F_{j+1}^3, U^j} &= \alpha_1 \varphi_{U_3^{j+1}} + \alpha_2 \varphi_{U_4^{j+1}} - \frac{1}{2}\varphi_{U_2^{j+1}} + \left(\frac{1}{2} - \alpha_1 - \alpha_2\right)\varphi_{U_1^{j+1}},
\end{aligned}$$

[2] Actually the equality holds at all the points of the sphere, except the vertices of the triangles of $\mathcal{U}^{j+1}$.

[3] The orthogonality is with respect to the norm $\|\cdot\|_{*,\mathbb{S}^2}$

with $\alpha_1,\ \alpha_2$ such that $4\left(\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2\right)-2\left(\alpha_1+\alpha_2\right)-1=0$.
An application of these wavelets in data compression, together with numerical examples can be found in [3]. A comparison of these classes of wavelets with respect to the reconstruction error was realized in [5].

3 Piecewise constant wavelets defined on triangulations of the closed surface $\mathcal{S}$

We project the spherical triangulations $\mathcal{U}^j$ onto the surface $\mathcal{S}$, obtaining the triangulations $\mathcal{Z}^j=\{\sigma\left(U^j\right),\ U^j\in\mathcal{U}^j\}$ of the closed surface $\mathcal{S}$. For a fixed j, to each triangle $Z_k^j\in\mathcal{Z}^j,\ k=1,2,\ldots,n\cdot 4^j$, the associated piecewise constant scaling functions defined on $\mathcal{S}$ will be defined as

$$\phi_{Z_k^j}\left(\eta\right)=\begin{cases}1, & \text{inside the triangle } Z_k^j\,,\\ 1/2, & \text{on the edges of } Z_k^j\,,\\ 0, & \text{in rest.}\end{cases}$$

and therefore the wavelets will have the following expressions.

$$\begin{aligned}
{}^1\Upsilon_{F^1_{j+1},Z^j} &= \alpha_1\phi_{Z_1^{j+1}}+\alpha_2\phi_{Z_3^{j+1}}+\frac{1}{2}\phi_{Z_2^{j+1}}-\left(\frac{1}{2}+\alpha_1+\alpha_2\right)\phi_{Z_4^{j+1}},\\
{}^1\Upsilon_{F^2_{j+1},Z^j} &= \alpha_1\phi_{Z_4^{j+1}}+\alpha_2\phi_{Z_1^{j+1}}+\frac{1}{2}\phi_{Z_2^{j+1}}-\left(\frac{1}{2}+\alpha_1+\alpha_2\right)\phi_{Z_3^{j+1}},\\
{}^1\Upsilon_{F^3_{j+1},Z^j} &= \alpha_1\phi_{Z_3^{j+1}}+\alpha_2\phi_{Z_4^{j+1}}+\frac{1}{2}\phi_{Z_2^{j+1}}-\left(\frac{1}{2}+\alpha_1+\alpha_2\right)\phi_{Z_1^{j+1}},
\end{aligned}\tag{3.1}$$

with $\alpha_1,\ \alpha_2$ such that $4\left(\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2\right)+2\left(\alpha_1+\alpha_2\right)-1=0$ and

$$\begin{aligned}
{}^2\Upsilon_{F^1_{j+1},Z^j} &= \alpha_1\phi_{Z_1^{j+1}}+\alpha_2\phi_{Z_3^{j+1}}-\frac{1}{2}\phi_{Z_2^{j+1}}+\left(\frac{1}{2}-\alpha_1-\alpha_2\right)\phi_{Z_4^{j+1}},\\
{}^2\Upsilon_{F^2_{j+1},Z^j} &= \alpha_1\phi_{Z_4^{j+1}}+\alpha_2\phi_{Z_1^{j+1}}-\frac{1}{2}\phi_{Z_2^{j+1}}+\left(\frac{1}{2}-\alpha_1-\alpha_2\right)\phi_{Z_3^{j+1}},\\
{}^2\Upsilon_{F^3_{j+1},Z^j} &= \alpha_1\phi_{Z_3^{j+1}}+\alpha_2\phi_{Z_4^{j+1}}-\frac{1}{2}\phi_{Z_2^{j+1}}+\left(\frac{1}{2}-\alpha_1-\alpha_2\right)\phi_{Z_1^{j+1}},
\end{aligned}\tag{3.2}$$

with $\alpha_1,\ \alpha_2$ such that $4\,(\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2)-2\,(\alpha_1+\alpha_2)-1=0$.

Before we establish the orthogonality of these wavelets defined on $\mathcal{S}$ and their Riesz stability, we need to establish some equivalencies between norms. These equivalencies will be presented and proved in the next section.

4 Inner products and norms in $L^2(\mathcal{S})$

Consider the following parametrization of the sphere $\mathbb{S}^2$

$$\eta(x,y,z)\in\mathbb{S}^2 \Leftrightarrow \begin{cases} x=x(u,v)=\sin v\cos u, \\ y=y(u,v)=\sin v\sin u, \\ z=z(u,v)=\cos v, \end{cases} \tag{4.1}$$

$(u,v)\in\overline{\Delta}=[0,2\pi]\times[0,\pi]$. Then we define the functions $r:\overline{\Delta}\to(0,\infty)$ and $X,Y,Z:\overline{\Delta}\to\mathbb{R}$ by

$$\begin{aligned} r(u,v) &= \rho(\sin v\cos u,\sin v\sin u,\cos v), \qquad (4.2)\\ X(u,v) &= r(u,v)x(u,v),\\ Y(u,v) &= r(u,v)y(u,v),\\ Z(u,v) &= r(u,v)z(u,v). \end{aligned}$$

The following proposition establishes the relation between the surface element of the sphere and the surface element of $\mathcal{S}$.

Proposition 4.1 *Let $d\omega$ be the surface element of $\mathbb{S}^2$ and $d\sigma$ be the surface element of $\mathcal{S}$. Then, the relation between $d\omega$ and $d\sigma$ is*

$$d\sigma^2=r^2\left(r^2+r_v^2+\frac{r_u^2}{\sin^2 v}\right)d\omega^2, \tag{4.3}$$

where $r=r(u,v)$ is defined in (4.2) and $r_u,\ r_v:\Delta=(0,2\pi)\times(0,\pi)\to\mathbb{R}$ denote its partial derivatives.

Proof. Let us denote

$$\begin{aligned} \eta(u,v) &= (x(u,v), y(u,v), z(u,v)), \\ R(u,v) &= (X(u,v), Y(u,v), Z(u,v)). \end{aligned}$$

An immediate calculation shows that

$$\begin{aligned} d\omega &= \|\eta_u \times \eta_v\|\, du\, dv = \sin v\, du\, dv, \\ d\sigma &= \|R_u \times R_v\|\, du\, dv = r \sin v \left(r^2 + r_v^2 + \frac{r_u^2}{\sin^2 v} \right)^{1/2} du\, dv, \end{aligned}$$

where $\|\cdot\|$ denotes the Euclidian norm and $u \times v$ stands for the cross product of the vectors u and v in $\mathbb{R}^3$. Therefore

$$d\sigma = E(u,v) d\omega,$$

where $E : (0, 2\pi) \times (0, \pi) \to \mathbb{R}$,

$$E(u,v) = r \left(r^2 + r_v^2 + \frac{r_u^2}{\sin^2 v} \right)^{1/2}. \tag{4.4}$$

■

Definition 4.1 *Let $F, G : \mathcal{S} \to \mathbb{R}$ be functions of $L^2(\mathcal{S})$. Then $\langle \cdot, \cdot \rangle_\sigma : L^2(\mathcal{S}) \times L^2(\mathcal{S}) \to \mathbb{R}$ defined by*

$$\langle F, G \rangle_\sigma = \langle F \circ \sigma, G \circ \sigma \rangle_{L^2(\mathbb{S}^2)} \tag{4.5}$$

is an inner product in $L^2(\mathcal{S})$. We also consider the induced norm

$$\|\cdot\|_\sigma = \langle \cdot, \cdot \rangle_\sigma^{1/2}. \tag{4.6}$$

Regarding this norm, the following norm-equivalency is true.

Proposition 4.2 *If there exist the constants $0 < m_\sigma \leq M_\sigma < \infty$ such that $m_\sigma \leq E(u,v) \leq M_\sigma$ for all $(u,v) \in \Delta$, then in $L^2(\mathcal{S})$ the norm $\|\cdot\|_{L^2(\mathcal{S})}$ is equivalent to the norm $\|\cdot\|_\sigma$.*

Proof. Let $F \in L^2(\mathcal{S})$. We have

$$\begin{aligned}
\|F\|^2_{L^2(\mathcal{S})} &= \int_{\mathcal{S}} F^2(\zeta) d\sigma = \int_{\mathcal{S}} F^2(\rho(\eta)\eta)\, d\sigma \\
&= \iint_{\Delta} F^2(X(u,v), Y(u,v), Z(u,v))\, E(u,v) \sin v \, du\, dv.
\end{aligned}$$

Taking into account the inequalities $m_\sigma \le E(u,v) \le M_\sigma$ for $(u,v) \in \Delta$, we can write

$$\begin{aligned}
m_\sigma \iint_{\Delta} F^2(X(u,v), Y(u,v), Z(u,v)) \sin v \, du\, dv \le \|F\|^2_{L^2(\mathcal{S})} \le \\
\le M_\sigma \iint_{\Delta} F^2(X(u,v), Y(u,v), Z(u,v)) \sin v \, du\, dv,
\end{aligned}$$

and therefore

$$\begin{gathered}
m_\sigma \int_{\mathbb{S}^2} F^2(\sigma(\eta))\, d\omega \le \|F\|^2_{L^2(\mathcal{S})} \le M_\sigma \int_{\mathbb{S}^2} F^2(\sigma(\eta))\, d\omega, \\
m_\sigma \|F \circ \sigma\|^2_{L^2(\mathbb{S}^2)} \le \|F\|^2_{L^2(\mathcal{S})} \le M_\sigma \|F \circ \sigma\|^2_{L^2(\mathbb{S}^2)}.
\end{gathered}$$

which means

$$\sqrt{m_\sigma}\|F\|_\sigma \le \|F\|_{L^2(\mathcal{S})} \le \sqrt{M_\sigma}\|F\|_\sigma. \tag{4.7}$$

■

Definition 4.2 *Let $F, G : \mathcal{S} \to \mathbb{R}$. Then $\langle \cdot, \cdot \rangle_{*,\sigma} : L^2(\mathcal{S}) \times L^2(\mathcal{S}) \to \mathbb{R}$ defined by*

$$\langle F, G \rangle_{*,\sigma} = \langle F \circ \sigma, G \circ \sigma \rangle_{*,\mathbb{S}^2} \tag{4.8}$$

is an inner product in $L^2(\mathcal{S})$. We also consider the induced norm

$$\|\cdot\|_{*,\sigma} = \langle \cdot, \cdot \rangle^{1/2}_{*,\sigma}. \tag{4.9}$$

Proposition 4.3 *In $L^2(\mathcal{S})$ the norm $\|\cdot\|_\sigma$ is equivalent to the norm $\|\cdot\|_{*,\sigma}$.*

Proof. Let $F \in L^2(\mathcal{S})$. Then $\|F\|^2_{*,\sigma} = \|F \circ \sigma\|^2_{*,\mathbb{S}^2}$. Using now the inequalities (2.3) we can write

$$m \|F \circ \sigma\|^2_{L^2(\mathbb{S}^2)} \le \|F\|^2_{*,\sigma} \le M \|F \circ \sigma\|^2_{L^2(\mathbb{S}^2)}$$

and therefore, using the definition 4.1 we obtain

$$m \|F\|^2_\sigma \le \|F\|^2_{*,\sigma} \le M \|F\|^2_\sigma,$$

which completes the proof. ■

5 Orthogonality and Riesz stability of the wavelets

The results established in the previous section allow us to establish the following results.

Proposition 5.1 *The wavelets* ${}^{\mathbf{i}}\Upsilon$ *given in (3.1) and (3.2) are orthonormal with respect to the inner product* $\langle \cdot, \cdot \rangle_{*,\sigma}$, *meaning that*

$$\left\langle 2^j \cdot {}^{\mathbf{i}}\,\Upsilon_{F^k_{j+1}, Z^j}, 2^j \cdot {}^{\mathbf{i}}\,\Upsilon_{F^l_{j+1}, Z^m} \right\rangle_{*,\sigma} = \delta_{lk}\delta_{jm}. \tag{5.1}$$

Proof. The wavelets ${}^{\mathbf{i}}\Psi$ were orthonormal with respect to the scalar product $\langle \cdot, \cdot \rangle_{*,\mathbb{S}^2}$, meaning that

$$\left\langle 2^j \cdot {}^{\mathbf{i}}\,\Psi_{F^k_{j+1}, U^j}, 2^j \cdot {}^{\mathbf{i}}\,\Psi_{F^l_{j+1}, U^m} \right\rangle_{*,\mathbb{S}^2} = \delta_{lk}\delta_{jm}. \tag{5.2}$$

Using the definition 4.2 and the fact that $\Psi = \Upsilon \circ \sigma$, we immediately obtain the relations (5.1). ■

Proposition 5.2 *If the numbers* $m_\sigma = \min_{(u,v)\in\Delta} E(u, v)$ *and* $M_\sigma = \max_{(u,v)\in\Delta} E(u, v)$ *are such that* $m_\sigma > 0$ *and* $M_\sigma < \infty$, *then the wavelets obtained in the previous section satisfy the Riesz stability property, meaning that there exist the constants* $0 < c \leq C < \infty$, *independent of the level* j, *such that*

$$c\sum_{l=1}^{3}\sum_{Z^j\in\mathcal{Z}^j} d^2_{l,Z^j} \leq \left\| \sum_{l=1}^{3}\sum_{Z^j\in\mathcal{Z}^j} d_{l,Z^j} 2^j\, {}^{\mathbf{i}}\Upsilon_{F^l_{j+1},Z^j} \right\|^2_{L^2(\mathcal{S})} \leq C\sum_{l=1}^{3}\sum_{Z^j\in\mathcal{Z}^j} d^2_{l,Z^j},$$

for $\mathbf{i} = 1, 2$ *and arbitrary real numbers* d_{l,Z^j}.

Proof. In [3] we proved the following inequalities

$$\frac{1}{M}\sum_{l=1}^{3}\sum_{U^j\in\mathcal{U}^j} d^2_{l,U^j} \leq \left\| \sum_{l=1}^{3}\sum_{U^j\in\mathcal{U}^j} d_{l,U^j} 2^j\, {}^{\mathbf{i}}\Psi_{F^l_{j+1},U^j} \right\|^2_{L^2(\mathbb{S}^2)} \leq \frac{1}{m}\sum_{l=1}^{3}\sum_{U^j\in\mathcal{U}^j} d^2_{l,U^j},$$

for $\mathbf{i} = 1, 2$ and arbitrary real numbers d_{l,U^j}, where the numbers m and M are given in Section 2. Combining these inequalities with the inequalities given in (4.7) we obtain, for $\mathbf{i} = 1, 2$ and arbitrary real numbers d_{l,Z^j},

$$\frac{m_\sigma}{M}\sum_{l=1}^{3}\sum_{Z^j\in\mathcal{Z}^j} d^2_{l,Z^j} \leq \left\| \sum_{l=1}^{3}\sum_{Z^j\in\mathcal{Z}^j} d_{l,Z^j} 2^j\, {}^{\mathbf{i}}\Upsilon_{F^l_{j+1},Z^j} \right\|^2_{L^2(\mathcal{S})} \leq \frac{M_\sigma}{m}\sum_{l=1}^{3}\sum_{Z^j\in\mathcal{Z}^j} d^2_{l,Z^j},$$

inequalities which prove the Riesz stability of our wavelets. ■

6 Some closed surfaces which assure the Riesz stability in $L^2(\mathcal{S})$

The question is now: *how should we choose the function ρ such that the hypotheses of Proposition 5.2 are satisfied.*

The supposition we have already made was that the function r defined in (4.2) is continuous and has partial derivatives on $\Delta = (0, 2\pi) \times (0, \pi)$. We want to see how the functions ρ or r should be taken to assure the boundness of the function $E : \Delta \to \mathbb{R}$,

$$E(u, v) = r\left(r^2 + r_v^2 + \frac{r_u^2}{\sin^2 v}\right)^{1/2}.$$

A natural choice is the following.

Proposition 6.1 *Let $\Omega \subseteq \mathbb{R}^3$ be a domain such that $\mathbb{S}^2 \subseteq int\Omega$. If the function $\rho : \Omega \to (0, \infty)$ is such that $\rho \in C^1(\Omega)$, then the function E is bounded on Δ.*

Proof. Let $m_0,\ M_0,\ M_1$ be real positive numbers such that

$$\begin{aligned} m_0 \le \rho(\eta) &\le M_0, \\ \max\{|\rho_x(\eta)|, |\rho_y(\eta)|, |\rho_z(\eta)|\} &\le M_1, \end{aligned}$$

for all $\eta \in \mathbb{S}^2$. Here ρ_x, ρ_y, ρ_z denote the partial derivatives of the function ρ. Evaluating r_v and r_u we obtain

$$\begin{aligned} r_v &= \rho_x \cos v \cos u + \rho_y \cos v \sin u - \rho_z \sin v, \\ \frac{r_u}{\sin v} &= -\rho_x \sin u + \rho_y \cos u \end{aligned}$$

and further, using the Cauchy-Schwarz inequality we get

$$\begin{aligned} r_v^2 &\le \left(\rho_x^2 + \rho_y^2 + \rho_z^2\right)\left(\cos^2 u \cos^2 v + \cos^2 v \sin^2 u + \sin^2 v\right) = \rho_x^2 + \rho_y^2 + \rho_z^2, \\ \left(\frac{r_u^2}{\sin v}\right)^2 &\le \left(\rho_x^2 + \rho_y^2\right)\left(\sin^2 u + \cos^2 u\right) = \rho_x^2 + \rho_y^2. \end{aligned}$$

With these inequalities we finally get

$$m_0^2 \leq E(u,v) \leq M_0\sqrt{M_0^2 + 5M_1^2},$$

for all $(u,v) \in \Delta$. ■

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.2,133-138,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

On Linear Differential Operators Whose Eigenfunctions are Legendre Polynomials

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Abstract. In this paper we consider linear differential operators whose eigenfunctions are Legendre polynomials. We find necessary and sufficient conditions imposed on eigenvalues for such an operator to be representable as linear combination of some standard linear differential operators.

Key words: Legendre polynomials, linear differential operator, eigenfunction, eigenvalue, Jacobi polynomials.

1. INTRODUCTION

We denote $\{P_m\}_{m=0}^{\infty}$ the system of Legendre polynomials standardized by the condition $P_m(1) = 1$ $(m = 0, 1, \ldots)$. We introduce linear differential operator $\mathcal{D}_{2r}(f; x) = \big((1 - x^2)^r f^{(r)}(x)\big)^{(r)}$ $(r \in N)$ of order $2r$ defined on the set $C_{2r} = \big\{f \colon f^{(2r)} \in C[-1, 1]\big\}$. It is easy to prove (see Lemma 1 below) that $\forall r \in N$ Legendre polynomials P_m $(m+1 \in N)$ are eigenfunctions of the operator $\mathcal{D}_{2r}$ (for $m < r$ it is obvious). It implies that for any differential operator

$$\overline{\mathcal{D}}_{2r} = \lambda_0 I + \sum_{k=1}^{r} \lambda_k \mathcal{D}_{2k}, \tag{1}$$

where $\lambda_i \in R$ $(i = 0, 1, \dots, r)$ and I is the identity operator, Legendre polynomials P_m $(m+1 \in N)$ are its eigenfunctions.

Suppose we have a linear differential operator

$$\underline{\mathcal{D}}_{2r}(f;x) = \sum_{i=0}^{2r} b_i(x) f^{(i)}(x), \quad b_i \in C[-1,1] \ \ (i = \overline{0, 2r}),$$

defined on C_{2r}. As we indicated before, for an operator $\underline{\mathcal{D}}_{2r}$ to be representable in the form (1), it is necessary for the Legendre polynomials to be eigenfunctions of $\underline{\mathcal{D}}_{2r}$, i.e. it is necessary that $\forall m$ $(m+1 \in N)$

$$\underline{\mathcal{D}}_{2r}(P_m) = \mu_m^{(r)} P_m, \quad \mu_m^{(r)} \in R \ (m+1 \in N). \tag{2}$$

We assume now that a linear differential operator $\underline{\mathcal{D}}_{2r}$ satisfies the conditions (2). In this note we find necessary and sufficient conditions imposed on the eigenvalues $\mu_m^{(r)}$ $(m+1 \in N)$ for such an operator to be representable in the form (1).

2. SOME AUXILIARY STATEMENTS

Lemma 1. *For $r, m \in N$, $m \geq r$ we have*

$$\mathcal{D}_{2r}(P_m) = (-1)^r \cdot \frac{(m+r)!}{(m-r)!} P_m. \tag{3}$$

Proof. We denote $P_m^{(\alpha,\beta)}(x)$ $(m+1 \in N)$ Jacobi polynomials orthogonal on $[-1,1]$ with the weight $(1-x)^\alpha (1+x)^\beta$ $(\alpha, \beta > -1)$ and standardized by the conditions

$$P_m^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+m+1)}{m!\Gamma(\alpha+1)} \quad (m+1 \in N).$$

Making use of the formulas

$$P_m^{(r)} = \frac{1}{2^r}(m+1)(m+2)\,\dots\,(m+r) P_{m-r}^{(r,r)} \quad (m \geq r)$$

and

$$\left((1-x^2)^r P_{m-r}^{(r,r)}(x)\right)^{(r)} = (-1)^r 2^r m(m-1)\,\dots\,(m-r+1) P_m(x) \ \ (m \geq r).$$

[2], pp. 75 and 107 respectively, we obtain (3). Lemma 1 is proved.

Lemma 2. *If $m-1 \in N$, $i \in N$, $1 \leq i < m$, then*

$$\sum_{k=i}^{m} (-1)^k \cdot \frac{(m+k)!}{(m-k)!(i+k+1)!(k-i)!} = 0. \tag{4}$$

Proof. We denote S the sum on the left side of (4). By introducing new index of summation $j = k - i$ we get

$$S = (-1)^i \sum_{j=0}^{m-i} (-1)^j \frac{(m+j+i)!}{(m-i-j)!(2i+j+1)!j!}\,. \tag{5}$$

If we denote $n = m + i + 1$, $M = m + i$, we obtain

$$S = (-1)^i \sum_{j=0}^{m-i} (-1)^j \frac{(M+j)!}{j!(m-i-j)!(n-m+i+j)!} \,. \tag{6}$$

The following formula holds true: if $p, m, n \in N$, $p \leq n$, then

$$\frac{n!}{m!} \sum_{i=0}^{p} (-1)^i \frac{(m+i)!}{i!(p-i)!(n-p+i)!} = \frac{(n-m-1)(n-m-2)\dots(n-m-p)}{p!} \,, \tag{7}$$

[3], p. 18. If we put in (6) $m - i = p$ and take into account (7) we get

$$S = (-1)^i \sum_{j=0}^{p} (-1)^j \frac{(M+j)!}{j!(p-j)!(n-p+j)!} = 0.$$

Lemma 2 is proved.

Lemma 3. *If* $m \in N$ *then*

$$\sum_{k=1}^{m} (-1)^k \frac{(m+k)!}{(m-k)!(k+1)!k!} = -1. \tag{8}$$

Proof. If we put in formula (7) $p = m, n = m + 1$, we get

$$\sum_{i=0}^{m} (-1)^i \frac{(m+i)!}{i!(m-i)!(i+1)!} = 0,$$

whence

$$\sum_{i=1}^{m} (-1)^i \frac{(m+i)!}{i!(m-i)!(i+1)!} = -1.$$

Lemma 3 is proved.

3. THE MAIN THEOREM

Theorem. *For a linear differential operator* $\underline{\mathcal{D}}_{2r}$, *defined on* C_{2r} *and satisfying conditions* (2), *to be representable in the form* (1), *it is necessary and sufficient that*

$$\mu_m^{(r)} = \mu_0^{(r)} + \sum_{k=1}^{r} \left(\sum_{i=0}^{k} \mu_i^{(r)} (-1)^i \frac{2i+1}{(i+k+1)!(k-i)!} \right) (-1)^k \cdot \frac{(m+k)!}{(m-k)!} \,, \quad m \geq r+1. \tag{9}$$

If the conditions (9) *hold then*

$$\underline{\mathcal{D}}_{2r} = \mu_0^{(r)} I + \sum_{k=1}^{r} \left(\sum_{i=0}^{k} \mu_i^{(r)} (-1)^i \frac{2i+1}{(i+k+1)!(k-i)!} \right) \mathcal{D}_{2k}. \tag{10}$$

Proof. We prove first that the differential operator

$$\widetilde{\mathcal{D}}_{2r}=\mu_0^{(r)}I+\sum_{k=1}^{r}\left(\sum_{i=0}^{k}\mu_i^{(r)}(-1)^i\frac{2i+1}{(i+k+1)!(k-i)!}\right)\mathcal{D}_{2k} \tag{11}$$

satisfies the conditions

$$\widetilde{\mathcal{D}}_{2r}(P_m)=\mu_m^{(r)}P_m,\quad 0\le m\le r. \tag{12}$$

Taking into consideration (3) and obvious equalities $\mathcal{D}_{2k}(P_m)=0$ $(k>m)$, we get

$$\widetilde{\mathcal{D}}_{2r}(P_m)=\mu_0^{(r)}P_m+\sum_{k=1}^{m}(-1)^k\frac{(m+k)!}{(m-k)!}\sum_{i=0}^{k}\mu_i^{(r)}(-1)^i\frac{2i+1}{(i+k+1)!(k-i)!}P_m. \tag{13}$$

If we change order of summation on the right side of (13), we derive

$$\begin{aligned}\widetilde{\mathcal{D}}_{2r}(P_m)&=\mu_0^{(r)}P_m+\mu_0^{(r)}\sum_{k=1}^{m}(-1)^k\frac{(m+k)!}{(m-k)!(k+1)!k!}P_m\\&\quad+\sum_{i=1}^{m}\mu_i^{(r)}(-1)^i(2i+1)\sum_{k=i}^{m}(-1)^k\frac{(m+k)!}{(m-k)!(i+k+1)!(k-i)!}P_m.\end{aligned}$$

By making use of (4) and (8) we obtain (12).

We will prove now that the conditions (9) are sufficient for the operator $\underline{\mathcal{D}}_{2r}$ to be representable in the form (1). To this end, we find $\widetilde{\mathcal{D}}_{2r}(P_m)$, $m>r$. Taking (3) and (9) into account, we obtain

$$\begin{aligned}\widetilde{\mathcal{D}}_{2r}(P_m)&=\mu_0^{(r)}P_m+\sum_{k=1}^{r}\left(\sum_{i=0}^{k}\mu_i^{(r)}(-1)^i\frac{2i+1}{(i+k+1)!(k-i)!}\right)\mathcal{D}_{2k}(P_m)\\&=\mu_0^{(r)}P_m+\sum_{k=1}^{r}\left(\sum_{i=0}^{k}\mu_i^{(r)}(-1)^i\frac{2i+1}{(i+k+1)!(k-i)!}\right)(-1)^k\frac{(m+k)!}{(m-k)!}P_m\\&=\left[\mu_0^{(r)}+\sum_{k=1}^{r}\left(\sum_{i=0}^{k}\mu_i^{(r)}(-1)^i\frac{2i+1}{(i+k+1)!(k-i)!}\right)(-1)^k\cdot\frac{(m+k)!}{(m-k)!}\right]P_m\\&=\mu_m^{(r)}P_m\quad(m\in N,\ m>r).\end{aligned} \tag{14}$$

It follows from (12) and (14) that

$$\underline{\mathcal{D}}_{2r}(P_m)=\widetilde{\mathcal{D}}_{2r}(P_m),\quad m=0,1,\dots,$$

which, in turn, implies that for any algebraic polynomial f we have

$$\underline{\mathcal{D}}_{2r}(f)=\widetilde{\mathcal{D}}_{2r}(f). \tag{15}$$

Since $\forall f \in C_{2r}$ and $\forall \varepsilon > 0$ there is algebraic polynomial Q such that $\|f^{(k)} - Q^{(k)}\|_{C[-1,1,]} < \varepsilon$ $(k = 0, 1, \ldots, 2r)$, [1], we conclude that (15) holds $\forall f \in C_{2r}$ so that $\underline{\mathcal{D}}_{2r} = \widetilde{\mathcal{D}}_{2r}$ and equality (10) is valid. Sufficiency is proved.

We will now prove necessity, that is we assume that $\underline{\mathcal{D}}_{2r}$ can be represented in the form (1) and we will deduce from this assumption that the relations (9) hold true. We observe first that a linear operator

$$\underline{\mathcal{D}}_{2r} = \nu_0 I + \sum_{k=1}^{r} \nu_k \mathcal{D}_{2k} \quad (\nu_0, \nu_1, \ldots, \nu_r \in R)$$

such that $\underline{\mathcal{D}}_{2r}(P_m) = \mu_m^{(r)} P_m$ $(m = \overline{0, r})$ is unique. This fact is obvious if we take into account (3). It follows from this observation and from relations (11), (12) that

$$\underline{\mathcal{D}}_{2r} = \mu_0^{(r)} I + \sum_{k=1}^{r} \left(\sum_{i=0}^{k} \mu_i^{(r)} (-1)^i \frac{2i+1}{(i+k+1)!(k-i)!} \right) \mathcal{D}_{2k}. \tag{16}$$

We derive from (16) and (3) that $\forall m \in N$, $m \geq r+1$ we have

$$\begin{aligned}\mu_m^{(r)} P_m = \underline{\mathcal{D}}_{2r}(P_m) &= \mu_0^{(r)} P_m + \sum_{k=1}^{r} \left(\sum_{i=0}^{k} \mu_i^{(r)} (-1)^i \frac{2i+1}{(i+k+1)!(k-i)!} \right) \mathcal{D}_{2k}(P_m) \\ &= \mu_0^{(r)} P_m + \sum_{k=1}^{r} \left(\sum_{i=0}^{k} \mu_i^{(r)} (-1)^i \frac{2i+1}{(i+k+1)!(k-i)!} \right) (-1)^k \cdot \frac{(m+k)!}{(m-k)!} P_m,\end{aligned}$$

which implies (9). This completes the proof of the theorem.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.2,139-149,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

REMARK ON THE THREE-STEP ITERATION FOR NONLINEAR OPERATOR EQUATIONS AND NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. The purpose of this paper is to show that the convergence of the three-step iterations suggested by Noor [10–13], Noor et al. [14] for solving nonlinear operator equations, general variational inequalities and multi-valued quasi-variational inclusions in Hilbert spaces or uniformly smooth Banach spaces are equivalent to the that of the Mann iteration.

AMS Mathematics Subject Classification : 49J40, 90C33.

Key words and phrases : Three-step iteration, strongly accretive mapping, strongly pseudocontractive mapping, uniformly smooth Banach space, Mann iteration.

1. Introduction and Preliminaries

Recently, much attention has been given to solve the nonlinear operator equations, general variational inequalities, multi-valued variational inequalities and multi-valued quasi varitional inclusions in uniformly smooth Banach spaces and Hilbert spaces by using the three-step iterative processes. Glowinski and Le Tallec [5] used the three-step iterative schemes for solving elastoviscoplasticity, liquid crystal and eigenvalue problems. Haubruge et al [6] have also studied the convergence analysis of the three-step iteration schemes of Glowwinski and Le Tallec [5] and applied these three-step iterations to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. For applications of the splitting techniques to partial differential equations, see [1] and the referees therein.

In recent years Noor [10–13] and Noor et al. [14] have suggested and analyzed three-step iterative methods for solving nonlinear operator equations, general variational inequalities, multi-valued quasi-variational inclusions and for finding the approximate solutions of the variational inclusions in Hilbert spaces or uniformly smooth Banach spaces.

The purpose of this paper is to prove that the convergence of the three-step iterations suggested by Noor [10–13] and Noor et al. [14] for solving nonlinear operator equations, general variational inequalities and multi-valued quasi-variational inclusions in Hilbert spaces or uniformly smooth Banach spaces are equivalent to the that of the one-step iteration.

For the purpose, we divide our paper into two parts. The first part is devoted to study the equivalence between three-step iteration and one-step iteration for nonlinear accretive operator equations in uniformly smooth Banach spaces. The second part is devoted to study the equivalence between three-step iteration and one-step iteration for general variational inequalities in Hilbert spaces.

2. Nonlinear operator equations in Banach spaces

Throughout this section we assume that E is a real uniformly smooth Banach space, E^* is the dual space of E, K is a nonempty closed convex subset of E, $F(T)$ is the set of fixed points of mapping T and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \ \|f\| = \|x\|\}, \quad x \in E.$$

Definition 2.1. Let $T : E \to E$ be a mapping.

(i) T is said to be *strongly accretive*, if there exists a constant $0 < c < 1$ such that for any $x, y \in E$ there exists a $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y)\rangle \geq c\|x - y\|^2. \tag{2.1}$$

(ii) T is said to be *strongly pseudo-contractive*, if there exists a constant $0 < k < 1$ such that for any $x, y \in E$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y)\rangle \leq k\|x - y\|^2. \tag{2.2}$$

The concept of accretive mapping was at first introduced independently by Browder [2] and Kato [7] in 1967. An early fundamental result in the theory of accretive mapping due to Browder states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0$$

is solvable, if T is locally Lipschitzian and accretive on E.

Definition 2.2. [14] Let $T : K \to K$ be a mapping, $x_0 \in K$ be a given point, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0, 1]$ satisfying some certain conditions. Then the sequence $\{x_n\} \in E$ defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \qquad \forall\, n \geq 0, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n; \end{cases} \tag{2.3}$$

is called the *three-step iteration process*, which was suggested and analyzed by Noor et al. [14] for nonlinear equations in uniformly smooth Banach spaces.

Definition 2.3. [8] For given $u_0 \in K$, the sequence $\{u_n\}$ defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \quad \forall\, n \geq 0 \tag{2.4}$$

is called the *one-step iteration process* (or Mann iteration process [8]), where the sequence $\{\alpha_n\}$ appeared in (2.4) is the same as in (2.3).

Remark 2.1. It is easy to see that if $u_0 = x_0$, $\beta_n = 0$ and $\gamma_n = 0$ for all $n \geq 0$, then the three-step iteration process (2.3) is reduced to the one-step iteration process (2.4).

The following lemmas will be needed in proving our main results.

Lemma 2.1. [4] *Let E be a real Banach space and let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall\, j(x+y) \in J(x+y). \tag{2.5}$$

Lemma 2.2. [3] *E is a uniformly smooth Banach space if and only if the normalized duality mapping J is single-valued and uniformly continuous on any bounded subset of E.*

Lemma 2.3. [15] *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition*:

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n, \quad \forall\, n \geq n_0,$$

where n_0 is some nonnegative integer and $\lambda_n \in [0, 1]$ is a sequence with $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.1. *Let E be a real uniformly smooth Banach space, $T : E \to E$ be a strongly pseudo-contractive mapping with a constant $0 < k < 1$ and the range $R(T)$ of T be bounded. Let $\{x_n\}$ and $\{u_n\}$ be the three-step iteration scheme and one-step iteration scheme defined by* (2.3) *and* (2.4) *respectively, p be a fixed point of T and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0, 1]$ satisfying the following conditions*:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0, \quad \lim_{n\to\infty} \gamma_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{2.6}$$

If $x_0 = u_0$, then $\{x_n\}$ converges strongly to $p \in F(T)$ if and only if $\{u_n\}$ converges strongly to $p \in F(T)$. Furthermore, p is the unique fixed point of T.

Proof. First we prove that $p \in E$ is the unique fixed point of T. In fact, let $p, q \in E$ be two fixed points of T. Since T is strongly pseudo-contractive with constant $0 < k < 1$, we have

$$\|p - q\|^2 = \langle p - q, J(p - q)\rangle = \langle Tp - Tq, J(p - q)\rangle \leq k\|p - q\|^2.$$

This implies that $\|p - q\| = 0$, i.e., $p = q$.

Next we prove the first statement.

Since E is uniformly smooth Banach space, by Lemma 2.2 we know that the normalized duality mapping J is single-valued and uniformly continuous on any bounded subset of E. Again since $T : E \to E$ is strongly pseudo-contractive with $0 < k < 1$, from Lemma 2.1 we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\|^2 &= \|(1 - \alpha_n)(x_n - u_n) + \alpha_n(Ty_n - Tu_n)\|^2 \\ &\leq (1 - \alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n\langle Ty_n - Tu_n, J(x_{n+1} - u_{n+1})\rangle \\ &\leq (1 - \alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n\langle Ty_n - Tu_n, J(y_n - u_n)\rangle \\ &\quad + 2\alpha_n\langle Ty_n - Tu_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\rangle \\ &\leq (1 - \alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n k\|y_n - u_n\|^2 \\ &\quad + 2\alpha_n\langle Ty_n - Tu_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\rangle.\end{aligned} \tag{2.7}$$

Now we consider the second term on the right side of (2.7). From (2.3) we have

$$\begin{aligned}\|y_n - u_n\|^2 &= \|(1 - \beta_n)(x_n - u_n) + \beta_n(Tz_n - u_n\|^2 \\ &\leq (1 - \beta_n)^2\|x_n - u_n\|^2 + 2\beta_n\langle Tz_n - u_n, J(y_n - u_n)\rangle \\ &\leq (1 - \beta_n)^2\|x_n - u_n\|^2 + 2\beta_n\|Tz_n - u_n\|\|y_n - u_n\| \\ &\leq (1 - \beta_n)^2\|x_n - u_n\|^2 + \beta_n\{\|Tz_n - u_n\|^2 + \|y_n - u_n\|^2\}.\end{aligned}$$

Simplifying, we have

$$(1 - \beta_n)\|y_n - u_n\|^2 \leq (1 - \beta_n)^2\|x_n - u_n\|^2 + \beta_n\|Tz_n - u_n\|^2. \tag{2.8}$$

Since $\beta_n \to 0$, there exists a nonnegative integer n_0 such that for $n \geq n_0$, $\beta_n < \frac{1}{2}$, and so $1 - \beta_n > \frac{1}{2}$, for all $n \geq n_0$. Therefore from (2.8) we have

$$\|y_n - u_n\|^2 \leq (1 - \beta_n)\|x_n - u_n\|^2 + 2\beta_n\|Tz_n - u_n\|^2, \quad \forall\, n \geq n_0. \tag{2.9}$$

Since the range $R(T)$ of T is bounded, there exists a constant $M_1 > 0$ such that $\sup_{x \in E}\|Tx\| \leq M_1$. Since $p = Tp$, we have

$$\sup_{n \geq 0}\{\|Tx_n\|, \|Ty_n\|, \|Tz_n\|, \|Tu_n\|, \|p\|\} \leq M_1. \tag{2.10}$$

Putting

$$M = \|x_0\| + M_1,$$

we can prove that

$$\sup_{n\geq 0}\{\|Tx_n\|, \|Ty_n\|, \|Tz_n\|, \|Tu_n\|, \|p\|, \|x_n\|, \|y_n\|, \|z_n\|, \|u_n\|\} \leq M. \quad (2.11)$$

In fact, for $n = 0$ we have

$$\|y_0\| = \|(1-\beta_0)x_0 + \beta_0 Tz_0\| \leq (1-\beta_0)\|x_0\| + \beta_0\|Tz_0\| \leq M;$$
$$\|z_0\| = \|(1-\gamma_0)x_0 + \gamma_0 Tx_0\| \leq (1-\beta_0)\|x_0\| + \gamma_0\|Tx_0\| \leq M.$$

For $n = 1$, noting that $x_0 = u_0$ we have

$$\begin{aligned}
\|x_1\| &= \|(1-\alpha_0)x_0 + \alpha_0 Ty_0\| \leq M;\\
\|y_1\| &= \|(1-\beta_1)x_1 + \beta_1 Tz_1\| \leq M;\\
\|z_1\| &= \|(1-\gamma_1)x_1 + \gamma_1 Tx_1\| \leq M;\\
\|u_1\| &= \|(1-\alpha_0)u_0 + \alpha_0 Tu_0\| \leq M.
\end{aligned}$$

By induction, we can prove that (2.11) is true.
It follows from (2.11)and (2.9) that

$$\begin{aligned}
\|y_n - u_n\|^2 &\leq (1-\beta_n)\|x_n - u_n\|^2 + 2\beta_n \cdot 4M^2\\
&= (1-\beta_n)\|x_n - u_n\|^2 + 8M^2\beta_n \qquad (2.12)\\
&\leq \|x_n - u_n\|^2 + 8M^2\beta_n, \quad \forall\, n \geq n_0.
\end{aligned}$$

Now we consider the third term on the right side of (2.7). We have

$$\begin{aligned}
&2\alpha_n\langle Ty_n - Tu_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\rangle\\
&\leq 2\alpha_n\|Ty_n - Tu_n\|\|J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\| \qquad (2.13)\\
&\leq 4M\alpha_n\|J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\beta_n \to 0$, we have

$$\begin{aligned}
&\|x_{n+1} - u_{n+1} - (y_n - u_n)\|\\
&= \|(1-\alpha_n)(x_n - u_n) + \alpha_n(Ty_n - Tu_n) - (1-\beta_n)(x_n - u_n) - \beta_n(Tz_n - u_n)\|\\
&\leq |\alpha_n - \beta_n|\|x_n - u_n\| + \alpha_n\|Ty_n - Tu_n\| + \beta_n\|Tz_n - u_n\|\\
&\leq 2M|\alpha_n - \beta_n| + 2M(\alpha_n + \beta_n) \to 0, \quad (n \to \infty). \qquad (2.14)
\end{aligned}$$

By the uniform continuity of J, it follows from (2.14) that

$$e_n := \|J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\| \to 0, \quad \text{as } n \to \infty. \quad (2.15)$$

Therefore, it follows from (2.15), (2.13), (2.12) and (2.7) that

$$
\begin{aligned}
\|x_{n+1} - u_{n+1}\|^2 &\le (1-\alpha_n)^2 \|x_n - u_n\|^2 \\
&\quad + 2\alpha_n k\{\|x_n - u_n\|^2 + 8M^2\beta_n\} + 4M\alpha_n e_n \\
&= (1 - 2\alpha_n(1-k))\|x_n - u_n\|^2 \\
&\quad + \alpha_n\{\alpha_n\|x_n - u_n\|^2 + 16M^2 k\beta_n + 4Me_n\} \\
&\le (1 - 2\alpha_n(1-k))\|x_n - u_n\|^2 \\
&\quad + \alpha_n\{4M^2\alpha_n + 16M^2 k\beta_n + 4Me_n\}, \quad \forall\, n \ge n_0.
\end{aligned} \tag{2.16}
$$

Taking $a_n = \|x_n - u_n\|^2$, $\lambda_n = 2\alpha_n(1-k)$ and $b_n = \alpha_n\{4M^2\alpha_n + 16M^2k\beta_n + 4Me_n\}$ in (2.16), we have

$$
a_{n+1} \le (1-\lambda_n)a_n + b_n, \quad \forall\, n \ge n_0.
$$

Since $\alpha_n \to 0$, there exists a positive integer $n_1 \ge n_0$ such that $\lambda_n \in [0,1]$, for all $n \ge n_1$. Again since $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have $\sum_{n=0}^{\infty} \lambda_n = \infty$. Moreover, we have that $b_n = o(\lambda_n)$. Therefore by Lemma 2.3 we know that $a_n \to 0$, as $n \to \infty$, and so $\|x_n - u_n\| \to 0$, as $n \to \infty$. Therefore if $x_n \to p$, then we have

$$
\|u_n - p\| \le \|u_n - x_n\| + \|x_n - p\| \to 0, \quad \text{as } \ n \to \infty.
$$

Conversely, if $u_n \to p$, then we have

$$
\|x_n - p\| \le \|x_n - u_n\| + \|u_n - p\| \to 0, \quad \text{as } \ n \to \infty.
$$

This completes the proof of Theorem 2.1. $\square$

Remark 2.2. We also, can easily prove that the one-step sequence $\{u_n\}$ converges to $p \in F(T)$. In fact, from (2.4) and Lemma 2.1 we have

$$
\begin{aligned}
\|u_{n+1} - p\|^2 &= \|(1-\alpha_n)(u_n - p) + \alpha_n(Tu_n - Tp)\|^2 \\
&\le (1-\alpha_n)^2\|u_n - p\|^2 + 2\alpha_n\langle Tu_n - Tp, J(u_{n+1} - p)\rangle \\
&\le (1-\alpha_n)^2\|u_n - p\|^2 + 2\alpha_n\langle Tu_n - Tp, J(u_n - p)\rangle \\
&\quad + 2\alpha_n\langle Tu_n - Tp, J(u_{n+1} - p) - J(u_n - p)\rangle \\
&\le (1-\alpha_n)^2\|u_n - p\|^2 + 2\alpha_n k\|u_n - p\| \\
&\quad + 2\alpha_n\|Tu_n - Tp\| \cdot \|J(u_{n+1} - p) - J(u_n - p)\| \\
&\le (1 - 2\alpha_n(1-k))\,\|u_n - p\|^2 + \alpha_n^2\|u_n - p\|^2 \\
&\quad + 4M\alpha_n\|J(u_{n+1} - p) - J(u_n - p)\| \\
&\le (1 - 2\alpha_n(1-k))\,\|u_n - p\|^2 + 4M\alpha_n^2 + 4M\alpha_n c_n \\
&\le (1 - 2\alpha_n(1-k))\,\|u_n - p\|^2 + 4M\alpha_n\{\alpha_n + c_n\},
\end{aligned} \tag{2.17}
$$

where $c_n := \|J(u_{n+1}-p) - J(u_n - p)\|$. Since

$$\begin{aligned}\|u_{n+1} - p - (u_n - p)\| &= \|u_{n+1} - u_n\| \\ &= \|\alpha_n(Tu_n - u_n)\| \\ &\le 2M\alpha_n \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

from the uniform continuity of J we know that

$$c_n \to 0, \quad \text{as } n \to \infty. \tag{2.18}$$

Taking $a_n = \|u_n - p\|^2$, $\lambda_n = 2\alpha_n(1-k))$ and $b_n = 4M\alpha_n\{\alpha_n + c_n\}$ in (2.17), we have

$$a_{n+1} \le (1-\lambda_n)a_n + b_n.$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\alpha_n \to 0$, as $n \to \infty$, there exists a positive integer n_2 such that $\lambda_n \in [0,1]$, for all $n \ge n_2$ and $\sum_{n=0}^{\infty}\lambda_n = \infty$. Again since $b_n = o(\lambda_n)$, by Lemma 2.3 we know that $a_n \to 0$, hence $u_n \to p \in F(T)$, as $n \to \infty$.

Remark 2.3. Theorem 2.1 shows that for solving nonlinear accretive operator equations in a uniformly smooth Banach space, we can use the simple one-step iteration to replace the complicated three-step iteration suggested and analyzed in Noor et al. [14].

3. General variational inequalities in Hilbert spaces

Throughout this section we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H.

For given nonlinear operator $T, g : H \to H$ consider the problem of finding $u \in H$, $g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \ge 0, \quad \forall\, g(u) \in K. \tag{3.1}$$

This kind of inequality is called a general variational inequality which was introduced and studied by Noor [9] in 1988.

Definition 3.1. Let $T : H \to H$ be a mapping.

(i) T is said to be *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|x-y\|^2, \quad \forall\, x, y \in H;$$

(ii) T is said to be *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \le \beta\|x - y\|, \quad \forall\, x, y \in H.$$

From (i) and (ii), we know that $\alpha \leq \beta$.

Recently, in [10] Noor suggested the following three-step iteration for solving the general variational inequality (3.1):

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{w_n - g(w_n) + P_K\left(g(w_n) - \rho T w_n\right)\}, \\ w_n = (1-\beta_n)x_n + \beta_n\{y_n - g(y_n) + P_K\left(g(y_n) - \rho T y_n\right)\}, \\ y_n = (1-\gamma_n)x_n + \gamma_n\{x_n - g(x_n) + P_K\left(g(x_n) - \rho T x_n\right)\}; \end{cases} \tag{3.2}$$

where $x_0 \in H$ is a given point and $P_K : H \to K$ is the projection operator, and proved the following theorem:

Theorem 3.1. [10] *Let the mappings $T, g : H \to H$ be both strongly monotone with constants $\alpha > 0$, $\sigma > 0$ and Lipschitz continuous with constants $\beta > 0$, $\delta > 0$, respectively.*

(1) *If*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{3.3}$$

where

$$k = 2\sqrt{1 - 2\sigma + \delta^2}, \tag{3.4}$$

then there exists a unique solution $p \in H$, $g(u) \in K$ of the general variational inequality (3.1)*.*

(2) *If $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the three-step iteration $\{x_n\}$ defined by* (3.2) *converges strongly to the exact solution p of the general variational inequality* (3.1)*.*

In the sequel, we shall prove that under the conditions given in Theorem 3.1, the convergence of the three-step iteration process $\{x_n\}$ defined by (3.2) is equivalent to the that of the one-step iteration $\{u_n\}$ defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_K\left(g(u_n) - \rho T u_n\right)\}, \tag{3.5}$$

where $u_0 \in H$ is a given point and $\{\alpha_n\}$ is the same as given in (3.2).

In order to prove our result, we need the following lemma:

Lemma 3.1. [9] *$p \in H$ is a solution of the general variational inequality* (3.1) *if and only if $p \in H$ satisfies the condition:*

$$g(p) = P_K\left(g(p) - \rho T p\right),$$

where $\rho > 0$ is a constant.

Theorem 3.2. *Let the mappings T, $g : H \to H$ satisfy all the assumptions of Theorem* 3.1. *Let $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$, for all $n \geq 0$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $x_0 = u_0$ and if the condition* (3.3) *is satisfied, then the convergence of the three-step iteration $\{x_n\}$ defined by* (3.2) *is equivalent to the that of the one-step iteration $\{u_n\}$ defined by* (3.5), *i.e., $x_n \to p$ (the solution of general variational inequality* (3.1)) *if and only if $u_n \to p$.*

Proof. Necessity. If $x_n \to p$, as $n \to \infty$, then take $\beta_n = \gamma_n = 0$, for all $n \geq 0$ in (3.2) and noting that $x_0 = u_0$, it is easy to see that $u_n \to p$.

Sufficiency. Let $u_n \to p$, next we prove that $x_n \to p$. In fact, since p is the unique solution of general variational inequality (3.1), by Lemma 3.1, we have

$$\begin{aligned} p &= (1-\alpha_n)p + \alpha_n\{p - g(p) + P_K\,(g(p) - \rho Tp)\} \\ &= (1-\beta_n)p + \beta_n\{p - g(p) + P_K\,(g(p) - \rho Tp)\} \\ &= (1-\gamma_n)p + \gamma_n\{p - g(p) + P_K\,(g(p) - \rho Tp)\}. \end{aligned} \tag{3.6}$$

From (3.2) and (3.5), we have

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|(1-\alpha_n)(x_n - u_n) + \alpha_n\{w_n - u_n - (g(w_n) - g(u_n))\} \\ &\quad + \alpha_n\{P_K\,(g(w_n) - \rho Tw_n) - P_K\,(g(u_n) - \rho Tu_n)\}\| \\ &\leq (1-\alpha_n)\|x_n - u_n\| + 2\alpha_n\|w_n - u_n - (g(w_n) - g(u_n))\| \\ &\quad + \alpha_n\|w_n - u_n - \rho(Tw_n - Tu_n)\|. \end{aligned} \tag{3.7}$$

First we consider the third term on the right side of (3.7). Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} \|w_n - u_n - \rho(Tw_n - Tu_n)\|^2 &= \|w_n - u_n\|^2 - 2\rho\langle Tw_n - Tu_n, w_n - u_n\rangle \\ &\quad + \rho^2\|Tw_n - Tu_n\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|w_n - u_n\|^2. \end{aligned}$$

Therefore, we have

$$\|w_n - u_n - \rho(Tw_n - Tu_n)\| \leq t(\rho)\|w_n - u_n\|, \tag{3.8}$$

where $t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}$.

Now we consider the second term on the right side of (3.7). Since the operator g is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$, in a similar way we have,

$$\|w_n - u_n - (g(w_n) - g(u_n))\|^2 \leq (1 - 2\sigma + \delta^2)\|w_n - u_n\|^2.$$

Therefore, we have

$$\|w_n - u_n - (g(w_n) - g(u_n))\| \leq \sqrt{1 - 2\sigma + \delta^2}\|w_n - u_n\|. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7) and simplifying, we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-\alpha_n)\|x_n-u_n\|+\alpha_n\left(t(\rho)+k\right)\|w_n-u_n\| \\ &= (1-\alpha_n)\|x_n-u_n\|+\alpha_n\theta\|w_n-u_n\|,\end{aligned} \tag{3.10}$$

where $k := 2\sqrt{1-2\sigma+\delta^2}$ and $\theta := k+t(\rho)$. By the condition (3.3), it is easy to prove that

$$0<\theta<1.$$

Finally we consider the last term on the right side of (3.10). We have

$$\|w_n-u_n\| \le \|w_n-p\|+\|u_n-p\|, \quad \forall\, n\ge 0. \tag{3.11}$$

In a similar way as given in the proof of (3.8), (3.9) and (3.10), it follows from (3.2) and (3.6) that

$$\begin{aligned}\|w_n-p\| &\le (1-\beta_n)\|x_n-p\|+2\beta_n\|y_n-p-(g(y_n-g(p))\| \\ &\quad+\beta_n\|y_n-p-\rho(Ty_n-Tp)\| \\ &\le (1-\beta_n)\|x_n-p\|+\beta_n\left(k+t(\rho)\right)\|y_n-p\| \\ &= (1-\beta_n)\|x_n-p\|+\beta_n\cdot\theta\|y_n-p\|.\end{aligned} \tag{3.12}$$

Similarly, it follows from (3.2) and (3.6) that

$$\begin{aligned}\|y_n-p\| &\le (1-\gamma_n)\|x_n-p\|+\gamma_n\theta\|x_n-p\| \\ &\le (1-\gamma_n)\|x_n-p\|+\gamma_n\|x_n-p\| \\ &= \|x_n-p\|.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13) we have

$$\begin{aligned}\|w_n-p\| &\le (1-\beta_n)\|x_n-p\|+\beta_n\cdot\theta\|x_n-p\| \\ &\le \|x_n-p\|.\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.11), it gets that

$$\begin{aligned}\|w_n-u_n\| &\le \|x_n-p\|+\|u_n-p\| \\ &\le \|x_n-u_n\|+2\|u_n-p\|.\end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.10), we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-\alpha_n)\|x_n-u_n\| \\ &\quad+\alpha_n\theta\left\{\|x_n-u_n\|+2\|u_n-p\|\right\} \\ &= \left(1-\alpha_n(1-\theta)\right)\|x_n-u_n\|+2\alpha_n\theta\|u_n-p\|.\end{aligned} \tag{3.16}$$

Let $a_n = \|x_n-u_n\|$, $\lambda_n = \alpha_n(1-\theta)$ and $b_n = 2\alpha_n\theta\|u_n-p\|$. Therefore (3.16) can be written as follows

$$a_{n+1} \le (1-\lambda_n)a_n+b_n, \quad \forall\, n\ge 0.$$

Since $0<\theta<1$ and by the assumption, $\sum_{n=0}^{\infty}\alpha_n=\infty$, hence $\lambda_n\in[0,1]$ and $\sum_{n=0}^{\infty}\lambda_n=\infty$. Again since $\|u_n-p\|\to 0$, as $n\to\infty$, we know that $b_n=o(\lambda_n)$. Therefore by Lemma 2.3, $\|x_n-u_n\|\to 0$, and so

$$\|x_n-p\| \le \|x_n-u_n\|+\|u_n-p\| \to 0, \quad \text{as } n\to\infty.$$

That is, $x_n\to p$, as $n\to\infty$. This completes the proof of Theorem 3.2. □

Remark 3.1.

(1) Theorem 3.2 shows that the convergence of the three-step iteration (3.2) suggested and analyzed by Noor [10] is equivalent to the that of the one-step iteration (3.5). Therefore in order to solve the general variational inequality (3.1) in Hilbert spaces, we can use the simple one-step iteration (3.5) to replace the complicated three-step iteration (3.2).

(2) At the end of this paper we would like to point out that the three-step iterations suggested and analyzed by Noor [11–13] for solving multi-valued quasi-variational inclusions and multi-valued variational inequalities also can be replaced by the corresponding one-step iterations. Owing to the methods of proof are similar, the details are omitted.

Acknowledgements. This work was supported by the Kyungnam University Research Fund. 2005.

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An Exponentially Fitted Finite Difference Scheme for Solving Boundary-Value Problems for Singularly-Perturbed Differential-Difference Equations: Small Shifts of Mixed Type with Layer Behavior

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Abstract : In this paper, we study a numerical approach to find the solution of the boundary-value problems for singularly perturbed differential-difference equations with small shifts. Similar boundary-value problems are associated with expected first-exit time problems of the membrane potential in models for activity of neuron [2-6] and in variational problems in control theory. Here we propose an exponentially fitted method based on finite difference to solve boundary-value problem for a singularly perturbed differential-difference equation with small shifts of mixed type, i.e., which contains both type of terms having negative shift as well as positive shift and consider the case in which the solution of the problem exhibits layer behavior. We calculate the fitting parameter for the exponentially fitted finite difference scheme corresponding to the problem and establish the error estimate which shows that the method converges to the solution of the problem. The effect of small shifts on the boundary layer solution is shown by considering the numerical experiments. The numerical results for several test examples demonstrate the efficiency of the method.

Key Words: Singular perturbation, differential-difference equation, fitting parameter, exponentially fitted, negative shift, positive shift, boundary layer.

1 Introduction

We continue the study of boundary-value problems for singularly-perturbed differential-difference equations with small shifts. There are many biological and physical models in which one can encounter such type of problems, e.g., in variational problem in control theory and first-exit time problem in modeling of determination of expected time for generation of action potentials in nerve cell by random synaptic inputs in the dendrites [2, 6].

On the theoretical side there have been many advanced models for activation of nerve membrane potential in the presence of random synaptic inputs in the dendrites. Reviews can be found in Segundo et. al [11], Fienberg and Holden [1]. Stein first gave a fairly realistic DDE model incorporating stochastic effects due to neuronal excitation, in which after refractory period, excitatory and inhibitory exponentially decaying inputs of constant size occur at random intervals and add up until a threshold value is reached. In [14], Stein generalized his model to handle a distribution of post-synaptic potential amplitudes and then approximating the solution using Monte-Carlo technique. Other methods for obtaining approximate solution have since been developed by Tuckwell and cope, Tuckwell and Richter [9, 10] and Wilber and Rinzel [15].

In [2], Lange and Miura considered the problem of determining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites. The general boundary-value problem for the linear second-order differential-difference equation that arises in the modeling of activation of neuron is

$$(\sigma^2/2)y''(x) + (\mu - x)y'(x) + \lambda_E y(x + a_E) + \lambda_I y(x - a_I) - (\lambda_E + \lambda_I)y(x) = -1,$$

where σ and μ are the variance and drift parameters, respectively and y is the expected first-exit time. The first order derivative term $-xy'(x)$ corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs modeled as Poisson process with mean rates λ_E and λ_I, respectively, and produce jumps in the membrane potential of amounts a_E and a_I, respectively, which are small quantities and could depend on voltage. The boundary condition is

$$y(x) = 0, \quad \forall\, x \notin (x_1, x_2),$$

where the values $x = x_1$ and $x = x_2$ correspond to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively.

This biological problem motivates the investigation of boundary-value problems for differential-difference equations with small shifts. The singular perturbation analysis of boundary-value problems for differential -difference equations with delay has been given in a series of papers by Lange and Miura [2-6] and they presented an asymptotic approach for solving such type of problems. In their papers [2] and [6], the study of BVPs for the two classes of singularly perturbed differential-difference equations with small shifts is given and the effect of small shifts is shown on the boundary layer solution [2] and on the oscillatory solution [6]. It is also pointed out that shifts affect oscillatory solution more than the boundary layer solution and the term having shift can be expanded using Taylor series, provided shift is of small order of parameter ε. In the above biological model, the shifts are due to the jumps in the potential membrane which are very small. There, the biologist Hutchinson [8] states "there is a tendency for the time lag to be reduced as much as possible by natural selection". Thus arguments for small delay problems are found through out the literature on epidemics and population [8]. Hence the small shift plays an important role in practical problems.

We make a numerical study for a class of BVPs for singularly-perturbed differential difference equations with small shifts of mixed type(i.e., which contains both the terms having negative as well as positive shifts) and present an exponentially fitted finite difference numerical scheme to solve such types of boundary-value problems. We show that the scheme is ε-uniform convergent of order h by proving the error estimate. In this method, we first approximate the terms containing shifts by Taylor series and then apply exponentially fitted finite difference scheme, provided the shifts are of small order of ε. The effect of small shifts on the boundary layer solution of the problem is shown by considering several numerical experiments.

2 Statement of the Problem

Here, we consider the boundary-value problem for a singularly perturbed differential-difference equation of mixed type (i.e. which contain both terms having positive shift and negative shift) with small shifts and with boundary layer behavior given by

$$\varepsilon^2 y''(x) + \alpha(x)y(x-\delta) + w(x)y(x) + \beta(x)y(x+\eta) = f(x), \tag{1}$$

on $[0,1]$,
under the boundary conditions

$$\begin{aligned} y(x) &= \phi(x), \quad -\delta \le x \le 0, \\ y(x) &= \psi(x), \quad 1 \le x \le 1+\eta. \end{aligned} \tag{2}$$

where ε is small parameter, $0 < \varepsilon \ll 1$, δ and η are also small shift parameters, $0 < \delta \ll 1$ and $0 < \eta \ll 1$; $\alpha(x)$, $\beta(x)$, $f(x)$, $\delta(\varepsilon)$, $\eta(\varepsilon)$, $\phi(x)$ and $\psi(x)$ are smooth functions. For a function $y(x)$ to be a smooth solution of the problem (1), it must satisfy Eq. (1) with the given boundary conditions (2), be continuous on $[0,1]$ and continuously differentiable on $(0,1)$. We present a numerical method to study the above problem under the condition $(\alpha(x)+\beta(x)+w(x)) < 0$, $x \in [0,1]$, i.e., when it exhibits a boundary layer solution.

2.1 Numerical Scheme

We have by Taylor series expansion

$$\begin{aligned} y(x-\delta) &\approx y(x) - \delta y'(x), \\ y(x+\eta) &\approx y(x) + \eta y'(x). \end{aligned} \tag{3}$$

From (1), (2) and (3), we obtain

$$\varepsilon^2 y''(x) + (\beta(x)\eta - \alpha(x)\delta)y'(x) + (\alpha(x)+\beta(x)+w(x))y(x) = f(x). \tag{4}$$

on $[0,1]$,
under the boundary conditions

$$\begin{aligned} y(0) &= \phi_0, \\ y(1) &= \psi_0. \end{aligned} \tag{5}$$

For discretizing the problem BVP (4), (5), we place an uniform mesh Ω_0^N of size $h = 1/N$ on the interval $[0,1]$. After discretization of the problem using exponentially fitted finite difference scheme, we obtain

$$\varepsilon^2 \rho_i(\tau) D_+ D_- y_i + (\eta\beta(x_i) - \delta\alpha(x_i))D_- y_i + (\alpha(x_i)+\beta(x_i)+w(x_i)) = f(x_i), \tag{6}$$
$$i = 1, \ldots, N-1.$$

with the boundary conditions

$$\begin{aligned} y(0) &= \phi_0, \\ y(1) &= \psi_0, \end{aligned} \tag{7}$$

where

$$\rho_i(\tau) = \frac{(\eta\beta(x_i) - \delta\alpha(x_i))\tau[1 - \exp(-\tau(\eta\beta(x_i) - \delta\alpha(x_i)))]}{4(\sinh(\tau(\eta\beta(x_i) - \delta\alpha(x_i))/2))^2}$$

is a fitting parameter with $\tau = h/\varepsilon^2$.

2.1.1 Calculation of Fitting Parameter

To compute the fitting parameter, we first prove the following Lemma.

Lemma. Let $\hat{y} = y_o + z_o$ be the zeroth order asymptotic approximation to the solution, where y_o represents the zeroth order approximate outer solution(i.e., the solution of the reduced problem) and z_o represents the zeroth order approximate solution in the boundary layer region. Also we assume that the scheme (6) is uniformly convergent, then for a fixed positive integer n

$$\lim_{h\to 0} y(nh) = y_o(0) + (\phi_0 - y_o(0)) \exp(-n(\eta\beta(0) - \delta\alpha(0))\tau). \tag{8}$$

Proof

We have

$$\begin{aligned} |L(y(x) - \hat{y}(x))| &\leq |L(y(x)) - L(y_o(x))| + |L(z_o(x))|, \\ &= |f(x) - \varepsilon^2 y_o''(x) - (\eta\beta(x) - \delta\alpha(x))y_o'(x) \\ &\quad -(\alpha(x) + \beta(x) + w(x))y_o(x)| \\ &+ |\frac{d^2 z_o(\nu)}{d\nu^2} + (\eta\beta(x) - \delta\alpha(x))\frac{dz_o(\nu)}{d\nu} \\ &\quad +(\alpha(x) + \beta(x) + w(x))z_o(\nu)|, \end{aligned}$$

where $\nu = x/\varepsilon^2$.
Since y_o and z_o are the solutions of the reduced problem

$$(\eta\beta(x_i) - \delta\alpha(x_i))y_o'(x) + (\alpha(x) + \beta(x) + w(x))y_o(x) = f(x),$$

$$y_o(1) = \gamma,$$

and of the boundary-value problem

$$\frac{d^2 z_o(\nu)}{d\nu^2} + (\eta\beta(0) - \delta\alpha(0))\frac{dz_o(\nu)}{d\nu} = 0,$$

$$z_o(0) = (\phi_0 - y_o(0))$$
$$z_o(\infty) = 0,$$

respectively and using Taylor series for $(\eta\beta(x) - \delta\alpha(x))$, we obtain

$$\begin{aligned} |L(y(x) - \hat{y}(x))| \quad & \leq \varepsilon^2 |y_o''(x)| + |\nu(\eta\beta'(\xi) - \delta\alpha'(\xi))(\xi)\frac{dz_o(\nu)}{d\nu} \\ & +(\alpha(x) + \beta(x) + w(x))z_o(\nu)|, \end{aligned}$$

where $\xi \in (0, 1)$.
Since $z_o(\nu) = (\phi_0 - y_o(0)) \exp(-\nu(\eta\beta(0) - \delta\alpha(0)))$, we get

$$\begin{aligned} |L(y(x) - \hat{y}(x))| \leq \varepsilon^2 |y_o''(x)| + |[-(\eta\beta'(\xi) - \delta\alpha'(\xi))\nu(\eta\beta(0) - \delta\alpha(0)) + (\alpha(x) \\ +\beta(x) + w(x))](\phi_0 - y_o(0)) \exp(-\nu(\eta\beta(0) - \delta\alpha(0)))|, \end{aligned}$$

now if $(\eta\beta(x) - \delta\alpha(x))$ is monotonically decreasing then $(\eta\beta'(\xi) - \delta\alpha'(\xi)) < 0$, using this and the fact that $t\exp(-t) \leq \exp(-t/2)$, in the above inequality, we obtain

$$\begin{aligned} |L(y(x) - \hat{y}(x))| \quad \leq \quad & \varepsilon^2 |y_o''(x)| + |(\eta\beta'(\xi) - \delta\alpha'(\xi))(\phi_0 - y_o(0))| \\ & . \exp\{(-x(\eta\beta(0) - \delta\alpha(0)))/2\varepsilon^2\}. \end{aligned}$$

Since $y_o''(x)$ is bounded independently of ε for sufficiently smooth $(\eta\beta(x) - \delta\alpha(x))$, $(\alpha(x) + \beta(x) + w(x))$ and $f(x)$, so there exists a positive constant C_1, s.t $|y_o''(x)| \leq C_1$ for $x \in (0, 1)$ using this fact in the above inequality, we obtain

$$|L(y(x) - \hat{y}(x))| \leq \varepsilon^2 C_2[C' + \frac{1}{\varepsilon^2} \exp(-x(\eta\beta(0) - \delta\alpha(0))/2\varepsilon^2)], \tag{9}$$

where $C_2 = |a'(\xi)(\phi_0 - y_o(0))|$ and $C' = C_1/C_2$.

Now let us introduce a barrier function

$$\psi(x) = (1 - x/2)A\varepsilon^2 + B\varepsilon^2 \exp\{-Mx/\varepsilon^2\} \pm (\hat{y}(x) - y(x)),$$

where A and B are positive constants. We have

$$\begin{aligned} L(\psi(x)) \quad &= \quad \varepsilon^2\psi''(x) + (\eta\beta(x) - \delta\alpha(x))\psi'(x) + (\alpha(x) + \beta(x) + w(x))\psi(x) \\ &= \quad -AM\varepsilon^2/2 + BM[M - (\eta\beta(x) - \delta\alpha(x))] \exp\{-Mx/\varepsilon^2)\} \\ & \quad +(\alpha(x) + \beta(x) + w(x))[(1 - x/2)A\varepsilon^2 + B\varepsilon^2 \exp\{-Mx/\varepsilon^2\}] \\ & \quad \pm L(\hat{y}(x) - y(x)), \end{aligned}$$

using assumption on $(\eta\beta(x)-\delta\alpha(x))$(i.e., $(\eta\beta(x)-\delta\alpha(x)) \geq M > 0$) and inequality (9), we obtain

$$\begin{aligned} L(\psi(x)) \;\; &\leq \;\; -AM\varepsilon^2/2 + (\alpha(x) + \beta(x) + w(x))[(1 - x/2)A\varepsilon^2 + B\varepsilon^2 \\ &\qquad . \exp\{-Mx/\varepsilon^2\}] + \varepsilon^2 C_2[C' + \{\exp(-x(\eta\beta(0) - \delta\alpha(0))/2\varepsilon^2)\}/\varepsilon^2]. \end{aligned}$$

Now since first and second terms are non-positive while third term is positive on right side of the above inequality, so we choose the constants A and B such that the total of the negative terms dominate the positive term. Thus we obtain

$$L(\psi(x)) \leq 0, \tag{10}$$

also one can easily show that $\psi(x) \geq 0$ at the both ends of the interval $[0, 1]$, then by maximum principle, we obtain

$$\psi(x) \geq 0,$$

after simplification, we obtain

$$|y(x) - y_o(0) - (\phi_0 - y_o(0)) \exp\{-x(\eta\beta(0) - \delta\alpha(0))/\varepsilon^2\}| \leq C\varepsilon^2. \tag{11}$$

Which gives the required result.

□

Now assume that the solution of (6), (7) converges ε uniformly to solution of BVP (4), (5). This implies that $f(x_i) - (\alpha(x_i) + \beta(x_i) + w(x_i))y_i$ is bounded. From (6), we have

$$\begin{aligned} \varepsilon^2\rho_i(\tau)(y_{i-1} - 2y_i + y_{i+1})/h^2 + (\eta\beta(x_i) - \delta\alpha(x_i))(y_{i+1} - y_i)/h \\ = f(x_i) - (\alpha(x_i) + \beta(x_i) + w(x_i))y_i, \end{aligned} \tag{12}$$

Now multiplying Eq. (12) by h for $i = n$ and then taking limit as $h \to 0$, we obtain

$$\lim_{h\to 0}[(\rho_n(\tau)/\tau)(y_{n-1} - y_n + y_{n+1}) + (\eta\beta(x_n) - \delta\alpha(x_n))(y_{n+1} - y_n)] = 0.$$

We use the assumption that the scheme (6) is uniformly convergent, so we replace y_{N-i} by $y((N-i)h)$ in the above equation, we obtain

$$\begin{aligned} &\lim_{h\to 0}[(\rho_n(\tau)/\tau)\{y((n-1)h) - y(nh) + y((n+1)h)\} \\ &+(\eta\beta(nh) - \delta\alpha(nh))\{y((n+1)h) - y(nh)\}] = 0. \end{aligned}$$

Now using the above Lemma in the above equation, we obtain

$$\lim_{h\to 0}(\rho_n(\tau)/\tau)(\phi_0 - y_o(0))\exp\{-n\tau(\eta\beta(0)-\delta\alpha(0))\}[\exp\{\tau(\eta\beta(0)-\delta\alpha(0))\} \\ -2+\exp\{-\tau(\eta\beta(0)-\delta\alpha(0))\}] + (\eta\beta(0)-\delta\alpha(0))(\phi_0 - y_o(0)) \\ .\exp\{-n\tau(\eta\beta(0)-\delta\alpha(0))\}[\exp\{-\tau(\eta\beta(0)-\delta\alpha(0))\}-1] = 0.$$

Which implies that

$$\lim_{h\to 0}\frac{\rho_n(\tau)}{\tau} = \frac{(\eta\beta(0)-\delta\alpha(0))[1-\exp(-\tau(\eta\beta(0)-\delta\alpha(0)))]}{4(\sinh(\tau(\eta\beta(0)-\delta\alpha(0))/2))^2}. \tag{13}$$

On simplification, Eq. (6) reduces to following tridiagonal system of difference equations

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{14}$$

where

$$\begin{aligned} E_i &= \varepsilon^2\rho_i(\tau)/h^2, \\ F_i &= \varepsilon^2\rho_i(\tau)/h^2 + (\eta\beta(x_i)-\delta\alpha(x_i))/h - (\alpha(x_i)+\beta(x_i)+w(x_i)), \\ G_i &= \varepsilon^2\rho_i(\tau)/h^2 + (\eta\beta(x_i)-\delta\alpha(x_i))/h, \\ H_i &= f(x_i), \qquad \mathrm{i} = 1,2,\ldots,N-1. \end{aligned}$$

The difference equations (14) form a tridiagonal system of $N-1$ equations with $N+1$ unknowns $y_0, y_1, \ldots, y_N$. The $N-1$ equations together with the given two boundary conditions are sufficient to solve the system. The coefficient matrix of such system of equations is non-singular, if it is either strictly diagonally dominant or irreducible diagonally dominant [13]. To solve this system of difference equations, we will use discrete invariant imbedding algorithm.

2.1.2 Discrete Invariant Imbedding Algorithm

Let us set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i, \tag{15}$$

where

$W_i = W(x_i)$ and $T_i = T(x_i)$ are to be determined.

From Eq. (15), we have

$$y_{i-1} = W_{i-1}y_i + T_{i-1}. \tag{16}$$

Using Eq. (16) in (14), we obtain

$$y_i = \frac{G_i}{(F_i - E_iW_{i-1})}y_{i+1} + \frac{E_iT_{i-1} - H_i}{(F_i - E_iW_{i-1})}. \tag{17}$$

By comparing Eq. (15) and Eq. (17), we get recurrence relations for W_i and T_i

$$W_i = G_i/(F_i - E_i W_{i-1}),$$

$$T_i = (E_i T_{i-1} - H_i)/(F_i - E_i W_{i-1}).$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1.$, we need the initial conditions for W_0 and T_0.
By the given boundary conditions, we have

$$y_0 = \phi_0 = W_0 y_1 + T_0,$$

if we choose $W_0 = 0$, then $T_0 = \phi_0$. Now by using these initial conditions, we can compute W_i and T_i for $i = 1, 2, \ldots, N-1$ and using these values of W_i and T_i in Eq. (15), we obtain y_i for $i = 1, 2, \ldots, N-1$.
Under the conditions

$$E_i > 0,\ G_i > 0,\ F_i \geq E_i + G_i \text{ and } |E_i| \leq |G_i|, \tag{18}$$

the discrete invariant imbedding algorithm is stable [12].
One can easily show that if the assumptions $(\eta\beta(x) - \delta\alpha(x)) > 0$, $(\alpha(x) + \beta(x) + w(x)) < 0$ and $(\varepsilon - \delta a(x)) > 0$ hold, then the above conditions (18) hold and thus the invariant imbedding algorithm is stable.

2.2 Error Estimate

Theorem 1. Suppose $(\eta\beta(x) - \delta\alpha(x)) \geq M > 0$ and $(\alpha(x) + \beta(x) + w(x)) < 0$, $\forall x \in [0, 1]$, and if y_i is the solution to the discretized problem (6), (7) and $y(x)$ is the solution of the corresponding continuous problem, then

$$|y(x_i) - y_i| \leq Ch, \tag{19}$$

where C is independent of i, h and ε.

Proof.

Let w_i be any mesh function defined on the uniform mesh Ω_0^N.
Suppose

$$w_i = y_i^h - y_{2i}^{h/2}, \qquad 0 \leq i \leq N, \tag{20}$$

then

$$w_0 = 0 = w_N,$$

and

$$|L_h(w_i)| \le D_1[h + \exp\{-M(x_i - h)/2\varepsilon^2\}], \tag{21}$$

where D_1 is independent of i, h and ε [7].
Then by discrete stability result [7], we obtain

$$|w_i| \le D_1\{|w_0^h| + |w_N^h| + \max_{1\le i\le N-1} |L_h(w_i^h)|\},$$

i.e.,

$$|y_i^h - y_{2i}^{h/2}| \le D_1\{h + \exp(-M(x_i - h)/2\varepsilon^2\}, \quad 0 \le i \le N. \tag{22}$$

Now to establish the estimate, let us construct a barrier function

$$\psi_i = h[1 - x_i + \exp\{-M(x_i - h)/2\varepsilon^2\}]. \tag{23}$$

We have

$$L_h(\psi_i) = E_i\psi_{i-1} - F_i\psi_i + G_i\psi_{i+1},$$

substituting for E_i, F_i and G_i from Eq. (14) and simplifying, we obtain

$$\begin{aligned} L_h(\psi_i) \;&=\; 4\varepsilon^2\rho_i(\tau)\sinh^2(Mh/2\varepsilon^2)\exp(-M(x_i - h)/2\varepsilon^2) \\ &\quad -h(\eta\beta(x_i) - \delta\alpha(x_i)) + (\eta\beta(x_i) - \delta\alpha(x_i))[-1 + \exp(-Mh/2\varepsilon^2)] \\ &\quad .\exp(-M(x_i - h)/2\varepsilon^2) \\ &\quad +(\alpha(x_i) + \beta(x_i) + w(x_i))h[1 - x_i + \exp(-M(x_i - h)/2\varepsilon^2)]. \end{aligned}$$

Now after omitting the positive terms on the right side of the above equation and simplification, we obtain

$$L_h(\psi_i) \ge -D_2[h + \exp\{-M(x_i - h)/2\varepsilon^2\}], \tag{24}$$

where

$$D_2 = (\|(\eta\beta - \delta\alpha - \alpha - \beta - w\|_{h,\infty}).$$

From inequalities (22) and (24), we obtain

$$L_h\{D\psi_i \pm (y_i^h - y_{2i}^{h/2})\} \le 0, \tag{25}$$

where $D = -D_1/D_2$, independent of i, h and ε. also one can easily show that $\psi_i \ge 0$ at the both end points, therefore by discrete maximum principle, we have

$$\begin{aligned} D\psi_i \pm (y_i^h - y_{2i}^{h/2}) \;&\ge\; 0, \\ |y_i^h - y_{2i}^{h/2}| \;&\le\; D\psi_i, \quad 0 \le i \le N. \end{aligned}$$

Which give the estimate (19), since this result is trivially true for i=0.

□

3 Numerical Results and Discussion

To demonstrate the efficiency of the method, we consider some numerical experiments. The exact solution of the BVP (1), (2) for constant coefficients (i.e.,$\alpha(x) = \alpha$, $\beta(x) = \beta$ and $w(x) = w$ are constant), $\phi(x) = 1 = \psi(x)$,
with $f(x) = 1$ is

$$\begin{aligned} y(x) &= (\alpha+\beta+w-1)[(\exp(m_2)-1)\exp(m_1 x) - (\exp(m_1)-1)\exp(m_2 x)] \\ &\quad /[(\alpha+\beta+w)(\exp(m_2)-\exp(m_1))] + 1/(\alpha+\beta+w), \end{aligned}$$

and if $f(x) = 0$ is

$$y(x) = [(1-\exp(m_2))\exp(m_1 x) - (1-\exp(m_1))\exp(m_2 x)]/(\exp(m_1)-\exp(m_2)),$$

where

$$\begin{aligned} m_1 &= [-(\beta\eta-\alpha\delta) + \sqrt{(\beta\eta-\alpha\delta)^2 - 4\varepsilon^2(\alpha+\beta+w)}]/2\varepsilon^2, \\ m_2 &= [-(\beta\eta-\alpha\delta) - \sqrt{(\beta\eta-\alpha\delta)^2 - 4\varepsilon^2(\alpha+\beta+w)}]/2\varepsilon^2. \end{aligned}$$

3.1 Case : $\beta(x) = 0$ i.e., The case when there is no terms containing positive shift in Eq.(1).

Example 1. $\varepsilon^2 y''(x) - y(x-\delta) + 0.5y(x) = 0$,
under the boundary conditions

$$\begin{aligned} y(x) &= 1, \quad -\delta \leq x \leq 0, \\ y(1) &= 1. \end{aligned}$$

We solve the example using the method presented and compare the results with exact solution and plot graphs of the computed and exact solution of the problem, which are represented by dotted and solid lines respectively, for $\varepsilon = 0.1$ and $\varepsilon = 0.01$ for different values of δ as shown in Figures (1) (2), respectively. We also compute the maximum error for $\varepsilon = 0.1$ and $\varepsilon = 0.01$ for different values of δ and grid size h as shown in Table (1).

Table 1 : The maximum error for example 1

$\varepsilon = 0.1$				
$\delta \downarrow \ N \rightarrow$	$E+01$	$E+02$	$E+03$	$E+04$
0.1ε	0.00780619	0.00008369	0.00000084	0.00000001
0.5ε	0.00979599	0.00010248	0.00000103	0.00000001
0.9ε	0.01105879	0.00011778	0.00000118	0.00000001
$\varepsilon = 0.01$				
0.1ε	0.02896148	0.00755964	0.00008200	0.00000082
0.5ε	0.08468024	0.00976235	0.00010214	0.00000102
0.9ε	0.13193658	0.01104920	0.00011772	0.00000118

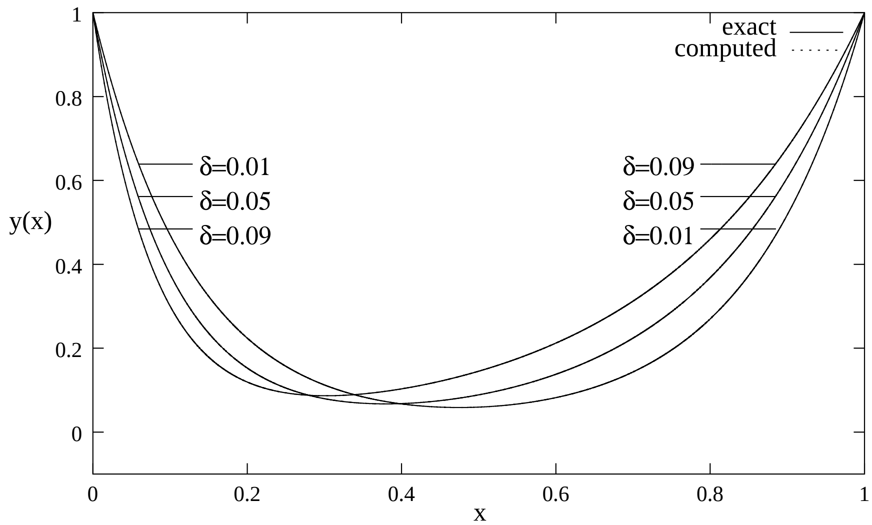

Figure 1: Comparison of exact and numerical solution for example 1 ($\varepsilon = 0.1$).

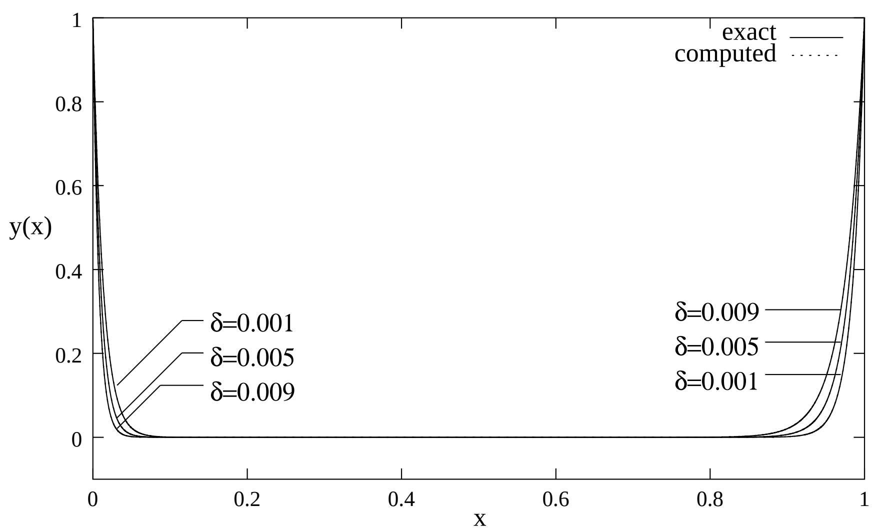

Figure 2: Comparison of exact and numerical solution for example 1 ($\varepsilon = 0.01$).

From the graphs of the solution of the above example, we observe that boundary layer solution depend on the both parameters δ as well as ε. For fixed ε, as δ increases thickness of boundary layer on the left side of the interval $[0,1]$ decreases while on the right side of the interval $[0,1]$ increases.

3.2 Case : $\alpha(x) = 0$ i.e., The case when there is no term containing negative shifts.

Example 2. $\varepsilon^2 y''(x) + y(x) - 1.25y(x+\eta) = 0$,
under boundary conditions

$$\begin{aligned} y(0) &= 1, \\ y(x) &= 1, \quad 1 \leq x \leq 1+\eta. \end{aligned}$$

In this case, we have considered an example in which there is no term containing negative shifts and solve the example using numerical scheme presented here. We plot the graphs for $\varepsilon = 0.1$, $\varepsilon = 0.01$ and for different values of η and compare the computed result with exact solution as shown in the Figures (3) and (4). Table (2) give the maximum error for $\varepsilon = 0.1$ and $\varepsilon = 0.01$ and for different η and grid size h.

Table 2 : The maximum error for example 2

$\varepsilon = 0.1$				
$\eta \downarrow \; N \rightarrow$	$E+01$	$E+02$	$E+03$	$E+04$
0.1ε	0.00463982	0.00004756	0.00000048	0.00000000
0.5ε	0.00577129	0.00005937	0.00000059	0.00000001
0.9ε	0.00642460	0.00006711	0.00000067	0.00000001
$\varepsilon = 0.01$				
0.1ε	0.05225074	0.00417902	0.00004306	0.00000043
0.5ε	0.13766296	0.00567762	0.00005861	0.00000059
0.9ε	0.16092712	0.00641204	0.00006693	0.00000067

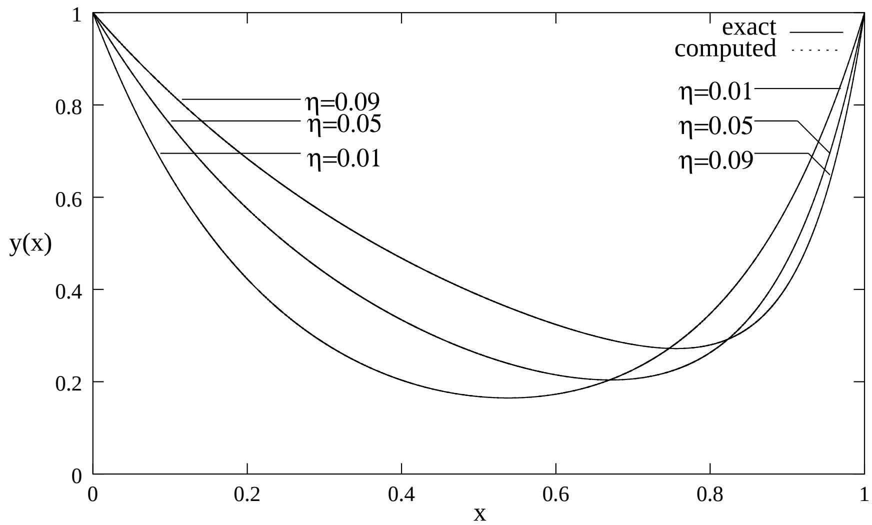

Figure 3: Comparison of exact and computed solution for example 2 ($\varepsilon = 0.1$).

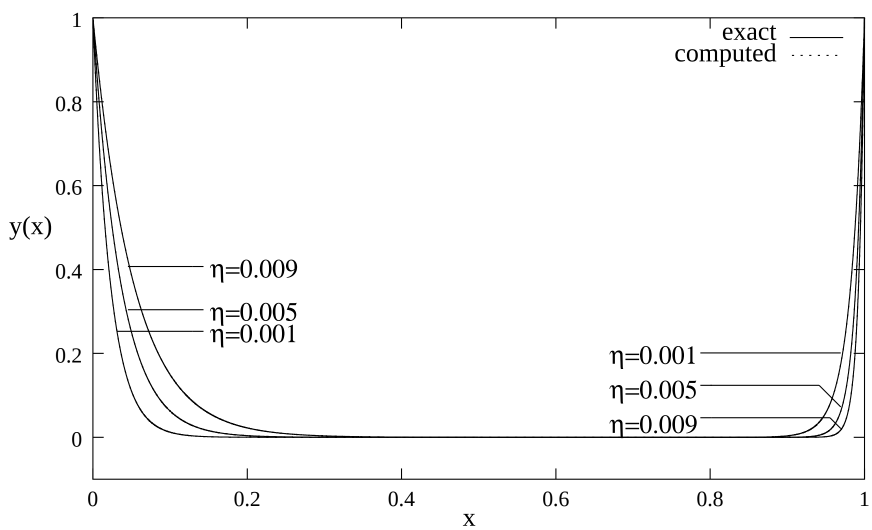

Figure 4: Comparison of exact and computed solution for example 2 ($\varepsilon = 0.01$).

As in the case when only negative shift occurs we have shown the effect on the boundary layer solution of negative small shift by considering numerical example. Similarly in this case we have considered the example in which only positive shift occur and observe from the graphs in figures (3) and (4), that for fixed, ε the thickness of the boundary layer on the right side decreases while on the left side increases of the solution as η increases.

3.3 Case : $\alpha(x) \neq 0$ and $\beta(x) \neq 0$, i.e., The case of mixed type (i.e., contains terms with both terms with negative as well as positive shifts).

Here, we consider the most general case where both type of shift *i.e.* positive as well as negative shift occur. To demonstrate the efficiency of the method, the following numerical experiments are carried out

Example 3. $y''(x) - y(x-\delta) + y(x) - 1.25y(x+\eta) = 1$,
under the boundary conditions

$$y(x) = 1, \quad -\delta \leq x \leq 0, \qquad y(x) = 1, \quad 1 \leq x \leq 1+\eta.$$

In this case to demonstrate the method we have solved the more general example in which both the negative as well as positive shifts occur using our method and compare computed results with the exacts solution by plotting the graphs for $\varepsilon = 0.01$ and for different values of δ and η as shown in Figures (5) and (6). The maximum error between computed and exact solution are shown in table (3) for $\varepsilon = 0.01$ and for different δ, η and grid size h.

Table 3 : The maximum error for example 3

$\varepsilon = 0.01;\ \eta = 0.9\varepsilon$			
$\delta \downarrow\ \ N \rightarrow$	$E+02$	$E+03$	$E+04$
0.1ε	0.04112737	0.00048798	0.00000489
0.5ε	0.03873699	0.00043715	0.00000438
0.8ε	0.03555902	0.00039383	0.00000394
$\eta \downarrow\ \ \varepsilon = 0.01;\ \delta = 0.9\varepsilon$			
0.1ε	0.03988186	0.00045703	0.00000458
0.5ε	0.03492230	0.00038652	0.00000387
0.8ε	0.03248877	0.00035962	0.00000360

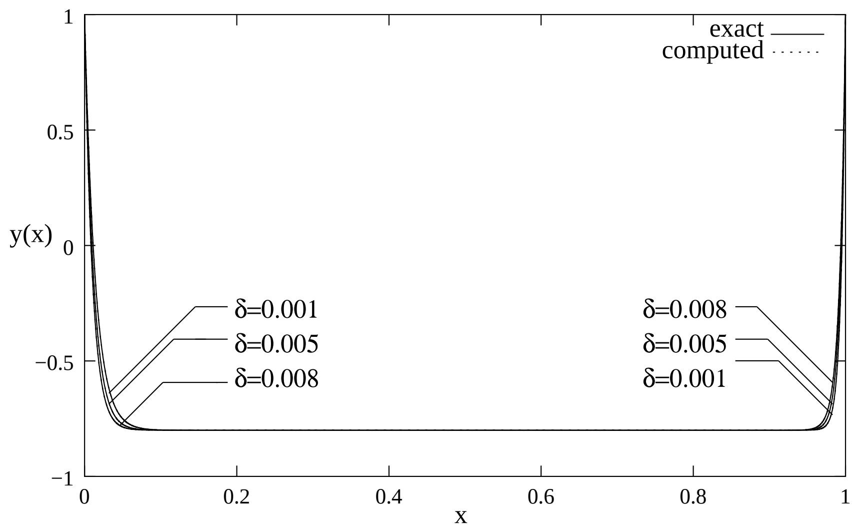

Figure 5: Comparison of exact and numerical solution for example 3 ($\varepsilon = 0.01$ and $\eta = 0.9\varepsilon$).

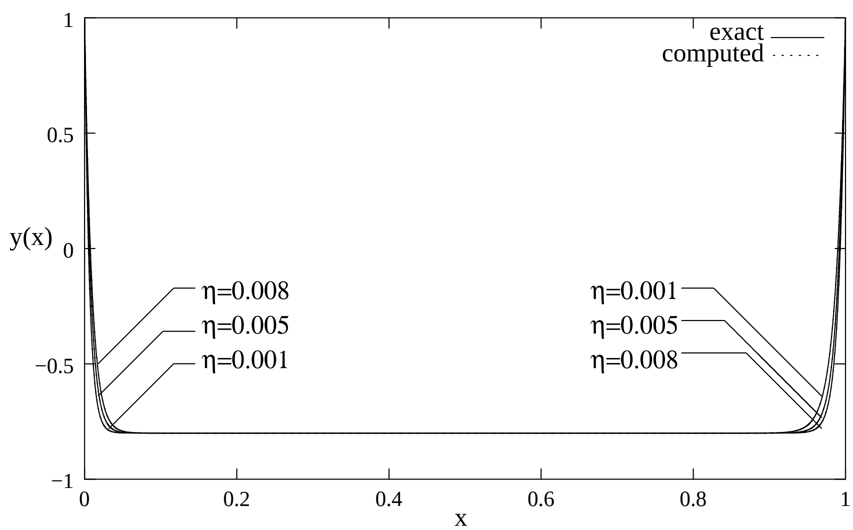

Figure 6: Comparison of exact and numerical solution for example 3 ($\varepsilon = 0.01$ and $\delta = 0.9\varepsilon$).

From the graphs of the solutions of the above example, we observe that the variations in boundary layer solution of the problem with the parameters ε, δ and η are similar to previous two cases i.e., when only one shift (negative or positive) is present at a time.

Finally we consider the following numerical examples with variable coefficients and solve these examples using the proposed numerical scheme.

Example 4. $y''(x) - \exp(x)y(x-\delta) + y(x) - (1+x)y(x+1) = 0$,
under boundary conditions

$$\begin{aligned} y(x) &= 1, \quad -\delta \le x \le 0, \\ y(x) &= 1, \quad 1 \le x \le 1+\eta. \end{aligned}$$

Example 5. $y''(x) - \exp(0.5)y(x-\delta) + xy(x) - (1+x^2)y(x+\eta) = 1$,
under boundary conditions

$$\begin{aligned} y(x) &= 1, \quad -\delta \le x \le 0, \\ y(x) &= 1, \quad 1 \le x \le 1+\eta. \end{aligned}$$

Since the exact solution for the examples 4 and 5 is not known, so we just compute the numerical solution and plotted in Figures 7 and 8, respectivelly.

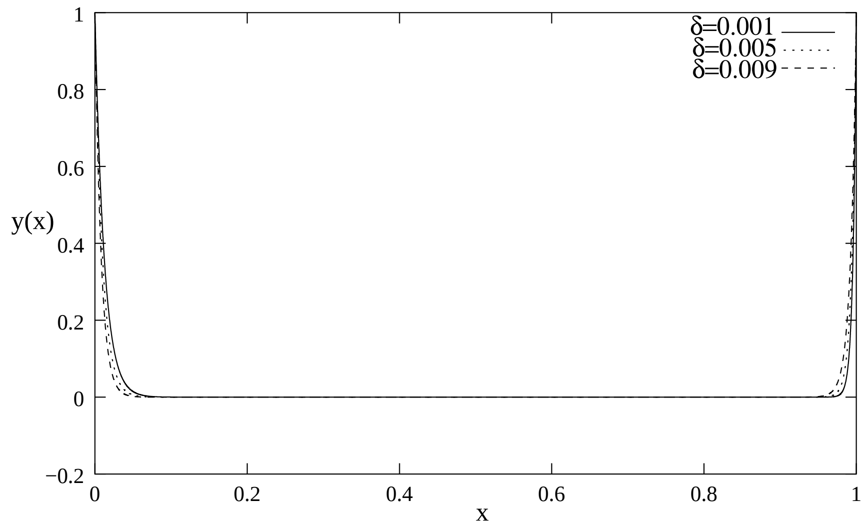

Figure 7: Numerical solution for example 4 ($\varepsilon = 0.01$ and $\eta = 0.5\varepsilon$).

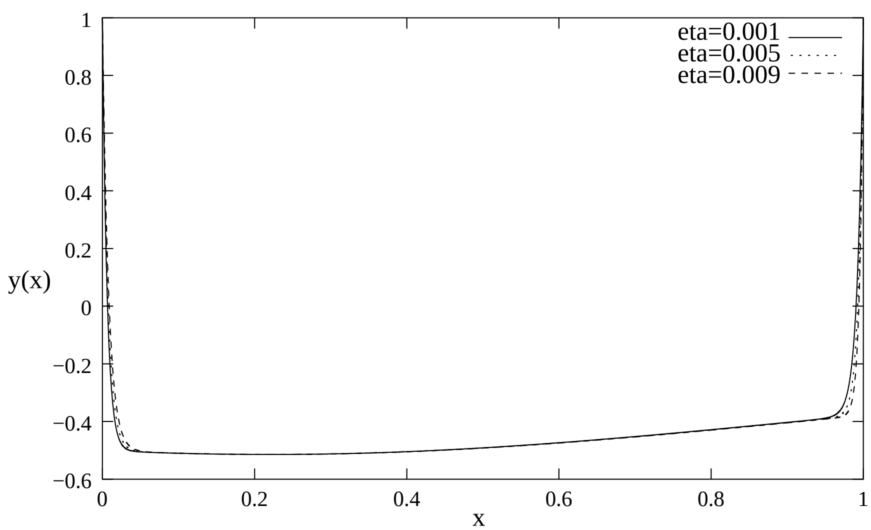

Figure 8: Numerical solution for example 5 ($\varepsilon = 0.01$ and $\delta = 0.5\varepsilon$).

4 conclusion

In this paper, we propose an exponentially fitted finite difference numerical scheme to solve boundary-value problems for singularly perturbed differential-difference equations with small shifts of mixed type (i.e., which contains both the terms having negative as well as positive shift). We observed from the numerical experiments discussed above that very small changes in shift affect the boundary layer solution by a considerable amount and does not affect the smooth solution. We also observe that as negative shift increases the thickness of the left side boundary layer decreases while of the right side boundary layer increases, whether the term containing positive shift occurs or not and the positive shift affects the boundary layer solution in the same form but reversely. This method works nicely for small shifts and easy for implementation.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.2,173-193,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

Existence and Asymptotic Stability for Viscoelastic Evolution Problems on Compact Manifolds

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Abstract. One considers the nonlinear viscoelastic evolution equation

$$u_{tt} + Au + F(x,t,u,u_t) - g * A\, u = 0 \quad \text{on } \Gamma \times (0,\infty)$$

where Γ is a compact manifold. When $F \neq 0$ and $g = 0$ we prove existence of global solutions as well as uniform (exponential and algebraic) decay rates. Furthermore, if $F = 0$ and $g \neq 0$ we prove that the dissipation introduced by the memory effect is strong enough to allow us to derive an exponential(or polynomial) decay rate provided the resolvent kernel of the relaxation function decays exponentially (or polynomially).

Key words: Asymptotic Stability, Viscoelastic Evolution Problem
2000 AMS Subject Classification 35G25, 37C75

1 Introduction

This manuscript is devoted to the study of the existence and uniform decay rates of solutions $u = u(x,t)$ of the evolution viscoelastic problem

$$(*) \qquad \begin{cases} u_{tt} + Au + F(x,t,u,u_t) - g * A\, u = 0 & \text{on } \Gamma \times (0,\infty) \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Gamma \end{cases}$$

where Γ is the boundary, assumed compact and smooth, of a domain Ω of $\mathbf{R}^n$, not necessarily bounded.

When $g = 0$, we will consider $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ a linear and continuous operator, that is, $A \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$, self-adjoint and such that verifies the coercivity condition

$$\langle Au, u \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \geq \alpha\, ||u||^2_{H^{1/2}(\Gamma)} \qquad \text{for all } u \in H^{1/2}(\Gamma), \tag{1.1}$$

for some $\alpha > 0$. In this case we prove global existence results and also exponential and algebraic decay rates of the energy associated to problem $(*)$, following the perturbed energy method; see, for instance, A. Haraux and E. Zuazua [3]. We observe that when $g = 0$ and the operator A verifies the above conditions, we have, as a particular example, that the existence of solutions of problem $(*)$ is related to the existence of solutions of the following one

$$\begin{cases} -\Delta y + ky = 0 \quad \text{in } \Omega \times (0,\infty), \quad k > 0 \\ \partial_\nu y + y_{tt} + F(x,t,y,y_t) = 0 \quad \text{on } \Gamma \times (0,\infty) \\ y(x,0) = u^0(x); \quad y_t(x,0) = u^1(x), \quad x \in \Gamma, \end{cases}$$

where ν is the outer unit vector normal to the boundary Γ; see J. L. Lions [[4] ,pp. 134-140] for details. In this situation the operator $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is defined as follows: Given $\varphi \in H^{1/2}(\Gamma)$, it is well known that the elliptic problem

$$\begin{cases} -\Delta w + kw = 0 \quad \text{in } \Omega \\ w = \varphi \quad \text{on } \Gamma \end{cases}$$

admits a unique solution $w \in \mathcal{H}(\Omega,\Delta) = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}$. Therefore, the operator

$$A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \quad \varphi \mapsto A\varphi = \partial_\nu w$$

is well defined and furthermore, $A \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$. On the other hand, making use of Green's formula we deduce that

$$0 = \int_\Omega (-\Delta w + kw)\, w dx = \int_\Omega |\nabla w|^2 \, dx + k \int_\Omega |w|^2 \, dx - \langle A\varphi, \varphi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

and consequently, $\langle A\varphi, \varphi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \geq C \, ||\varphi||^2_{H^{1/2}(\Gamma)}$ for some $C > 0$. In this direction is important to mention the work from the authors M. M. Cavalcanti and V. N. Domingos Cavalcanti [2] who proved global existence and asymptotic behaviour for degenerate equations on manifolds. On the other hand, when $F = 0$ and $g \neq 0$, we will assume that A is the self-adjoint operator, not necessarily bounded, defined by the triple $\{H^{1/2}(\Gamma), L^2(\Gamma), ((\cdot,\cdot))_{H^{1/2}(\Gamma)}\}$. In this case, A is characterized by

$$\begin{aligned} D(A) = \{u \in H^{1/2}(\Gamma);\ &\text{there exists } f_u \in L^2(\Gamma) \text{ such that} \qquad (1.2) \\ &(f_u, v)_{L^2(\Gamma)} = ((u,v))_{H^{1/2}(\Gamma)}\,; \text{ for all } v \in H^{1/2}(\Gamma)\}, \quad f_u = Au \\ (Au, v)_{L^2(\Gamma)} = ((u,v))_{H^{1/2}(\Gamma)}\,;\ &\text{ for all } u \in D(A) \text{ and for all } v \in H^{1/2}(\Gamma). \end{aligned}$$

Since the embedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact, we recall that the spectral theorem for self-adjoint operators guarantees the existence of a complete orthonormal system $\{\omega_\nu\}_{\nu \in \mathbf{N}}$ of $L^2(\Gamma)$ given by eigen-functions of A. If $\{\lambda_\nu\}_{\nu \in \mathbf{N}}$ are the corresponding eigenvalues of A, then $\lambda_\nu \to +\infty$ as $\nu \to +\infty$. Besides,

$$\begin{aligned} D(A) &= \{u \in L^2(\Gamma); \Sigma_{\nu=1}^{+\infty} \lambda_\nu^2 |\, (u,\omega_\nu)_{L^2(\Gamma)}\, |^2 < +\infty\}, \\ Au &= \Sigma_{\nu=1}^{+\infty} \lambda_\nu \, (u,\omega_\nu)_{L^2(\Gamma)} \, \omega_\nu; \quad \text{for all } u \in D(A). \end{aligned}$$

Considering in $D(A)$ the norm $|Au|_{L^2(\Gamma)}$, it turns out that $\{\omega_\nu\}$ is a complete system in $D(A)$. In fact, it is known that $\{\omega_\nu\}$ is also a complete system in $H^{1/2}(\Gamma)$. Now, since A is positive, given $\delta > 0$ one has

$$D(A^\delta) = \left\{ u \in L^2(\Gamma); \Sigma_{\nu=1}^{+\infty} \lambda_\nu^{2\delta} \left| (u, \omega_\nu)_{L^2(\Gamma)} \right|^2 < +\infty \right\},$$
$$A^\delta u = \Sigma_{\nu=1}^{+\infty} \lambda_\nu^\delta \, (u, \omega_\nu)_{L^2(\Gamma)} \, \omega_\nu; \quad \text{for all } u \in D(A^\delta).$$

In $D(A^\delta)$ we consider the topology given by $|A^\delta u|_{L^2(\Gamma)}$. We observe that from the spectral theory, such operators are also self-adjoint, that is,

$$(A^\delta u, v)_{L^2(\Gamma)} = (u, A^\delta v)_{L^2(\Gamma)}; \quad \text{for all } u, v \in D(A^\delta)$$

and, in particular,

$$D(A^{1/2}) = H^{1/2}(\Gamma). \tag{1.3}$$

At this point it is convenient to observe that, according to J. L. Lions and E. Magenes [[5], Remark 7.5] one has

$$H^{1/2}(\Gamma) = D[(-\Delta_\Gamma)^{1/2}], \tag{1.4}$$

where Δ_Γ is the Laplace-Beltrami operator on Γ. Then, from (1.2), (1.3) and (1.4) we deduce that

$$(Au, v)_{L^2(\Gamma)} = (-\Delta_\Gamma u, v)_{L^2(\Gamma)}; \quad \text{for all } u \in D(A), \ \text{for all } v \in H^{1/2}(\Gamma), \tag{1.5}$$

that is, $Au = -\Delta_\Gamma u$ for all $u \in D(A)$ which implies that $A \leq -\Delta_\Gamma$. This means that when A is the operator defined by the above triple, problem $(*)$ can also be viewed like the wave operator on the compact manifold Γ.

Now, if one considers the extension $\tilde{A} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ of A defined by

$$< \tilde{A}u, v >_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = ((u, v))_{H^{1/2}(\Gamma)}; \quad \text{for all } u, v \in H^{1/2}(\Gamma) \tag{1.6}$$

it is well known that $\tilde{A}$ is bijective, self-adjoint, coercive and continuous (indeed isometry). Then this extension satisfies the assumptions of the operator A introduced in the beginning of this introduction, more precisely in (1.1).

When $F \neq 0$ and $g = 0$ we derive exponential and algebraic decay rates. Finally when $F = 0$ and $g \neq 0$ we show that the energy associated to the related problem decays exponentially (or algebraically) assuming that the kernel of the memory also decays exponentially (or algebraically). In other words, the unique dissipative mechanism is due to the memory term. For this end we follow ideas introduced by J. Muñoz Rivera in [6].

Our paper is organized as follows: In section 2 we present some notations, the assumptions on g and F and state our main result. In section 3 we prove existence and uniqueness for regular and weak solutions and in section 4 we give the proof of the uniform decay.

2 Assumptions and Main Result

Define $(u,v) = \int_\Gamma u(x)v(x)\,dx$; $|u|^2 = (u,u)$, $||u||_p^p = \int_\Gamma |u(x)|^p\,dx$. The precise assumptions on the function $F(x,t,u,u_t)$ and on the memory term g of $(*)$ are given in the sequel.

(A.1) Assumptions on $F(x,t,u,u_t)$

We represent by (x,t,ξ,η) a point of $\Gamma\times[0,\infty)\times\mathbf{R}^2$. Let

$$F:\Gamma\times[0,\infty)\times\mathbf{R}^2\to\mathbf{R}$$

satisfying the conditions

$$F\in C^1\left(\Gamma\times[0,\infty)\times\mathbf{R}^2\right). \tag{$H.1$}$$

There exist positive constants C, D and $\beta>0$ such that

$$|F(x,t,\xi,\eta)|\le C\left(1+|\xi|^{\gamma+1}+|\eta|^{\rho+1}\right), \tag{$H.2$}$$

where $0<\xi,\rho\le\frac{1}{n-2}$ if $n\ge3$ and $\xi,\rho>0$ if $n=1,2$;

$$F(x,t,\xi,\eta)\zeta\ge|\xi|^\gamma\,\xi\zeta+\beta\,|\eta|^{\rho+1}\,|\zeta|;\quad\text{for all }\zeta\in\mathbf{R}; \tag{$H.3$}$$

$$|F_t(x,t,\xi,\eta)|\le C\left(1+|\eta|^{\rho+1}+|\xi|^{\gamma+1}\right); \tag{$H.4$}$$

$$|F_\xi(x,t,\xi,\eta)|\le C\left(1+|\eta|^{\rho}+|\xi|^{\gamma}\right); \tag{$H.5$}$$

$$F_\eta(x,t,\xi,\eta)\ge\beta\,|\eta|^\rho; \tag{$H.6$}$$

$$\left(F(x,t,\xi,\eta)-F(x,t,\hat\xi,\hat\eta)\right)\left(\zeta-\hat\zeta\right) \tag{$H.7$}$$

$$\ge -D\left(|\xi|^\gamma+\left|\hat\xi\right|\right)\left|\xi-\hat\xi\right|\left|\zeta-\hat\zeta\right|\quad\text{for all }\zeta,\hat\zeta\in\mathbf{R}.$$

A simple variant of the above function is given by the following example $F(x,t,\xi,\eta)=\beta\,|\eta|^\rho\,\eta+|\xi|^\gamma\,\xi$.

(A.2) Assumptions on the Kernel

We assume that $g:\mathbf{R}_+\to\mathbf{R}_+$ is a bounded C^2 function satisfying

$$1-\int_0^\infty g(s)\,ds=l>0 \tag{$H.8$}$$

and such that there exist positive constants ξ_1, ξ_2 and ξ_3 verifying

$$-\xi_1 g(t)\le g'(t)\le-\xi_2 g(t);\quad\text{for all }t\ge0, \tag{$H.9$}$$

$$0\le g''(t)\le\xi_3 g(t);\quad\text{for all }t\ge0, \tag{$H.10$}$$

$$0\ge g'''(t)\ge\xi_4 g'(t);\quad\text{for all }t\ge0. \tag{$H.11$}$$

Next, we present two technical lemmas that will play an essential role when establishing the existence of weak solutions.

For this end let us consider V and H Hilbert spaces with V dense in H and the imbedding $V \hookrightarrow H$ is continuous. Let $A : V \to V'$ be a linear operator such that $A \in \mathcal{L}(V, V')$. Suppose that A is self-adjoint and verifies the coercivity condition:

$$\langle Av, v\rangle_{V',V} \geq \alpha \, ||v||_V^2 \quad \text{for all } v \in V, \tag{2.1}$$

for some $\alpha > 0$.

Lemma 2.1. *A is bijective.*

Proof. The condition (2.1) implies immediately that A is injective. Then, it remains to prove that A is onto. First, we are going to prove that

$$AV \text{ is closed in } V'. \tag{2.2}$$

Indeed, let $\{v_\nu\} \subset V$ and $w \in V'$ such that

$$Av_\nu \to w \text{ in } V' \text{ as } \nu \to +\infty. \tag{2.3}$$

From (2.1) we obtain, for all $\nu, \mu \in \mathbf{N}$

$$\langle Av_\nu - Av_\mu, v_\nu - v_\mu\rangle_{V',V} \geq \alpha \, ||v_\nu - v_\mu||_V^2$$

which implies that

$$||Av_\nu - Av_\mu||_{V'} \geq \alpha \, ||v_\nu - v_\mu||_V$$

and consequently, from (2.3) we deduce that $\{v_\nu\}$ is a sequence of Cauchy in V. Therefore, there exists $v \in V$ such that $v_\nu \to v$ in V. Since A is continuous, it results that

$$Av_\nu \to Av \text{ in } V' \text{ as } \nu \to +\infty. \tag{2.4}$$

Taking (2.3) and (2.4) into account, we conclude that $Av = w$ and consequently AV is closed in V' as we desired to prove in (2.2).

On the other hand, since V' is a Hilbert space, we can write, in view of (2.2), that

$$V' = AV \oplus AV^\perp.$$

Next, we are going to prove that

$$AV^\perp = \{0\}. \tag{2.5}$$

In fact, since V is a Hilbert space, we can write

$$AV^\perp = \{f \in V; \langle f, u\rangle_{V,V'} = 0 \quad \text{for all } u \in AV\}$$

or, in other words

$$AV^\perp = \{f \in V; \langle f, Av\rangle_{V',V} = 0 \quad \text{for all } v \in V\}.$$

We argue by contradiction. So let us suppose that there exists $f_0 \in V$; $f_0 \neq 0$ such that

$$\langle f_0, Av\rangle_{V',V} = 0; \quad \text{for all } v \in V.$$

Then, the last identity and (2.1) yield

$$0 = \langle f_0, Af_0\rangle_{V',V} \geq \alpha \, ||f_0||_V^2 \quad \text{implies} \quad f_0 = 0.$$

But this is a contradiction and consequently (2.5) holds, which implies that A is onto. ♢

Identifying $H \equiv H'$ one has the following embedding

$$V \hookrightarrow H \equiv H' \hookrightarrow V'$$

with H' dense in V', see H. Brézis [1].

Lemma 2.2. *The space* $\mathcal{H} = \{u \in V; Au \in H\}$ *is dense in* V.

Proof. Let $T \in V'$ such that

$$\langle T, w\rangle_{V',V} = 0 \quad \text{for all } w \in \mathcal{H}. \tag{2.6}$$

We will prove that

$$\langle T, w\rangle_{V',V} = 0 \quad \text{for all } w \in V. \tag{2.7}$$

Indeed, let $v \in V$. Then $Av \in V'$ and since H is dense in V', there exists $\{y_\mu\} \subset H$ such that

$$y_\mu \to Av \quad \text{in } V'. \tag{2.8}$$

But, for each $\mu \in \mathbf{N}$, according to lemma 2.1, $y_\mu = Ax_\mu$ with $x_\mu \in V$. Then, from (2.1) and for all $\nu, \mu \in \mathbf{N}$, we have

$$\langle Ax_\nu - Ax_\mu, x_\nu - x_\mu\rangle_{V',V} \geq \alpha \, ||x_\nu - x_\mu||_V^2 \, ,$$

that is,

$$||Ax_\nu - Ax_\mu||_{V'} \geq \alpha \, ||x_\nu - x_\mu||_V \, .$$

The last inequality and the convergence given in (2.8) yield that $\{x_\nu\}$ is a sequence of Cauchy in V. Consequently, there exists $x \in V$ such that $x_\nu \to x$ in V and therefore

$$y_\nu = Ax_\nu \to Ax \quad \text{in } V'. \tag{2.9}$$

From (2.8) and (2.9) we deduce that

$$Av = Ax \;\text{ which implies that }\; v = x \;\text{ and }\; Ax_\nu \to Av \;\text{ in }\; V'. \tag{2.10}$$

However, from (2.6) we have

$$\langle T, x_\nu\rangle_{V',V} = 0 \;\text{ since }\; x_\nu \in V \;\text{ and }\; Ax_\nu = y_\nu \in H. \tag{2.11}$$

Passing to the limit in (2.11) we obtain (2.7) as we desired to show. $\diamondsuit$

In the sequel we will consider $V = H^{1/2}(\Gamma)$, $H = L^2(\Gamma)$ and $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ the linear, continuous, self-adjoint and coercive operator above mentioned in this section.

We define

$$\mathcal{H} = \{u \in H^{1/2}(\Gamma); Au \in L^2(\Gamma)\}. \tag{2.12}$$

Then, $\mathcal{H}$ is a Hilbert space endowed with the natural inner product

$$(u, v)_{\mathcal{H}} = (u, v)_{H^{1/2}(\Gamma)} + (Au, Av)\,. \tag{2.13}$$

Moreover, according to lemma 2.1, $\mathcal{H}$ is dense in $L^2(\Gamma)$.

Now we are in a position to state our main result.

Theorem 2.1. *Let the initial data $\left\{u^0, u^1\right\}$ belong to $\mathcal{H} \times H^{1/2}(\Gamma)$ and assume that the assumptions in (A.1) hold and $g = 0$. Then, problem $(*)$ possesses a unique regular solution u in the class*

$$u \in L^\infty(0, \infty; \mathcal{H}), \quad u' \in L^\infty(0, \infty; H^{1/2}(\Gamma), \quad u'' \in L^\infty(0, \infty; L^2(\Gamma)). \tag{2.14}$$

Moreover, the energy

$$E(t) = \frac{1}{2}\{|u'(t)|^2 + \langle Au(t), u(t)\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} + \frac{2}{\gamma+2}\,||u(t)||_{\gamma+2}^{\gamma+2}\} \tag{2.15}$$

has the following decay rate

$$E(t) \leq (\varepsilon\theta t + [E(0)]^{-\rho/2})^{-2/\rho}, \quad \text{for all } t \geq 0, \quad \text{for all } \varepsilon \in (0, \varepsilon_0], \tag{2.16}$$

where θ and ε_0 are positive constants.

When $\rho = 0$ and therefore we have a linear dissipation, exponential decay rates are also obtained, namely

$$E(t) \leq CE(0)e^{-\varepsilon\omega t} \quad \text{for all }, t \geq 0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0], \tag{2.17}$$

where C, ω and ε_0 are positive constants.

Theorem 2.2. *Let the initial data belong to $H^{1/2}(\Gamma) \times L^2(\Gamma)$ and assume the same hypotheses of theorem 2.1 hold. Then, problem $(*)$ possesses a unique weak solution u in the class*

$$u \in C^0([0, \infty), H^{1/2}(\Gamma)) \cap C^1([0, \infty); L^2(\Gamma)). \tag{2.18}$$

Besides, the weak solution has the same decays given in (2.16) and (2.17).

Theorem 2.3. *Suppose that the assumptions in (A.2) hold and $F = 0$. Then, given $\left\{u^0, u^1\right\} \in D(A) \times D(A^{1/2})$, problem $(*)$ possesses a unique solution u in the class*

$$u \in C^0\left([0, \infty; D(A)\right) \cap C^1\left([0, \infty; D(A^{1/2})\right). \tag{2.19}$$

Moreover, the energy

$$E(t) := \frac{1}{2}\left\{|u'(t)|^2 + \left(1 - \int_0^t g(s)\,ds\right)|A^{1/2}u(t)|^2 + g\diamond A^{1/2}u(t)\right\} \quad (2.20)$$

decays exponentially, that is, there exist positive constants C and γ such that

$$E(t) \leq Ce^{-\gamma t}, \quad \text{for all } t \geq 0. \quad (2.21)$$

When, instead of hypothesis (H.9) we consider

$$-C_0 g^{1+1/p}(t) \leq g'(t) \leq -C_1 g^{1+1/p}(t) \quad \text{for all } t \geq 0, \quad (H.12)$$

for some positive constants C_0, C_1 and $p > 2$, we have the following decay

$$E(t) \leq CE(0)\,(1+t)^{-p} \quad \text{for all } \geq 0. \quad (2.22)$$

3 Existence and Uniqueness of Solutions

In this section we first prove existence and uniqueness of regular solutions to problem $(*)$ making use of Faedo-Galerkin method. Then, we extend the same result to weak solutions using a density argument.

3.1 Regular Solutions: First of all we consider the case $g = 0$ and $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the linear, continuous, self-adjoint and coercive operator mentioned before.

Let $\{\omega_\nu\}$ be a basis in $\mathcal{H}$ and let us consider V_m the space generated by $\omega_1, \cdots, \omega_m$. Let

$$u_m(t) = \Sigma_{j=1}^m \delta_{jm}(t)\omega_j \quad (3.1)$$

the solution of the approximate Cauchy problem

$$(u_m''(t), w) + (Au_m(t), w) + (F(x, t, u_m(t), u_m'(t)), w) = 0 \text{ for all } w \in V_m, \quad (3.2)$$
$$u_m(0) = u_{0m} \to u^0 \text{ in } \mathcal{H}, \quad u_m'(0) = u_{1m} \to u^1 \text{ in } H^{1/2}(\Gamma). \quad (3.3)$$

We observe that, in view of assumptions $(H.1) - (H.2)$ and noting that

$$H^{1/2}(\Gamma) \hookrightarrow L^{2(\gamma+1)}(\Gamma) \quad \text{and} \quad H^{1/2}(\Gamma) \hookrightarrow L^{2(\rho+1)}(\Gamma) \quad (3.4)$$

the nonlinear term in (3.2) is well defined, that is, belong to $L^2(\Gamma)$. By standard methods in differential equations, we prove the existence of solutions to the approximate problem on some interval $[0, t_m)$ and this solution can be extended to the closed interval [0,T] by using the first estimate below.

3.1.1 - A Priori Estimates. *The First Estimate:* Setting $w = u_m'(t)$ in (3.2), observing that A is self adjoint and taking the assumption $(H.3)$ into account, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|u_m'(t)|^2 + (Au_m(t), u_m(t)) + \frac{2}{\gamma+2}\,||u_m(t)||_{\gamma+2}^{\gamma+2}\right\} + \beta\,||u_m'(t)||_{\rho+2}^{\rho+2} \leq 0. \quad (3.5)$$

Integrating (3.5) over (0,t) taking (1.1) into account, we deduce

$$\begin{aligned}&|u_m'(t)|^2+\alpha\,||u_m(t)||^2_{H^{1/2}(\Gamma)}+\tfrac{1}{\gamma+2}\,||u_m(t)||^{\gamma+2}_{\gamma+2}+2\beta\textstyle\int_0^t||u_m'(s)||^{\rho+2}_{\rho+2}\,ds\\&\quad\le|u_{1m}|^2+||Au_{0m}||_{H^{-1/2}(\Gamma)}\,||u_{0m}||_{H^{1/2}(\Gamma)}+\tfrac{1}{\gamma+2}\,||u_{0m}||^{\gamma+2}_{\gamma+2}\,.\end{aligned}$$

From the last inequality, from the convergence in (3.3), observing the embedding in (3.4) and employing Gronwall's lemma, we obtain the first estimate

$$|u_m'(t)|^2+||u_m(t)||^2_{H^{1/2}(\Gamma)}+||u_m(t)||^{\gamma+2}_{\gamma+2}+\textstyle\int_0^t||u_m'(s)||^{\rho+2}_{\rho+2}\,ds\le L_1 \tag{3.6}$$

where L_1 is a positive constant independent of $m\in\mathbf{N}$ and $t\in[0,T]$.

The Second Estimate: First of all we are going to estimate $u_m''(0)$ in $L^2(\Gamma)$ norm. Considering $w=u_m''(0)$ and $t=0$ in (3.2) and considering the hypothesis $(H.2)$, it holds that

$$|u_m''(0)|\le[|Au_{0m}|+C(meas(\Gamma)^{1/2}+||u_{0m}||^{\gamma+1}_{2(\gamma+1)}+||u_{1m}||^{\rho+1}_{2(\rho+1)})]. \tag{3.7}$$

Considering the convergence in (3.3) and the embedding in (3.4) we conclude that

$$|u_m''(0)|\le L_2 \tag{3.8}$$

where L_2 is a positive constant independent of $m\in\mathbf{N}$.

Taking the derivative of (3.2) with respect to t, substituting $w=u_m''(t)$ in the obtained expression, and taking the assumptions $(H.3)-(H.6)$ into account, we obtain

$$\begin{aligned}&\tfrac12\tfrac{d}{dt}\left\{|u_m''(t)|^2+(Au_m'(t),u_m'(t))\right\}+\beta\textstyle\int_\Gamma|u_m'|^\rho\,(u_m'')^2\,d\Gamma\\&\quad\le C\textstyle\int_\Gamma\left(1+|u_m'|^{\rho+1}+|u_m|^{\gamma+1}\right)|u_m''|\,d\Gamma\\&\quad+C\textstyle\int_\Gamma\left(1+|u_m'|^{\rho}+|u_m|^{\gamma}\right)|u_m'|\,|u_m''|\,d\Gamma.\end{aligned} \tag{3.9}$$

Next, we going to analyze the two terms on the right hand side of (3.9).

Estimate for $I_1:=\int_\Gamma\left(1+|u_m'|^{\rho+1}+|u_m|^{\gamma+1}\right)|u_m''|\,d\Gamma$.

From Cauchy-Schwartz inequality and considering the inequality $ab\le\frac{1}{4\eta}a^2+\eta b^2$, where η is an arbitrary positive number, we deduce

$$\begin{aligned}|I_1|&\le meas(\Gamma)+|u_m''(t)|^2+\eta\textstyle\int_\Gamma|u_m'|^\rho\,|u_m''|^2\,d\Gamma+\frac{1}{4\eta}\,||u_m'(t)||^{\rho+2}_{\rho+2}\\&\quad+\tfrac12\,||u_m(t)||^{2(\gamma+1)}_{2(\gamma+1)}+\tfrac12\,|u_m''(t)|^2\,.\end{aligned} \tag{3.10}$$

Estimate for $I_2:=\int_\Gamma\left(1+|u_m'|^{\rho}+|u_m|^{\gamma}\right)|u_m'|\,|u_m''|\,d\Gamma$.

From Cauchy-Schwarz inequality, making use of the inequality $ab\le\frac{1}{4\eta}a^2+\eta b^2$ above mentioned and the generalized Hölder inequality observing that $\frac{\gamma}{2(\gamma+1)}+\frac{1}{2(\gamma+1)}+\frac12=1$, we have

$$\begin{aligned}|I_2|&\le\tfrac12\,|u_m'(t)|^2+\tfrac12\,|u_m''(t)|^2+\eta\textstyle\int_\Gamma|u_m'|^\rho\,|u_m''|^2\,d\Gamma+\frac{1}{4\eta}\,||u_m'(t)||^{\rho+2}_{\rho+2}\\&\quad+||u_m(t)||^{\gamma}_{2(\gamma+1)}\,||u_m'(t)||_{2(\gamma+1)}\,|u_m''(t)|\,.\end{aligned} \tag{3.11}$$

Integrating (3.9) over (0,t), considering (1.1), (3.4), (3.10) and (3.11), we deduce

$$\begin{aligned} &\tfrac{1}{2}\left|u_m''(t)\right|^2 + \tfrac{\alpha}{2}\left|\left|u_m'(t)\right|\right|^2_{H^{1/2}(\Gamma)} + (\beta - 2C\eta)\int_0^t\int_\Gamma \left|u_m'\right|^\rho \left|u_m''\right|^2 d\Gamma ds \\ &\le \tfrac{1}{2}\left|u_m''(0)\right|^2 + \tfrac{1}{2}\left|\left|Au_{1m}\right|\right|_{H^{-1/2}(\Gamma)}\left|\left|u_{1m}\right|\right|_{H^{1/2}(\Gamma)} + CT\, meas(\Gamma) \qquad (3.12)\\ &+C_1\int_0^t \left(\left|\left|u_m(s)\right|\right|^{2(\gamma+1)}_{H^{1/2}(\Gamma)} + \left|u_m'(s)\right|^2\right) ds + \tfrac{C}{2\eta}\int_0^t \left|\left|u_m'(s)\right|\right|^{\rho+2}_{\rho+2} ds \\ &+C\int_0^t \left|u_m''(s)\right|^2 ds + C_2\int_0^t \left|\left|u_m(s)\right|\right|^\gamma_{H^{1/2}(\Gamma)}\left|\left|u_m'(s)\right|\right|_{H^{1/2}(\Gamma)}\left|u_m''(t)\right| ds, \end{aligned}$$

where C_1 and C_2 are positive constants.

From (3.12), making use of the first estimate (3.6), considering the convergence in (3.3), taking (3.8) into account, choosing $\eta > 0$ sufficiently small and applying Gronwall's lemma, we obtain the second estimate

$$\left|u_m''(t)\right|^2 + \left|\left|u_m'(t)\right|\right|^2_{H^{1/2}(\Gamma)} + \int_0^t\int_\Gamma \left|u_m'\right|^\rho \left|u_m''\right|^2 d\Gamma ds \le L_3, \qquad (3.13)$$

where L_3 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0,T]$.

3.1.2 - Passage to the Limit: From the estimates (3.6) and (3.13) we deduce that there exists $\{u_\mu\}$, subsequence of $\{u_m\}$, which from now on will be represented by the same notation, and a function u, such that

$$\begin{aligned} u_\mu' &\rightharpoonup u' \quad \text{weak-star in } L^\infty_{loc}(0,\infty; L^2(\Gamma)), &(3.14)\\ u_\mu &\rightharpoonup u \quad \text{weak-star in } L^\infty_{loc}(0,\infty; H^{1/2}(\Gamma)), &(3.15)\\ u_\mu'' &\rightharpoonup u'' \quad \text{weak-star in } L^\infty_{loc}(0,\infty; L^2(\Gamma)), &(3.16)\\ u_\mu' &\rightharpoonup u' \quad \text{weak-star in } L^\infty_{loc}(0,\infty; H^{1/2}(\Gamma)). &(3.17) \end{aligned}$$

On the other hand, from the assumption $(H.2)$ and having in mind the embedding in (3.4), we infer

$$\begin{aligned} &\int_0^T\int_\Gamma \left|F\left(x,t,u_\mu,u_\mu'\right)\right|^2 d\Gamma dt \qquad (3.18)\\ &\le 4C\left\{meas(\Gamma)T + \int_0^T \left|\left|u_\mu(t)\right|\right|^{2(\gamma+1)}_{2(\gamma+1)} dt + \int_0^T \left|\left|u_\mu'(t)\right|\right|^{2(\rho+1)}_{2(\rho+1)} dt\right\} \le C' \end{aligned}$$

where C' is a positive constant independent of $\mu \in \mathbf{N}$ and $t \in [0,T]$.

On the other hand, from the a priori estimates, we also deduce that

$$\begin{aligned} &\{u_\mu\} \quad \text{is bounded in } L^2_{loc}(0,\infty; H^{1/2}(\Gamma)),\\ &\{u_\mu'\} \quad \text{is bounded in } L^2_{loc}(0,\infty; H^{1/2}(\Gamma)),\\ &\{u_\mu''\} \quad \text{is bounded in } L^2_{loc}(0,\infty; L^2(\Gamma)). \end{aligned}$$

Since the embedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact, using the Aubin-Lions theorem, see J. L. Lions [4], pp. 57-58, we conclude

$$\begin{aligned} u_\mu &\to u \quad \text{strongly in } L^2_{loc}(0,\infty; L^2(\Gamma)), &(3.19)\\ u_\mu' &\to u' \quad \text{strongly in } L^2_{loc}(0,\infty; L^2(\Gamma)). &(3.20) \end{aligned}$$

Consequently,

$$F(x,t,u_\mu,u'_\mu) \to F(x,t,u,u') \quad \text{a.e.} \quad \text{in } \Gamma\times(0,T). \tag{3.21}$$

Then, combining (3.18) and (3.21) we conclude, applying Lions' lemma, see J. L. Lions [[4], pp. 12-13], that

$$F(x,t,u_\mu,u'_\mu) \rightharpoonup F(x,t,u,u') \quad \text{weakly in } L^2_{loc}(0,\infty;L^2(\Gamma)). \tag{3.22}$$

Finally, since $A\in\mathcal{L}(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))$, from (3.15) we infer

$$Au_\mu \rightharpoonup Au \quad \text{weak-star in } L^\infty_{loc}(0,\infty;H^{-1/2}(\Gamma)). \tag{3.23}$$

The above convergence are sufficient to pass to the limit in (3.2) to obtain

$$u''+Au+F(x,t,u,u')=0 \quad \text{in } L^2_{loc}(0,\infty;L^2(\Gamma)). \tag{3.24}$$

3.1.3 Uniqueness: Let u and $\hat{u}$ be two regular solutions of $(*)$ satisfying theorem 2.1. Defining $z=u-\hat{u}$, from assumption $(H.7)$ we obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\left\{|z'(t)|^2+(Az(t),z(t))\right\} &\le D\textstyle\int_\Gamma(|u|^\gamma+|\hat{u}|^\gamma)\,|z|\,|z'|\,d\Gamma \\ &\le C(\gamma)\left(||u(t)||^\gamma_{2(\gamma+1)}+||\hat{u}(t)||^\gamma_{2(\gamma+1)}\right)||z(t)||_{2(\gamma+1)}\,|z'(t)|\end{aligned} \tag{3.25}$$

where the last inequality comes from the generalized Hölder inequality.

Integrating (3.25) over (0,t), observing (1.1), (3.4), (3.6) and (3.15), we deduce

$$\begin{aligned}&\tfrac{1}{2}\,|z'(t)|^2+\tfrac{\alpha}{2}\,||z(t)||^2_{H^{1/2}(\Gamma)} \\ &\le C_1(\gamma)L_1^\gamma\textstyle\int_0^t\left(\tfrac{1}{2}\,||z(s)||^2_{H^{1/2}(\Gamma)}+\tfrac{1}{2}\,|z'(s)|^2\right)ds.\end{aligned} \tag{3.26}$$

Employing Gronwall's lemma, from (3.26) we obtain that $|z'(t)|^2=||z(t)||^2_{H^{1/2}(\Gamma)}$ $=0$, which concludes the proof of uniqueness. $\diamondsuit$

Now, let us consider the existence of regular solutions for $(*)$ when $F=0$ and $g\neq 0$ making use of the special basis $\{\omega_j\}$ formed by eigen-functions of the operator A defined by the triple $\left\{H^{1/2}(\Gamma),L^2(\Gamma),((\cdot,\cdot))_{H^{1/2}(\Gamma)}\right\}$ whose properties were mentioned in the introduction of this paper. So, put $V_m=[\omega_1,\cdots,\omega_m]$ and $u_m(t)=\sum_{j=1}^m\delta_{jm}(t)\omega_j$ satisfying the Cauchy problem

$$(u''_m(t),w)+(Au_m(t),w)-\textstyle\int_0^t g(t-\tau)\,(Au_m(\tau),w)\,d\tau=0,\ \forall w\in V_m, \tag{3.27}$$

$$u_m(0)=u_{0m}\to u^0 \text{ in } D(A);\ u'_m(0)=u_{1m}\to u^1 \quad \text{in } D(A^{1/2}). \tag{3.28}$$

3.1.7 - A Priori Estimates: Considering $w=Au'_m(t)$ in (3.27), it holds that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left\{|A^{1/2}u'_m(t)|^2+|Au_m(t)|^2\right\}+g(0)|Au_m(t)|^2 \\ &=-\textstyle\int_0^t g'(t-\tau)\,(Au_m(\tau),Au_m(t))\,d\tau \\ &+\tfrac{d}{dt}\left\{\textstyle\int_0^t g(t-\tau)\,(Au_m(\tau),Au_m(t))\,d\tau\right\}.\end{aligned} \tag{3.29}$$

But, from assumption $(H.9)$ and making use of the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, $\eta > 0$, we have

$$\begin{aligned} &\int_0^t g'(t-\tau)\,(Au_m(\tau), Au_m(t))\,d\tau \\ &\leq \frac{\xi_1^2}{4\eta}\,||g||_{L^1(0,\infty)} \int_0^t g(t-\tau)\,|Au_m(\tau)|^2\,d\tau + \eta\,|Au_m(t)|^2 . \end{aligned} \tag{3.30}$$

Integrating (3.29) over (0,t) taking (3.30) into account, we deduce

$$\begin{aligned} &\tfrac{1}{2}\left|A^{1/2}u_m'(t)\right|^2 + \tfrac{1}{2}\,|Au_m(t)|^2 + (g(0)-\eta)\int_0^t |Au_m(s)|^2\,ds \\ &\leq \tfrac{1}{2}\,|Au_{1m}|^2 + |Au_{0m}|^2 + \frac{\xi_1^2}{4\eta}\,||g||^2_{L^1(0,\infty)} \int_0^t |Au_m(s)|^2\,ds \\ &+ \int_0^t g(t-\tau)\,(Au_m(\tau), Au_m(t))\,d\tau. \end{aligned} \tag{3.31}$$

We notice that

$$\begin{aligned} &\int_0^t g(t-\tau)\,(Au_m(\tau), Au_m(t))\,d\tau \\ &\leq \eta\,|Au_m(t)|^2 + \frac{1}{4\eta}\,||g||_{L^1(0,\infty)}\,||g||_{L^\infty(0,\infty)} \int_0^t |Au_m(\tau)|^2\,d\tau. \end{aligned} \tag{3.32}$$

Combining (3.31)-(3.32), choosing $\eta > 0$ small enough, observing the convergence in (3.28) and employing Gronwall's lemma we conclude the estimate

$$|A^{1/2}u_m'(t)|^2 + |Au_m(t)|^2 \leq L_7, \tag{3.33}$$

where L_7 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0,T]$.

3.1.8 - Passage to the Limit: From the estimate (3.33) we are able to pass to the limit in (3.27) in order to obtain

$$u'' + Au - g * Au = 0 \quad \text{in } L^2_{loc}(0,\infty; L^2(\Gamma)). \tag{3.34}$$

The proof of the uniqueness is similar to the above case. Thus, it will be omitted.

Let us consider, now, m_2 and m_1 positive natural numbers such that $m_2 > m_1$ and let us define in $u_m(t) = \sum_{j=1}^m \delta_{jm}(t)\omega_j$ the following

$$\delta_{jm_1} = 0 \quad \text{for } m_1 \leq j \leq m_2.$$

Under this assumption we can conclude that both u_{m_2} and u_{m_1} are approximated solutions of system (3.27), since it is a linear one. Denoting $z_m = u_{m_2} - u_{m_1}$ the Cauchy difference, we deduce, proceeding as we have done in the uniqueness of solutions that

$$|A^{1/2}z_m'(t)|^2 + |Az_m(t)|^2 \leq C\left\{|A^{1/2}z_m'(0)|^2 + |Az_m(0)|^2\right\}, \tag{3.35}$$

where C is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0,T]$.

We can write (3.35) as $\left|\left|u_{m_2}'(t) - u_{m_1}'(t)\right|\right|^2_{H^{1/2}(\Gamma)} + ||u_{m_2}(t) - u_{m_1}(t)||^2_{D(A)}$

$$\leq C\left\{||u_{1m_2} - u_{1m_1}||^2_{H^{1/2}(\Gamma)} + ||u_{0m_2} - u_{0m_1}||^2_{D(A)}\right\}.$$

The last inequality yields $\{u_m\}$ is a sequence of Cauchy in $C^0([0,T];D(A))$, $\{u_m\}$ is a sequence of Cauchy in $C^0([0,T];H^{1/2}(\Gamma))$. Therefore, there exists a function u such that

$$u_m \to u \quad \text{strongly in } C^0([0,T];D(A)); \quad \text{for all } T>0, \tag{3.36}$$

$$u_m' \to u' \quad \text{strongly in } C^0([0,T];H^{1/2}(\Gamma)), \quad \text{for all } T>0. \tag{3.37}$$

This concludes the proof of existence of regular solutions of Theorems 2.1 and 2.3.

3.2 Weak Solutions: We begin this section considering the case $F \neq 0$ and $g=0$ and $A: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the linear, continuous, self-adjoint and coercive operator mentioned in the introduction. So, let $\left\{u^0,u^1\right\} \in H^{1/2}(\Gamma) \times L^2(\Gamma)$. Then, according to lemma 2.2, $\mathcal{H} = \left\{u \in H^{1/2}(\Gamma); Au \in L^2(\Gamma)\right\}$is dense in $H^{1/2}(\Gamma)$ and since $H^{1/2}(\Gamma)$ is also dense in $L^2(\Gamma)$, there exists $\left\{u_\mu^0, u_\mu^1\right\} \in \mathcal{H} \times H^{1/2}(\Gamma)$ such that

$$u_\mu^0 \to u^0 \quad \text{in } H^{1/2}(\Gamma) \quad \text{and} \quad u_\mu^1 \to u^1 \quad \text{in } L^2(\Gamma). \tag{3.38}$$

For each $\mu \in \mathbf{N}$, let u_μ be the regular solution of $(*)$ with $g=0$, and initial data $\left\{u_\mu^0, u_\mu^1\right\}$, that is

$$\begin{cases} u_\mu'' + Au_\mu + F(x,t,u_\mu,u_\mu') = 0 \quad \text{a.e. in } \Gamma \times (0,\infty) \\ u_\mu(0) = u_\mu^0; \quad u_\mu'(0) = u_\mu^1. \end{cases}$$

Repeating analogous arguments used in section 3.1.1, we deduce as in (3.6) that

$$|u_\mu'(t)|^2 + ||u_\mu(t)||^2_{H^{1/2}(\Gamma)} + ||u_\mu(t)||^{\gamma+2}_{\gamma+2} + \int_0^t \left|\left|u_\mu'(s)\right|\right|^{\rho+2}_{\rho+2} ds \le C_1 \tag{3.39}$$

for all $t \ge 0$ and for all $\mu \in \mathbf{N}$, where C_1 is a positive constant independent of μ and t.

Now, defining $z_{\mu,\sigma} = u_\mu - u_\sigma$; $\mu,\sigma \in \mathbf{N}$, taking (3.39) into account and making use of the same arguments considered in the proof of the uniqueness, section 3.1.3, we obtain

$$\begin{aligned} &|u_\mu'(t) - u_\sigma'(t)|^2 + ||u_\mu(t) - u_\sigma(t)||^2_{H^{1/2}(\Gamma)} \\ &\le C(\gamma,T)\left(\left|u_\mu^1 - u_\sigma^1\right|^2 + \left|\left|u_\mu^0 - u_\sigma^0\right|\right|^2_{H^{1/2}(\Gamma)}\right) \end{aligned} \tag{3.40}$$

where $C(\gamma,T)$ is a positive constant independent of $\mu \in \mathbf{N}$.

From the last inequality and considering (3.38) we conclude that there exists a function u such that , for all $T>0$, we have

$$u_\mu \to u \quad \text{strongly in } C^0([0,T],H^{1/2}(\Gamma)), \tag{3.41}$$

$$u_\mu' \to u' \quad \text{strongly in } C^0([0,T];L^2(\Gamma)). \tag{3.42}$$

The strong convergence in (3.41)-(3.42) and the weak ones which came from (3.39) are sufficient to pass to the limit using arguments of compactness in order to obtain a weak solution to problem $(*)$ with $g = 0$. More precisely, one has

$$\begin{cases} u'' + \tilde{A}u + F(x,t,u,u') = 0 \quad \text{in } L^2_{loc}(0,\infty; H^{-1/2}(\Gamma)) \\ u(0) = u^0, \quad u'(0) = u^1 \end{cases} \tag{3.43}$$

The uniqueness of weak solutions requires a regularization procedure and can be obtained using the standard method of Visik-Ladyzhenkaya, c.f. J. L. Lions [[4], pp. 14-16]. The case $F = 0$ and $g \neq 0$ is similar. Then, its proof will be omitted. So, the proof of existence of weak solutions of Theorem 2.2 and 2.3 is concluded.

4 Asymptotic Stability

In this section we obtain the uniform decay of the energy for regular solutions, since the same occurs for weak solutions using standard density arguments.

Let us consider, initially, $g = 0$ and $F \neq 0$ according to theorem 2.1.

From (3.24) and taking the assumption $(H.3)$ into account, we deduce that

$$E'(t) \leq -\beta \, ||u'(t)||^{\rho+2}_{\rho+2}\,, \tag{4.1}$$

where $E(t)$ is defined in (2.15). Let us define the Liapunov functional

$$\psi(t) = [E(t)]^{\rho/2} \,(u'(t), u(t))\,. \tag{4.2}$$

Taking the derivative of $\psi(t)$ with respect t and substituting $u'' = -Au - F(x,t,u,u')$ in the obtained expression, it follows that

$$\begin{aligned} \psi'(t) \quad &= \quad \frac{\rho}{2}\,[E(t)]^{\frac{\rho-2}{2}}\,E'(t)\,(u'(t),u(t)) \\ &+ \quad [E(t)]^{\rho/2}\left\{-\,(Au(t),u(t)) - (F(x,t,u(t),u'(t)),u(t)) + |u'(t)|^2\right\}. \end{aligned} \tag{4.3}$$

On the other hand, from (1.1) and noting that $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ it holds that

$$\begin{aligned} |(u'(t),u(t))| \quad &\leq \quad k_1\,|u'(t)|\,||u(t)||_{H^{1/2}(\Gamma)} \\ &\leq \quad k_1\alpha\,|u'(t)|\,(Au(t),u(t))^{1/2} \leq CE(t), \end{aligned} \tag{4.4}$$

where k_1 and C are positive constants.

The inequality in (4.4) yields

$$-\frac{\rho}{2}\,[E(t)]^{\frac{\rho-2}{2}}\,(u'(t),u(t)) \leq \frac{C\rho}{2}\,[E(0)]^{\rho/2}\,, \tag{4.5}$$

and since $-E'(t) \geq 0$, we deduce

$$\frac{\rho}{2}\,[E(t)]^{\frac{\rho-2}{2}}\,(u'(t),u(t))\,E'(t) \leq -C_1E'(t) \tag{4.6}$$

where $C_1 = \frac{C\rho}{2}\left[E(0)\right]^{\rho/2}$. Combining (4.3), (4.6) and considering the hypothesis $(H.3)$, we infer

$$\psi'(t) \le -C_1 E'(t) \tag{4.7}$$
$$+\left[E(t)\right]^{\rho/2}\left\{-\left(Au(t),u(t)\right) - ||u(t)||_{\gamma+2}^{\gamma+2} - \beta\int_\Gamma |u'|^{\rho+1}|u|\,d\Gamma + |u'(t)|^2\right\}.$$

Estimate for $J_1 := \beta\int_\Gamma |u'|^{\rho+1}|u|\,d\Gamma$. Noting that $\frac{\rho+1}{\rho+2}+\frac{1}{\rho+2}=1$, having in mind that $H^{1/2}(\Gamma)\hookrightarrow L^{\rho+2}(\Gamma)$, taking (1.1) into account and applying Hölder and Young inequalities, we obtain

$$\begin{aligned} |J_1| &\le \beta\,||u'(t)||_{\rho+2}^{\rho+1}\,||u(t)||_{\rho+2} \le k_2\,||u'(t)||_{\rho+2}^{\rho+1}\,||u(t)||_{H^{1/2}(\Gamma)} \\ &\le k_2\alpha\,||u'(t)||_{\rho+2}^{\rho+1}\left(Au(t),u(t)\right)^{1/2} \\ &\le k_3(\eta)\,||u'(t)||_{\rho+2}^{\rho+2} + \eta\left(Au(t),u(t)\right)^{\frac{\rho+2}{2}} \end{aligned} \tag{4.8}$$

where $\eta > 0$ is arbitrary and $k_3(\eta)$ is a positive constant which depends on η.
But,

$$\left(Au(t),u(t)\right)^{\frac{\rho+2}{2}} \le 2^{\rho/2}\left[E(0)\right]^{\rho/2}\left(Au(t),u(t)\right). \tag{4.9}$$

Then, from (4.7), (4.8) and (4.9) we arrive at

$$\begin{aligned} \psi'(t) &\le -C_1E'(t) \\ &+ \left[E(t)\right]^{\rho/2}\Big\{-\left(1-\eta 2^{\rho/2}\left[E(0)\right]^{\rho/2}\right)\left(Au(t),u(t)\right) + k_3(\eta)\,||u'(t)||_{\rho+2}^{\rho+2} \\ &\qquad - ||u(t)||_{\gamma+2}^{\gamma+2} - |u'(t)|^2\Big\}. \end{aligned} \tag{4.10}$$

Choosing $\eta > 0$ such that $1-\eta 2^{\rho/2}\left[E(0)\right]^{\rho/2} = \frac{1}{2}$, from (4.10) we obtain

$$\begin{aligned} \psi'(t) &\le -C_1E'(t) + k_3\left[E(0)\right]^{\rho/2}||u'(t)||_{\rho+2}^{\rho+2} \\ &+ \left[E(t)\right]^{\rho/2}\left\{-\frac{1}{2}\left(Au(t),u(t)\right) - \frac{1}{\gamma+2}\,||u(t)||_{\gamma+2}^{\gamma+2} + |u'(t)|^2\right\}. \end{aligned} \tag{4.11}$$

From (4.1) and (4.11), we have

$$\begin{aligned} \psi'(t) &\le -(C_1+C_2)E'(t) \\ &+ \left[E(t)\right]^{\rho/2}\left\{-\frac{1}{2}\left(Au(t),u(t)\right) - \frac{1}{\gamma+2}\,||u(t)||_{\gamma+2}^{\gamma+2}\right\} + \left[E(t)\right]^{\rho/2}|u'(t)|^2, \end{aligned} \tag{4.12}$$

where $C_2 = \beta^{-1}k_3\left[E(0)\right]^{\rho/2}$.
Defining the perturbed energy by

$$E_\varepsilon(t) = \left(1+\varepsilon\left(C_1+C_2\right)\right)E(t) + \varepsilon\psi(t) \tag{4.13}$$

then, there exists, in view of (4.4), $L = L(E(0))$ such that

$$|E_\varepsilon(t) - E(t)| \le \varepsilon L\, E(t); \quad \text{for all } \varepsilon > 0. \tag{4.14}$$

Considering $\varepsilon \in (0, 1/2L]$, from (4.14) we deduce

$$\tfrac{1}{2} E(t) \le E_\varepsilon(t) \le 2E(t) \tag{4.15}$$

and consequently

$$2^{-\frac{\rho+2}{2}} \left[E(t)\right]^{\frac{\rho+2}{2}} \le \left[E_\varepsilon(t)\right]^{\frac{\rho+2}{2}} \le 2^{\frac{\rho+2}{2}} \left[E(t)\right]^{\frac{\rho+2}{2}}; \quad \varepsilon \in (0, 1/2L]. \tag{4.16}$$

Getting the derivative of (4.13) with respect to t, taking (4.1) and (4.12) into account, we infer

$$\begin{aligned} E'_\varepsilon(t) &\le -\beta\, ||u'(t)||_{\rho+2}^{\rho+2} \\ &+\varepsilon \left[E(t)\right]^{\rho/2} \left\{ -\tfrac{1}{2} \left(Au(t), u(t)\right) - \tfrac{1}{\gamma+2} ||u(t)||_{\gamma+2}^{\gamma+2} \right\} + \varepsilon \left[E(t)\right]^{\rho/2} \left|u'(t)\right|^2. \end{aligned} \tag{4.17}$$

Having in mind that $-\frac{1}{2}\left(Au(t), u(t)\right) = -E(t) + \frac{1}{2}\left|u'(t)\right|^2 + \frac{1}{\gamma+2} ||u(t)||_{\gamma+2}^{\gamma+2}$ and since $L^{\rho+2}(\Gamma) \hookrightarrow L^2(\Gamma)$, from (4.17) it holds that

$$E'_\varepsilon(t) \le -\beta C_0 \left|u'(t)\right|^{\rho+2} - \varepsilon \left[E(t)\right]^{\frac{\rho+2}{2}} + \frac{3}{2} \varepsilon \left[E(t)\right]^{\rho/2} \left|u'(t)\right|^2 \tag{4.18}$$

where C_0 comes from the imbedding $L^{\rho+2}(\Gamma) \hookrightarrow L^2(\Gamma)$.

Observing that $\frac{\rho}{\rho+2} + \frac{2}{\rho+2} = 1$ the Hölder inequality yields

$$\begin{aligned} \left[E(t)\right]^{\rho/2} \left|u'(t)\right|^2 &\le \frac{\rho}{\rho+2} \left(\mu \left[E(t)\right]^{\rho/2}\right)^{\frac{\rho+2}{\rho}} + \frac{2}{\rho+2} \left(\frac{1}{\mu} \left|u'(t)\right|^2\right)^{\frac{\rho+2}{2}} \\ &\le \mu^{\frac{\rho+2}{\rho}} \left[E(t)\right]^{\frac{\rho+2}{2}} + \frac{1}{\mu^{\frac{\rho+2}{2}}} \left|u'(t)\right|^2, \end{aligned} \tag{4.19}$$

where $\mu > 0$ is arbitrary.

Combining (4.18) and (4.19), we obtain

$$E'_\varepsilon(t) \le -\left(\beta C_0 - \frac{3}{2} \varepsilon \frac{1}{\mu^{\frac{\rho+2}{2}}}\right) \left|u'(t)\right|^2 - \varepsilon \left(1 - \frac{3}{2} \mu^{\frac{\rho+2}{\rho}}\right) \left[E(t)\right]^{\frac{\rho+2}{2}}. \tag{4.20}$$

Choosing $\mu > 0$ sufficiently small such that $\theta = 1 - \frac{3}{2}\mu^{\frac{\rho+2}{\rho}} > 0$ and ε small enough in order to have $\beta C_0 - \frac{3}{2}\varepsilon \frac{1}{\mu^{\frac{\rho+2}{2}}} \ge 0$ from (4.20) we conclude that

$$E'_\varepsilon(t) \le -\varepsilon\theta \left[E(t)\right]^{\frac{\rho+2}{2}}. \tag{4.21}$$

Combining (4.16) and (4.21) we infer that $E'_\varepsilon(t) \le -\frac{N}{2^{\frac{\rho+2}{2}}} \left[E(t)\right]^{\frac{\rho+2}{2}}$, where $N = \varepsilon\theta$. Therefore

$$E'_\varepsilon(t) \left[E(t)\right]^{-\frac{\rho+2}{2}} \le -\frac{N}{2^{\frac{\rho+2}{2}}}. \tag{4.22}$$

But since $\frac{d}{dt}\left[E_\varepsilon(t)\right]^{-\rho/2} = -\frac{\rho}{2}\left[E(t)\right]^{-\frac{\rho+2}{2}} E'_\varepsilon(t)$ from (4.22) it holds that $\frac{d}{dt}\left[E_\varepsilon(t)\right]^{-\rho/2} \geq \frac{\rho N}{2^{\frac{\rho+4}{2}}}$.

Integrating the above inequality over (0,t), it follows that

$$\left[E_\varepsilon(t)\right]^{-\rho/2} \geq \left[E_\varepsilon(0)\right]^{-\rho/2} + \frac{\rho N}{2^{\frac{\rho+4}{2}}} t. \tag{4.23}$$

Finally, from (4.23) and (4.16) we deduce that

$$E_\varepsilon(t) \leq \left\{\left[E_\varepsilon(0)\right]^{-\rho/2} + \frac{\rho N}{2^{\frac{\rho+4}{2}}} t\right\}^{-2/\rho} \leq \left\{2^{\rho/2}\left[E(0)\right]^{-\rho/2} + \frac{\rho N}{2^{\frac{\rho+4}{2}}} t\right\}^{-2/\rho}$$

which implies

$$E(t) \leq \left\{\left[E(0)\right]^{-\rho/2} + \frac{\rho N}{2^{\rho+2}} t\right\}^{-2/\rho}. \tag{4.24}$$

We observe that when $\rho = 0$ then, from (4.15) and (4.18) the exponential decay holds easily. The proof of theorems 2.1 and (by density arguments) theorem 2.2 is completed. $\diamondsuit$ From now on, we will consider the last case, that is, $F = 0$ and $g \neq 0$, according to Theorem 2.3. We will prove that the kernel is strong enough to derive an exponential (or polynomial) decay provided the kernel decays exponentially (or polynomially).

As $F = 0$, equation $(*)$ becomes

$$u_{tt} + Au - \int_0^t g(t-\tau)Au(\tau)\,d\tau = 0 \quad \text{on } \Gamma \times (0,\infty). \tag{4.47}$$

Taking the duality product between equation (4.47) and $u'(t)$ and using identity (4.26) we obtain

$$E'(t) = \frac{1}{2}(g' \diamond A^{1/2}u(t) - g(t)|A^{1/2}u(t)|^2). \tag{4.48}$$

Now, let us introduce the following functional

$$R_1(t) := \left\{-\big(u', (g*u)'\big) - \frac{1}{2}g'' \diamond u + \frac{1}{2}g'(t)|u|^2 + \frac{1}{2}|g * A^{1/2}u|^2\right\}.$$

The duality product between the equation (4.47) and $(g*u)'$ together with identity (4.26) imply that

$$R_1'(t) = -g(0)|u'|^2 - \frac{1}{2}g''' \diamond u + \frac{1}{2}g''(t)|u|^2 + (A^{1/2}u, (g * A^{1/2}u)').$$

Similarly, for the functional

$$R_2(t) := \big(u', u\big),$$

we have, by considering the duality product between equation (4.47) and u, that

$$R_2'(t) = |u'|^2 - \beta(t)|A^{1/2}u|^2 - \big(A^{1/2}u, g\triangle A^{1/2}u\big),$$

where the function β and the binary operator $\triangle$ are given by

$$\beta(t) := 1 - \textstyle\int_0^t g(s)\,ds, \qquad (g\triangle h)(t) := \textstyle\int_0^t g(t-s)\big(h(t)-h(s)\big)\,ds,$$

Note that, from definition and Hölder inequality, this binary operator has the following properties

$$\begin{aligned} (g\triangle h)(t) &= \left(\int_0^t g(s)\,ds\right) h(t) - (g*h)(t), \\ |g\triangle h|^2(t) &\leq \left(\int_0^t g(s)\,ds\right)(g\diamond h)(t). \end{aligned} \tag{4.49}$$

In these conditions, for the functional

$$\mathcal{R}(t) := R_1(t) + \frac{g(0)}{2}R_2(t),$$

we have that, from the previous estimates

$$\begin{aligned} \mathcal{R}'(t) &= -\frac{g(0)}{2}|u'|^2 - \frac{g(0)}{2}\beta(t)|A^{1/2}u|^2 - \frac{g(0)}{2}\big(A^{1/2}u, g\triangle A^{1/2}u\big) \\ &\quad -\frac{1}{2}g''' \diamond u + \frac{1}{2}g''(t)|u|^2 + \big(A^{1/2}u, (g*A^{1/2}u)'\big). \end{aligned} \tag{4.50}$$

The term $(g*A^{1/2}u)'$ of the above identity can be written as

$$(g*A^{1/2}u)' = g(t)A^{1/2}u + g\triangle A^{1/2}u.$$

Applying Young's inequality to (4.50), using the above identity, hypothesis $(H.10)-(H.11)$, inequality (4.49) and adding the term $g\diamond A^{1/2}u$ we get

$$\mathcal{R}'(t) \leq -\frac{g(0)}{2}E(t) + C\big\{g(t)|A^{1/2}u(t)|^2 - g'\diamond A^{1/2}u(t) + g\diamond A^{1/2}u(t)\big\}. \tag{4.51}$$

In this point we will see that the rate of decay of the energy will depend of a appropriate estimate of the last term of the above inequality.

Exponential Decay: We consider hypothesis $(H.9)$ which implies that inequality (4.51) can be written as

$$\mathcal{R}'(t) \leq -\frac{g(0)}{2}E(t) + C\big\{g(t)|A^{1/2}u(t)|^2 - g'\diamond A^{1/2}u(t)\big\}.$$

Let us consider the perturbed energy $E_\delta(t) := E(t) + \delta\mathcal{R}(t)$. It is easy to verify using Young's inequality that this functional satisfies, for $\delta > 0$ small

$$\frac{1}{2}E(t) \leq E_\delta(t) \leq 2E(t). \tag{4.52}$$

The definition of E_δ and inequalities (4.48), (4.51) imply that, for δ small

$$E'_\delta(t) \leq -\frac{\delta g(0)}{2} E(t) \leq -\frac{\delta g(0)}{4} E_\delta(t),$$

from where follows that $E_\delta(t) \leq E_\delta(0) e^{-\frac{\delta g(0)}{4} t}$. Therefore, in view of inequality (4.52) we conclude that $E(t) \leq 4E(0) e^{-\frac{\delta g(0)}{4} t}$. This proves the first part of Theorem 2.3.

Polynomial Decay: In this case we consider hypothesis $(H.12)$. For to estimate the last term of inequality (4.51) we will need some technical lemmas.

Lemma 4.1. *Suppose that $g \in C([0,\infty[)$, $w \in L^1_{loc}(0,\infty)$ and $0 \leq \theta \leq 1$, then we have that*

$$\int_0^t |g(\tau)w(\tau)|\, d\tau \leq \left\{\int_0^t |g(\tau)|^{1-\theta} |w(\tau)|\, d\tau\right\}^{\frac{1}{\sigma+1}} \left\{\int_0^t |g(\tau)|^{1+\frac{\theta}{\sigma}} |w(\tau)|\, d\tau\right\}^{\frac{\sigma}{\sigma+1}}.$$

Proof. For any fixed t we have

$$\int_0^t |g(\tau)w(\tau)|\, d\tau = \int_0^t \underbrace{|g(\tau)|^{\frac{1-\theta}{\sigma+1}} |w(\tau)|^{\frac{1}{\sigma+1}}}_{:=w_1} \underbrace{|g(\tau)|^{1-\frac{1-\theta}{\sigma+1}} |w(\tau)|^{\frac{\sigma}{\sigma+1}}}_{:=w_2}\, d\tau.$$

Note that $w_1 \in L^s_{loc}(0,\infty)$, $w_2 \in L^{s'}_{loc}(0,\infty)$, where $s = \sigma+1$ and $s' = \frac{\sigma+1}{\sigma}$. Using Hölder's inequality, we get

$$\int_0^t |g(\tau)w(\tau)|\, d\tau \leq \left\{\int_0^t |g(\tau)|^{1-\theta} |w(\tau)|\, d\tau\right\}^{\frac{1}{\sigma+1}} \left\{\int_0^t |g(\tau)|^{1+\frac{\theta}{\sigma}} |w(\tau)|\, d\tau\right\}^{\frac{\sigma}{\sigma+1}}.$$

This completes the proof. $\diamondsuit$

Lemma 4.2. *Let us suppose that $v \in L^\infty(0,T;D(A^{1/2})$ and g is a continuous function. Then, there exists $C > 0$ such that*

$$g \diamond A^{1/2}v \leq C\left\{\int_0^t |A^{1/2}v|^2\, d\tau + t|A^{1/2}v|^2\right\}^{\frac{1}{p+1}} \left\{g^{1+\frac{1}{p}} \diamond A^{1/2}v\right\}^{\frac{p}{p+1}}.$$

Moreover, If there exists $0 < \theta < 1$ such that $\displaystyle\int_0^\infty g^{1-\theta}(s)\, ds < \infty$, then we have

$$g \diamond A^{1/2}v \leq C\left\{\left(\int_0^\infty g^{1-\theta}\, d\tau\right) \|A^{1/2}v\|^2_{L^\infty(0,T;L^2)}\right\}^{\frac{1}{\theta p+1}} \left\{g^{1+\frac{1}{p}} \diamond A^{1/2}v\right\}^{\frac{\theta p}{\theta p+1}}.$$

Proof. From the hypothesis on v and Lemma 4.1 we get

$$\begin{aligned} g \diamond A^{1/2}v &= \int_0^t g(t-\tau) \underbrace{|A^{1/2}v(t) - A^{1/2}v(\tau)|^2}_{=w(\tau)}\, d\tau \\ &\leq \left\{\int_0^t g^{1-\theta}(t-\tau)w(\tau)d\tau\right\}^{\frac{1}{\theta p+1}} \left\{\int_0^t g^{1+\frac{1}{p}}(t-\tau)w(\tau)d\tau\right\}^{\frac{\theta p}{\theta p+1}} \\ &\leq \left\{g^{1-\theta} \diamond A^{1/2}v\right\}^{\frac{1}{\theta p+1}} \left\{g^{1+\frac{1}{p}} \diamond A^{1/2}v\right\}^{\frac{\theta p}{\theta p+1}} \end{aligned} \tag{4.53}$$

Now, for $0 < \theta < 1$ we have

$$\begin{aligned} g^{1-\theta} \diamond A^{1/2}v &= \int_0^t g^{1-\theta}(t-\tau)|A^{1/2}v(t) - A^{1/2}v(\tau)|^2 d\tau \\ &\leq C\left(\int_0^t g^{1-\theta}(\tau)\, d\tau\right) \|A^{1/2}v\|^2_{L^\infty(0,T;L^2)}. \end{aligned}$$

From where the second inequality of this Lemma follows. when $\theta = 1$ we get

$$\begin{aligned} 1 \diamond A^{1/2}v &= \int_0^t |A^{1/2}v(t) - A^{1/2}v(\tau)|^2\, d\tau \\ &\leq C\left\{t|A^{1/2}v(t)|^2 + \int_0^t |A^{1/2}v(\tau)|^2\, d\tau\right\}. \end{aligned}$$

Substitution of this inequality into (4.53) yields the first inequality. The proof is now complete. $\diamondsuit$

Next, we will estimate the term $g \diamond A^{1/2}u$. From hypothesis $(H.12)$ it's easy to verify that $g(t) \leq C(1+t)^{-p}$ for some $C > 0$. Let us fix $\theta = 1/2$, then $(1-\theta)p > 1$, from where follows that

$$\int_0^\infty g^{1-\theta}(s)\, ds \leq C\int_0^\infty \tfrac{1}{(1+s)^{(1-\theta)p}}\, ds < \infty.$$

Using this estimate in the second part of Lemma 4.2 we get

$$g \diamond A^{1/2}u \leq CE(0)^{\frac{1}{\theta p+1}}\left(g^{1+\frac{1}{p}} \diamond A^{1/2}u\right)^{\frac{\theta p}{\theta p+1}}. \tag{4.54}$$

Substitution of this inequality into (4.51) we arrive at

$$\mathcal{R}'(t) \quad \leq \quad -\frac{g(0)}{2}E(t) + C\left\{g(t)|A^{1/2}u|^2 - g' \diamond A^{1/2}u + \left(g^{1+\frac{1}{p}} \diamond A^{1/2}u\right)^{\frac{\theta p}{\theta p+1}}\right\}.$$

Since $\mathcal{R}(t) \leq CE(t)$ for some $C > 0$, the above inequality implies that

$$\begin{aligned} [E^{\frac{1}{\theta p}}\mathcal{R}]'(t) &= \frac{1}{\theta p}\mathcal{R}(t)E^{\frac{1}{\theta p}-1}(t)E'(t) + E^{\frac{1}{\theta p}}(t)\mathcal{R}'(t) \\ &\leq -CE^{\frac{1}{\theta p}}(t)E'(t) + E^{\frac{1}{\theta p}}(t)\mathcal{R}'(t) \\ &\leq -k_1\left(E^{1+\frac{1}{\theta p}}\right)'(t) - \frac{g(0)}{2}E^{1+\frac{1}{\theta p}}(t) + CE^{\frac{1}{\theta p}}(0)\left\{g(t)|A^{1/2}u|^2 - g' \diamond A^{1/2}u\right\} \\ &\quad + CE^{\frac{1}{\theta p}}(t)\left(g^{1+\frac{1}{p}} \diamond A^{1/2}u\right)^{\frac{\theta p}{\theta p+1}}, \end{aligned} \tag{4.55}$$

for some positive constant k_1. Now, we will estimate the last term of the above inequality. Applying Young's inequality yields, for $\epsilon > 0$

$$E^{\frac{1}{\theta p}}(t)\left(g^{1+\frac{1}{p}} \diamond A^{1/2}u\right)^{\frac{\theta p}{\theta p+1}} \quad \leq \quad \epsilon E^{\frac{\theta p+1}{\theta p}}(t) + C_\epsilon g^{1+\frac{1}{p}} \diamond A^{1/2}u. \tag{4.56}$$

Substitution of (4.56) into (4.55) and taking ϵ small we arrive at

$$[E^{\frac{1}{\theta p}}(\mathcal{R} + k_1E)]'(t) \leq -\frac{g(0)}{4}E^{1+\frac{1}{\theta p}}(t) + C\left\{g(t)|A^{1/2}u|^2 - g' \diamond A^{1/2}u\right\}. \tag{4.57}$$

We consider a perturbed energy $E_\delta(t) := E(t) + \delta E^{\frac{1}{\theta p}}(t)\big(\mathcal{R}(t) + k_1 E(t)\big)$. Using Young's inequality we can verify that, for $\delta > 0$ small

$$\frac{1}{2}E(t) \leq E_\delta(t) \leq 2E(t). \tag{4.58}$$

From definition of the functional E_δ and inequalities (4.48), (4.57) we get, for δ small $E'_\delta(t) \leq -\frac{\delta g(0)}{4} E^{1+\frac{1}{\theta p}}(t)$, from where follows, in view of (4.58), that

$$E'_\delta(t) \quad \leq \quad -k_2 E_\delta^{1+\frac{1}{\theta p}}(t), \tag{4.59}$$

for some $k_2 > 0$. Hence, we obtain

$$E_\delta(t) \leq \frac{C}{(1+t)^{\theta p}} \quad \Longrightarrow \Big(\text{by } (4.58)\Big) \quad E(t) \leq \frac{C}{(1+t)^{\theta p}}.$$

Since $p > 2$ e $\theta = 1/2$ we have that $\theta p > 1$. Therefore

$$\int_0^\infty |A^{1/2}u(\tau)|^2 \, d\tau + t|A^{1/2}u(t)|^2 \leq C\left\{\int_0^\infty E(\tau)\, d\tau + t\, E(t)\right\} < \infty.$$

From the first part of Lemma 4.2 we get the following estimate

$$g \diamond A^{1/2}u \leq C\left(g^{1+\frac{1}{p}} \diamond A^{1/2}u\right)^{\frac{p}{p+1}}.$$

Using this inequality instead of (4.54) and repeating the same calculations and changing θp by p, we conclude that

$$E(t) \leq \frac{C}{(1+t)^p}.$$

This completes the proof. $\diamondsuit$

Acknowledgements. The authors are deeply grateful to the referee for fruitful comments and would like to thank Jerome Goldstein for his kind attention during the refereeing process.

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Volume 8,Number 3 July 2006
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

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Symmetries, Invariances, and Boundary Value Problems for the Hamilton-Jacobi Equation

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Abstract:

Even solutions, odd solutions, skew odd solutions, and periodic solutions to a perturbed Hamilton-Jacobi equation in N dimension are established via the theory of invariant sets for semigroups of nonlinear operators. These solutions are related to the Neumann, Dirichlet, and periodic initial-boundary value problems in the first quadrant. Lipschitz regularity of the solutions are also explored.

2000 AMS Subject Classification: 35F30, 35D10, 47H20

Keywords and phrases: symmetry, boundary value problem, Hamilton-Jacobi, m-dissipative operator, invariant set

1. Introduction

Of concern is the study of solutions to some initial and boundary value problems (IBVP) of the perturbed Hamilton-Jacobi equation

$$u_t + H(\nabla_x u) + G(\cdot, u) = 0,\ (x \in \mathbb{R}^N_+, t \geq 0), \tag{1.1}$$

on the first quadrant $\mathbb{R}^N_+ = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_i \geq 0\}$ of $\mathbb{R}^N$. The Hamiltonian H and the perturbation term G satisfy assumptions which will be stated later. In particular, we consider the Dirichlet, Neumann, and periodic boundary conditions for the perturbed problem (1.1). We will show that the corresponding initial-boundary value problems are governed by nonexpansive semigroups on certain closed subsets of the space of bounded and uniformly continuous functions on the first quadrant $BUC(\mathbb{R}^{\mathrm{N}}_+)$. The Crandall-Liggett theorem can be used to prove that the above problem is governed by a non linear semigroup, hence well-posedness of the problems follows.

The initial boundary value problems are related to the existence of solutions of the corresponding Cauchy problem on the whole space $\mathbb{R}^N$ having certain symmetries. The even solution on the whole space will solve the Neumann problem on the first quadrant. The skew-odd solution on the whole space will solve the Dirichlet problem on the first quadrant. Due to some sign counting problem this case in particular (i.e. skew-odd solutions) will only work in even spatial dimension. By the same token, periodic solutions on the whole space will correspond to the periodic problem on the first quadrant.

As a preparation, in Section 2, we will establish the notion of skew-odd and even functions (solutions) in $\mathbb{R}^N$. In Section 3 we will apply the Crandall-Liggett theorem to the problems. More specifically, we apply the Crandall-Liggett theorem to closed (invariant) subspaces of $BUC(\mathbb{R}^{\mathbb{N}})$. The boundary conditions are already built into the definition of the domain of the operator. In Section 4, we explore Lipschitz regularity of the solutions in questions; here we employ some recent results of Goldstein and Goldstein [9].

The Cauchy problem for a related perturbation of the Hamilton-Jacobi equation was addressed in Goldstein-Soeharyadi [10]. Interplay among even solutions, periodic solutions, Dirichlet, Neumann, periodic, and mixed boundary value problems for the Hamilton-Jacobi equation in the positive ray was explored in Burch-Goldstein [5]. Boundary value problems of the Hamilton-Jacobi equation are also discussed, for example, in Ishii [11], Aizawa [3], Lions [12], and Tataru [13]. A semigroup treatment of the Hamilton-Jacobi equation can be found in Crandall-Lions [6], Crandall-Evans-Lions [7], Burch [4], Aizawa [1, 2]. For nonlinear semigroup treatments of partial differential equations we refer to Goldstein [8].

2. Symmetries

Let ϵ be a function on $\mathbb{R}^N$ defined by

$$\epsilon(x) = (\epsilon_1 x_1, \ldots, \epsilon_N x_N),$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, and ϵ_i is either 1 or -1. Obviously, there are 2^N such functions. Let E be the collection of all such functions. A real valued function f on $\mathbb{R}^N$ is said to be *E-invariant* if

$$f(\epsilon(x)) = f(x), \text{ for all } x \in \mathbb{R}^N, \text{and all } \epsilon \in E.$$

We can easily see that f is E-invariant if and only if it is even with respect to each variable. We regard E-invariance as a higher dimensional notion of evenness. We say f is *skew* E-invariant if

$$f(\epsilon(x)) = (-1)^{\sigma(\epsilon)} f(x), \text{ for all } x \in \mathbb{R}^N, \text{ and all } \epsilon \in E;$$

here $\sigma(\epsilon)$ is the number of the ϵ_i's which have value -1.

In one dimension an odd function naturally satisfies the Dirichlet boundary condition at the origin, while a differentiable, even function satisfies the Neumann boundary condition at the origin. We have a similar situation in higher dimensions, with the notions of E-invariance and skew E-invariance playing the role of even and odd.

Lemma 2.1. *If f is skew E-invariant on $\mathbb{R}^N$, then $f = 0$ on $\partial\mathbb{R}^N_+$.*

Proof. Let y be in $\partial\mathbb{R}^N_+$. Then $y_i = 0$ for some $i, 1 \leq i \leq N$. For

$$\epsilon(x) = (x_1, \ldots, -x_i, \ldots, x_N), \ x \in \mathbb{R}^N,$$

we have $y = \epsilon(y)$, and thus

$$f(y) = f(\epsilon(y)) = -f(y),$$

forcing $f(y) = 0$. □

If f is E-invariant, then the first partial derivative of f in each variable is odd since f is even with respect to each of its variables. This is in fact the property of the derivative of an even function being odd, in one dimension. However, the derivatives (partial and hence the divergence) stop short of being skew E-invariant, as shown by this easy example in $N = 2$, $f(x, y) = x^2y^2$. It is E-invariant, but none of its first derivatives is skew E-invariant.

Suppose f is E-invariant, and y is a point on $\partial\mathbb{R}^N_+$, such that for exactly one index i, $y_i = 0$. The outer unit normal to the point y is $(0, \ldots, 0, -1, 0, \ldots, 0)$, where -1 sits in the i-th position. Then

$$\begin{aligned}\frac{\partial f}{\partial n}(y) &= \nabla f(y) \cdot n \\ &= -\frac{\partial f}{\partial x_i}(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_N) \\ &= 0,\end{aligned}$$

since $\partial f/\partial x_i$ is odd in the i-th variable. This shows the following:

Proposition 2.2. *If f is E-invariant and differentiable on a neighborhood of $\partial\mathbb{R}^N_+$, then f satisfies Neumann boundary condition on $\partial\mathbb{R}^N_+$.*

In this paper we will say that a continuous, E-invariant function u satisfies *generalized Neumann boundary condition* on $\partial\mathbb{R}^N_+$. An E-invariant function is simply called even, a skew E-invariant function is called skew-odd. Note that any skew-odd function f satisfies $f(0) = 0$.

A *spherically odd* function f (or simply called odd), is a function which satisfies

$$f(-x) = -f(x), \text{ for all } x \in \mathbb{R}^N.$$

There is an extensive literature which exploits this notion of oddness. However, the notion of (spherically) odd and our notion of skew-odd are distinct. The example $f(x, y) = x + y$ shows an odd function which is not skew-odd. While another example $g(x, y) = xy$ exhibits a skew-odd function which is not odd. But a skew-odd function needs to be odd in an odd dimensional space.

In any dimension, there exists a unique decomposition of any real valued function, $f = g + h$, where g is an even function, and h is an odd function. This is not true anymore with even, skew-odd functions decomposition, as shown by the function $f(x, y) = x + y$. Suppose it is true, that is

$$x + y = g(x, y) + h(x, y),$$

where g and h are even and skew-odd functions, respectively. Then for all $x \in \mathbb{R}^N$ we have

$$2x = h(x, x) + g(x, x) = h(-x, -x) + g(-x, -x) = -2x,$$

which is a contradiction for all $x \neq 0$.

3. Invariant sets

Using the notion of E-invariance and skew E-invariance as above we construct subsets of the Banach space $X = BUC(\mathbb{R}^N)$ which are invariant under the semigroup action the Hamilton-Jacobi equation. Recall that Goldstein-Soeharyadi [10] dealt with the Cauchy problem

$$\begin{aligned} u_t + H(\nabla_x u) + G(\cdot, u) = 0,\ x \in \mathbb{R}^N,\ t > 0, \\ u(x, 0) = u_0(x),\ u_0 \in X, \end{aligned} \tag{3.1}$$

where $H \in C^2(\mathbb{R}^N)$, is weakly convex, (i.e. $\sum_{i,j}^N H_{x_i x_j}(x)\xi_i\xi_j \geq 0$, for all $x, \xi \in \mathbb{R}^N$), and satisfies $H(0) = 0$. The perturbation term $G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, assumes the following:

$$(\gamma_1) \qquad G \in C^2(\mathbb{R}^N \times \mathbb{R})$$

$$(\gamma_2) \qquad | G(x,u) | \leq K_1 \mid u \mid, \text{ for all } x \in \mathbb{R}^N, \text{and } u \in \mathbb{R}$$

$$(\gamma_3) \qquad | G(x,u) - G(y,v) | \leq K_2(| x-y | + | u-v |),$$

$$\text{for all } x,y \in \mathbb{R}^N, \text{and } u,v \in \mathbb{R}$$

$$(\gamma_4) \qquad \|G_{ij}\| \leq K_3,$$

$$\text{for any entry } G_{ij} \text{of the matrix } D^2_{x,u}G.$$

Burch [4] (see also [1, 2]) showed that the operator $A_0 = H \circ \nabla_x$ defined by $A_0u = -H(\nabla_x u)$ (on a suitable domain) is densely defined and m-dissipative on X. Crandall and Lions [6], and later Crandall, Evans, and Lions [7] generalized this substantially. Using their notion of viscosity solutions, they were able to reduce the hypotheses on H to mere continuity, i.e. $H \in C(\mathbb{R}^N)$. Hence by the Crandall-Liggett theorem, the problem (without perturbation) is governed by a strongly continuous nonexpansive (or contractive) nonlinear semigroup $T_0 = \{T_0(t) : t \geq 0\}$ on X. Again, this is under the assumption that H is a real continuous function on $\mathbb{R}^N$. In particular $u(t) = T_0(t)u_0$ is the unique mild solution of the Cauchy problem, for any initial data $u_0 \in X$. Using perturbation theory, one can show that the Cauchy problem of the perturbed Hamilton-Jacobi (3.1) is well-posed. Let $A_1u = -G(\cdot,u)$, $A_2u = A_0u + A_1u = -H(\nabla_x u) - G(\cdot,u)$, as mappings from X to X. A_1 is (globally) Lipschitzian with Lipschitz constant K_2, assuming (γ_3) holds. Thus A_i generates a nonlinear semigroup T_i, $i = 1,2$, satisfying $\|T_i(t)\|_{Lip} \leq \exp(tK_2)$, for each $t \geq 0$. In addition $A_i - K_2 I$ is m-dissipative, that is

$$\text{Range}(\lambda I - A_i) = X, \text{ for } 0 < \lambda < 1/K_2, \tag{3.2}$$

$$\|u_1 - u_2\| \leq (1 - \lambda K_2)^{-1} \|h_1 - h_2\|, \tag{3.3}$$

$$\text{for } h_j \in X, \text{and } u_j - \lambda A_i u_j = h_j, (i, j = 1, 2),\ 0 < \lambda < 1/K_2.$$

For $i = 0$, (3.2), (3.3) hold with K_2 replaced by zero. The solution to the Cauchy problem is given by the action of the semigroup on the initial data

$$u(t) = T_2(t) u_0 = \lim_{n \to \infty} (I - \frac{t}{n} A_2)^{-n} u_0.$$

This result is actually obtained assuming only (γ_3). Goldstein-Soeharyadi [10] assumed $(\gamma_1, \gamma_2, \gamma_4)$ for showing some regularity in the context of Burch's result [4]. We shall return to the question of regularity in Section 5.

We now exhibit subsets of the Banach space X which are invariant under the action of the semigroup. For $p \in \mathbb{R}^N$, let us define a function to be p-periodic if $f(x+p) = f(x)$, for all $x \in \mathbb{R}^N$. We also define the following.

$$X_e := \{u \in X : \ u \text{ is even }\}$$

$$X_{os} := \{u \in X : \ u \text{ is skew-odd }\}$$

$$X_p := \{u \in X : \ u \text{ is } p\text{-periodic}\}$$

Recall that skew-odd means skew E-invariant. We can now state one of our main results.

Theorem 3.1. *Let H be real and continuous on $\mathbb{R}^N$ and assume (γ_3). Let $0 < \lambda < 1/K_2$.*

(1) *Assume H is even, and G is even with respect to its first N variables, then*

$$(I - \lambda A_2)^{-1} X_e \subseteq X_e.$$

(2) *Assume the dimension N is even, the Hamiltonian H is skew-odd, G is even with respect to its first N variables, and odd with respect to the last variable, then*

$$(I - \lambda A_2)^{-1} X_{os} \subseteq X_{os}.$$

(3) *Assuming G is p-periodic in the first N variables, we have*

$$(I - \lambda A_2)^{-1} X_p \subseteq X_p.$$

Proof. For (1), we let $f \in X_e$. We seek a unique $u \in X_e$ solving the resolvent equation, i.e., u satisfying $(I - \lambda A_2)u = f$. That means

$$u(x) + \lambda H(\nabla_x u(x)) + G(x, u(x)) = f(x). \tag{3.4}$$

The existence of such a u in X is guaranteed by the quasi m-dissipativity of the operator A_2. We shall show that indeed $u \in X_e$. Let $\epsilon \in E$ and $v(x) = u(\epsilon(x))$. For simplicity, let $y = \epsilon(x)$. Then

$$\begin{aligned}
& v(x) + \lambda H(\nabla_x v(x)) + G(x, v(x)) \\
&\quad = u(y) + \lambda H(\epsilon(\nabla_y u(y)) + G(x, u(y)) \\
&\quad = u(y) + \lambda H(\nabla_y (u(y)) + G(y, u(y)) \\
&\quad = f(y) \\
&\quad = f(x),
\end{aligned}$$

and therefore v solves the resolvent equation (3.4). By uniqueness, we have

$$u(x) = v(x) = u(\epsilon(x)),$$

for $x \in \mathbb{R}^N$ and $\epsilon \in E$. Thus $u \in X_e$.

For (2), first we can find a unique u in X satisfying (3.4), given $f \in X_{os}$. We shall show that $u \in X_{os}$. Let $\epsilon \in E$ and set $v(x) = (-1)^\sigma u(\epsilon(x))$. Here $\sigma = \sigma(\epsilon)$. Then

$$\begin{aligned} & v(x) + \lambda H(\nabla_x v(x)) + G(x, v(x)) \\ & \quad = (-1)^\sigma u(y) + \lambda H((-1)^\sigma \epsilon(\nabla_y u(y))) + G(x, (-1)^\sigma u(y)). \end{aligned}$$

We now examine the case when σ is an even number. The last equality becomes

$$\begin{aligned} & u(y) + \lambda H(\epsilon(\nabla_y u(y))) + G(y, u(y)) \\ & \quad = u(y) + \lambda H(\nabla_y u(y)) + G(y, u(y)) \\ & \quad = f(y) \\ & \quad = f(x). \end{aligned}$$

If σ is odd, we have

$$\begin{aligned} & -u(y) + \lambda H(-\epsilon(\nabla_y u(y))) - G(y, u(y)) \\ & \quad = -u(y) + (-1)^{N-\sigma} \lambda H(\nabla_y u(y)) - G(y, u(y)) \\ & \quad = -u(y) - \lambda H(\nabla_y u(y)) - G(y, u(y)) \\ & \quad = -f(y) \\ & \quad = f(x) \end{aligned}$$

since $N - \sigma$ is odd. In both cases v satisfies the resolvent equation (3.4). Again, by uniqueness, $v(x) = u(x)$, and thus

$$u(x) = v(x) = (-1)^\sigma u(\epsilon(x)),$$

for $x \in \mathbb{R}^N, \epsilon \in E$, and hence u is skew E-invariant. The proof of (3) follows from a straightforward substitution of $v(x) = u(x+p)$ into the resolvent equation. A uniqueness argument as the above finishes the proof. $\square$

The above theorem shows that the restriction

$$S_\alpha(t) : X_\alpha \to X_\alpha,$$

($\alpha = e, os, p$) of the semigroup $T_2(t)$ to the set X_α, is itself a quasicontractive semigroup on X_α, generated by $A_\alpha = A_2 \mid_{X_\alpha}$. In turn, this establishes well-posedness of the Cauchy problem (3.1) in the spaces X_e, X_{os}, and X_p.

4. Boundary Value Problems

Symmetries and invariance give us the main result for initial-boundary value problems for perturbed Hamilton - Jacobi equation, in the first quadrant $\mathbb{R}^N_+$ of $\mathbb{R}^N$.

Theorem 4.1. *Let H be a continuous real function on $\mathbb{R}^N$. Let G be jointly Lipschitzian (i.e., (γ_3)). Consider the initial value problem*

$$u_t + H(\nabla_x u) + G(\cdot, u) = 0, \ \text{ a.e. for } x \in \mathbb{R}^N_+,\ t > 0, \tag{4.1}$$

$$u(x, 0) = u_0(x), \ \ x \in \mathbb{R}^N_+; \tag{4.2}$$

we consider the following boundary conditions

$$u(x, t) = 0, \ \ x \in \partial\mathbb{R}^N_+,\ t > 0, \tag{4.3}$$

$$\partial u/\partial n(x, t) = 0, \ \ x \in \partial\mathbb{R}^N_+,\ t > 0, \tag{4.4}$$

$$u(x + p, t) = u(x), \ \ x \in \partial\mathbb{R}^N_+,\ t > 0. \tag{4.5}$$

Then the following conclusions hold.

(1) *The generalized Neumann problem (4.1), (4.2), (4.4), is governed by a strongly continuous quasicontractive semigroup* $\{S_e(t)\}$ *on* $BUC(\mathbb{R}^N_+)$.

(2) *The Dirichlet problem (4.1)-(4.3) is governed by a strongly continuous quasicontractive semigroup* $\{S_{os}(t)\}$ *on* $Y = \{u \in BUC(\mathbb{R}^N_+) : u(x) = 0, \text{for}\, x \in \partial\mathbb{R}^N_+\}$, *if the spatial dimension* N *is even.*

(3) *If* N *is even, the periodic problem (4.1),(4.3),(4.5) is governed by a strongly continuous quasi contractive semigroup* $\{S_p(T)\}$ *on* $Z = \{u \in BUC(\mathbb{R}^N_+) : u(x+p) = u(x), \text{for}\, x \in \mathbb{R}^N\}$.

While (4.1) and (4.4) are satisfied in a certain generalized sense, (4.2), (4.3) and (4.5) are satisfied in strong sense. We outline the proof of the theorem. For problems in the first quadrant, we first extend the Hamiltonian H, the perturbation term G, and the initial data according what is required (even, odd, or periodic) to the whole space $\mathbb{R}^N$. We apply the invariance result of Theorem 3.1. The boundary conditions are built into the domains of the corresponding operators, and are thus automatically satisfied. The corresponding operators are quasi m-dissipative (quasidissipativity being inherited from the unrestricted A_2). We apply the Crandall-Liggett theorem to obtain the associated semigroups.

Remark 4.2. Some mixed problems are possible, for example, the periodic problem (4.1)-(4.3), (4.5) in even dimensions is governed by the semigroup $\{S_\beta(t)\}$. Here, S_β is the restriction

$$S_\beta(t) = S_{os}(t)\,|_{X_{os}\cap X_p}: X_{os} \cap X_p \to X_{os} \cap X_p.$$

5. Regularity

Let $Lip(\Omega)$ denote the space of real-valued Lipschitz functions on Ω, with its usual seminorm $\|\cdot\|_{Lip}$. The metric space $\Omega = (\Omega, \rho)$ is assumed to satisfy a certain geometric property, namely, there is a $K_0 > 0$ such that for all $x, y \in \Omega$, there is a uniformly continuous $\tau : \Omega \to \Omega$ satisfying $\tau(x) = y$ and $\rho(\tau(z), z) \leq K_0 \rho(x, y)$ for all $z \in \Omega$. This holds for any ellipsoid (or ball, or the whole space) in a Hilbert space with structure constant $K_0 = 1$. See Goldstein and Goldstein [9]. For $k > 0$, let

$$Lip_k = \{f \in Lip(\Omega) : \|f\|_{Lip} \leq k\}.$$

Goldstein and Goldstein [9] showed, for an operator (possibly multivalued) B with $Dom(B) \subseteq BUC(\Omega)$, and which is quasi m-dissipative (so that the Crandall-Liggett theorem holds), the following holds:

$$T(t)(Lip_k) \subseteq Lip_s,$$

for all $t > 0$, $k > 0$, and a suitable $s = s(t, k, K_0)$. Here $\{T(t) : t \geq 0\}$ is the semigroup generated by B. Further, they conjectured that this is the case with perturbed Hamilton-Jacobi equation. In this section we shall show that this is true. We shall compute a bound for the Lipschitz norm of a solution at any $t > 0$. This result can be interpreted as a regularity result for the perturbed Hamilton-Jacobi equation. While the analysis we carry out is for the whole space $\mathbb{R}^N$, the result applies also to the boundary problems in the first quadrant $\mathbb{R}^N_+$ of $\mathbb{R}^N$.

In addition, let us assume that $G \in C^1(\mathbb{R}^N \times \mathbb{R})$. Recall also that from (γ_3), K_2 is a bound for Lipschitz constant of G; hence it is a bound for $| \nabla_{x,u} G |$.

Lemma 5.1. *Let u solve the resolvent equation with data h, i.e.,*

$$u + \lambda H(\nabla_x u) = h - \lambda G(\cdot, u),$$

for some $\lambda > 0$. Let $l \in \mathbb{R}^N$ and $\|h\|_{Lip} = k$. Then

$$| u(\cdot + l) - u(\cdot) | \leq \frac{k + \lambda K_2}{1 - \lambda K_2} | l | .$$

Proof. We observe that $u_l(\cdot) := u(\cdot + l)$ satisfies a translated resolvent equation

$$u_l + \lambda H(\nabla_x u_l) = h_l - \lambda G(\cdot + l, u_l).$$

By dissipativity of the unperturbed problem,

$$\begin{aligned}\|u - u_l\| &\leq \|h - \lambda G(\cdot, u) - (h_l - \lambda G(\cdot + l, u_l))\| \\ &\leq \|h - h_l\| + \lambda \|G(\cdot, u) - G(\cdot + l, u_l)\|.\end{aligned}$$

However

$$\|G(\cdot + l, u_l) - G(\cdot, u)\| \leq K_2(| l | + \|u - u_l\|).$$

Combining the previous inequalities,

$$\begin{aligned}\|u - u_l\| &\leq \|h - h_l\| + \lambda K_2(| l | + \|u - u_l\|) \\ &\leq k | l | + \lambda K_2(| l | + \|u - u_l\|).\end{aligned}$$

Thus

$$| u(x + l) - u(x) | \leq \|u - u_l\| \leq \frac{k + \lambda K_2}{1 - \lambda K_2} | l | .$$

□

Lemma 5.2. *For any positive integer n we have*

$$\|(I-(t/n)A_2)^{-n}u_0-(I-(t/n)A_2)^{-n}u_{0l}\| \leq \frac{k+1-(1-\lambda K_2)^n}{(1-\lambda K_2)^n}\,|\,l\,|\,.$$

Proof. Assuming the initial data is u_0, with $\|u_0\|_{Lip}\leq k$, Lemma 5.1 gives

$$\|(I-\lambda A_2)^{-1}u_0-(I-\lambda A_2)^{-1}u_{0l}\| \leq \frac{k+\lambda K_2}{1-\lambda K_2}\,|\,l\,|\,. \tag{5.1}$$

We now use $(I-\lambda A_2)^{-1}u_0$ as initial data, and repeating the process, iterate the bound (5.1)

$$\|(I-\lambda A_2)^{-2}u_0-(I-\lambda A_2)^{-2}u_{0l}\| \leq \frac{k+\lambda K_2+(1-\lambda K_2)\lambda K_2}{(1-\lambda K_2)^2}\,|\,l\,|\,.$$

After n iterations this yields

$$\begin{aligned}\|(I-\lambda A_2)^{-n}u_0-(I-\lambda A_2)^{-n}u_{0l}\| &\leq \frac{k+\lambda K_2\sum_{s=0}^{n-1}(1-\lambda K_2)^s}{(1-\lambda K_2)^n}\,|\,l\,| \\ &= \frac{k+1-(1-\lambda K_2)^n}{(1-\lambda K_2)^n}\,|\,l\,|\,.\end{aligned}$$

Noting that $\sum_{s=0}^{n-1}(1-\lambda K_2)^s=(1-(1-\lambda K_2)^n)/\lambda K_2$, and $\lambda=t/n$, the conclusion of the lemma is then confirmed. $\square$

Thanks to the Crandall-Liggett theorem, as $n\to\infty$, we have from the above lemmas,

$$\|u(t)\|_{Lip}\leq(k+1)e^{tK_2}-1,$$

for any $t>0$, and thus,

$$T(t)(Lip_k)\subseteq Lip_{s(t)},$$

with $s(t)=(k+1)e^{tK_2}-1$.

6. Remarks

In our analysis, the Dirichlet problem in one dimension is not covered by our semigroup generation results. Burch and Goldstein [5] obtained a generation result for a mixed Dirichlet and nonnegativity condition on the nonnegative ray in $\mathbb{R}$. It seems that nonnegativity is the significant condition there.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.3,223-227,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

On the absolute summability factors of Fourier series

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Abstract

In this paper a main theorem on $\mid \bar{N}, p_n \mid_k$ summability factors of Fourier series has been proved. Also some new results have been obtained.

2000 AMS Subject Classification: 40D15, 40G99, 42A24, 42B15.

Keywords and Phrases: Absolute summability, Fourier series.

1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see[3]).

The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{3}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid$ (resp. $\mid \bar{N}, p_n \mid$) summability. Also if we take $k = 1$ and $p_n = 1/n + 1$ summability $\mid \bar{N}, p_n \mid_k$, is equivalent to the summability $\mid R, logn, 1 \mid$. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of f(t) is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{4}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} A_n(t). \tag{5}$$

Bor [2] has proved the following theorem concerning the $\mid \bar{N}, p_n \mid_k$ summability factors of Fourier series.

Theorem A. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty \quad as \quad$ n$\to \infty$, and $\sum_{v=1}^{n} P_v A_v(t) = O(P_n)$, then the series $\sum A_n(t)P_n\lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

2. The aim of this paper is to prove a more general theorem in the following form.

Theorem. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty \quad as \quad$ n$\to \infty$, and $\sum_{v=1}^{n} P_v A_v(t) = O(P_n)$, then the series $\sum A_n(t)P_n\lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

It should be noted that the conditions on the sequence (λ_n) in our theorem, are somewhat more general than in Theorem A.

We need the following lemma for the proof of our theorem.

Lemma. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty \quad as \quad$ n$\to \infty$, then $P_n\lambda_n = O(1) \quad$ as $\quad n \to \infty \quad and \quad \sum P_n\Delta\lambda_n < \infty$.

Proof. Since (λ_n) is non-increasing, we have that

$$P_m\lambda_m = \lambda_m \sum_{n=0}^{m} p_n = O(1)\sum_{n=0}^{m} p_n\lambda_n = O(1) \quad as \quad m\to\infty.$$

Applying the Abel transform to the sum $\sum_{n=0}^{m} p_n\lambda_n$, we have that

$$\sum_{n=0}^{m} p_n\lambda_n = \sum_{n=0}^{m-1} P_n\Delta\lambda_n + P_m\lambda_m = \sum_{n=0}^{m} P_n\Delta\lambda_n - P_m\Delta\lambda_m + P_m\lambda_m = \sum_{n=0}^{m} P_n\Delta\lambda_n + P_m\lambda_{m+1}.$$

Hence

$$\sum_{n=0}^{m} P_n\Delta\lambda_n = \sum_{n=0}^{m} p_n\lambda_n - P_m\lambda_{m+1}.$$

Since $\lambda_n \geq \lambda_{n+1}$, we obtain that

$$\sum_{n=0}^{m} P_n\Delta\lambda_n \leq P_m\lambda_m + \sum_{n=0}^{m} p_n\lambda_n = O(1) + O(1) = O(1) \quad as \quad m\to\infty.$$

This completes the proof of the Lemma.

Proof of the Theorem. Let $T_n(t)$ denotes the $(\bar{N}, p_n)$ means of the series $\sum A_n(t)P_n\lambda_n$.Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} A_r(t)P_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})A_v(t)\lambda_v P_v.$$

Then, for $n \geq 1$, we have

$$T_n(t) - T_{n-1}(t) \quad = \quad \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}P_vA_v(t)\lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned}
T_n(t) - T_{n-1}(t) &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta(P_{v-1}\lambda_v)\sum_{r=1}^{v} P_rA_r(t) + \frac{p_n}{P_n}\lambda_n\sum_{v=1}^{n} P_vA_v(t)\\
&= O(1)\{\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v - p_v\lambda_v - P_v\lambda_{v+1})P_v\} + O(1)p_n\lambda_n\\
&= O(1)\{\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vP_v\Delta\lambda_v - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vp_v\lambda_v + p_n\lambda_n\}\\
&= O(1)\{T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t)\}, \quad say.
\end{aligned}$$

Since

$$\mid T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t) \mid^k \leq 3^k\{\mid T_{n,1}(t)\mid^k + \mid T_{n,2}(t)\mid^k + \mid T_{n,3}(t)\mid^k\},$$

to complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1} \mid T_{n,r}(t) \mid^k< \infty, \quad for \quad r=1,2,3. \tag{6}$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, and since

$$\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v \leq P_{n-1}\sum_{v=1}^{n-1} P_v\Delta\lambda_v$$

it follows by the Lemma that

$$\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v \leq \sum_{v=1}^{n-1} P_v\Delta\lambda_v = O(1) \quad as \quad m\to\infty, \tag{7}$$

we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{k-1} \mid T_{n,1}(t) \mid^k &\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\{\sum_{v=1}^{n-1} P_vP_v\Delta\lambda_v\}\times\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\}^{k-1}.\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\\
&= O(1)\sum_{v=1}^{m}P_vP_v\Delta\lambda_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}P_v\Delta\lambda_v = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by the Lemma. Again

$$\begin{aligned}
\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{k-1} \mid T_{n,2}(t) \mid^k &\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\{\sum_{v=1}^{n-1}(P_v\lambda_v)^kp_v\}\times\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\}^{k-1}\\
&= O(1)\sum_{v=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v)^kp_v\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^kp_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^k\frac{p_v}{P_v}\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^{k-1}p_v\lambda_v\\
&= O(1)\sum_{v=1}^{m}p_v\lambda_v = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and the Lemma. Finally as in $T_{n,1}(t)$, we have that

$$\begin{aligned}\sum_{n=1}^{m}(\frac{P_n}{p_n})^{k-1} \mid T_{n,3}(t) \mid^k &= \sum_{n=1}^{m}(P_n\lambda_n)^{k-1}p_n\lambda_n \\ &= O(1)\sum_{n=1}^{m}p_n\lambda_n = O(1) \quad as \quad m \to \infty.\end{aligned}$$

Therefore, we get that

$$\sum_{n=1}^{m}(\frac{P_n}{p_n})^{k-1} \mid T_{n,r}(t) \mid^k = O(1) \quad as \quad m \to \infty, for \quad r = 1, 2, 3.$$

This completes the proof of the Theorem.

As special cases of this Theorem, one can obtain the following results.

1. If we take $p_n = 1$ for all values of n, then we get a result concerning the $\mid C, 1 \mid_k$ summability factors of Fourier series.

2. If we take $k = 1$ and $p_n = 1/(n+1)$, then we get another new result related to $\mid R, logn, 1 \mid$ summability factors of Fourier series.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.3,229-248,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

On Hahn Polynomials and Continuous Dual Hahn Polynomials

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Abstract: This paper deals with the hypergeometric orthogonal polynomials with special emphasis on the Hahn polynomials and the continuous dual Hahn polynomials. New representations, generating functions, and summation formulas are derived. The addition theorems for the Krawtchouk and the Meixner-Pollaczek polynomials are also included.

AMS Mathematics Subject Classification (2000): Primary 33C45, 33D45. Secondary 42C05.

Key Words: Hahn polynomials, continuous dual Hahn polynomials, Krawtchouk polynomials, Meixner-Pollaczek polynomials, Poisson-Charlier polynomials, generating function, summation formulas, Dirichlet averages, addition theorems

1. Introduction

The Hahn polynomials were introduced by Hahn [15] as limiting cases of some general systems of orthogonal polynomials. They provide a useful tool in some problems of genetics [16–18].

For $\alpha > -1$, $\beta > -1$, N a non-negative integer, and $n = 0, 1, \ldots, N$ the Hahn polynomials Q_n are defined by [16]

$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -n, n+\mu, -x \\ \alpha+1, -N \end{matrix} \;\Big|\; 1\right), \tag{1.1}$$

where $\mu := \alpha+\beta+1$. These polynomials are orthogonal on $\{0,1,\dots,N\}$ with the weights

$$\rho(x) = \binom{x+\alpha}{x}\binom{N-x+\beta}{N-x}\Big/\binom{N+\mu}{N}.$$

Continuous dual Hahn polynomials are defined by

$$S_n(x^2;a,b,c) = (a+b,n)(a+c,n)\,{}_3F_2\left(\begin{matrix}-n, a+ix, a-ix\\ a+b, a+c\end{matrix}\ \Big|\ 1\right) \tag{1.2}$$

(see [2, p. 47], [19, (5.2)], [26, p. 697]). Here we have used the Appell symbol (z,n) which is defined by $(z,0)=1$, $(z,n)=z(z+1)\cdots(z+n-1)$, $n=1,2,\dots$. The parameters a,b,c have positive real parts. Clearly $S_n(x^2;a,b,c)$ is a polynomial of degree n in x^2. These polynomials are orthogonal on $[0,\infty)$ with the weight function

$$x \to \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2.$$

The interest in this class of orthogonal polynomials also stems from the observation made by Koornwinder. He has shown that the Jacobi functions

$$\varphi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1\left(\begin{matrix}(\mu+\lambda i)/2, (\mu-\lambda i)/2\\ \alpha+1\end{matrix}\ \Big|\ -\sinh^2 t\right)$$

may be obtained as limiting forms of the continuous dual Hahn polynomials by means of the relation

$$\lim_{n\to\infty}\frac{S_n(\lambda^2/4;\mu/2, n/\sinh^2 t, (\alpha-\beta+1)/2)}{((\mu/2)+n/\sinh^2 t, n)(\alpha+1,n)} = \varphi_\lambda^{(\alpha,\beta)}(t)$$

(see [19, (5.14)]). The Jacobi functions constitute a complicated system of orthogonal functions.

This paper is organized as follows. Notation and definitions are introduced in Section 2. The contour integrals for polynomials under discussion are derived in Section 3. The bilinear generating functions for the continuous dual Hahn polynomials and generating functions for Q_n and S_n are given in Section 4. Some summation formulas are discussed in Section 5. In the next section we demonstrate how some known results for the Jacobi polynomials can be generalized easily to the case of Hahn polynomials. Examples include Gasper's projection formula [12, (1.4)] and a formula for the symmetric Hahn polynomials [13, (3.6)]. In Section 7 we deal with the addition theorems for the Krawtchouk polynomials, the Meixner-Pollaczek polynomials, and the Poisson-Charlier polynomials.

2. Notation and Definitions

Throughout the sequel we will employ the notation used in [8]. By $\mathbb{C}_>$ we will denote the open right half-plane in $\mathbb{C}$, U will stand for the complex plane punctured at the non-positive integers, i.e., $U = \{z \in \mathbb{C} : z \neq 0, -1, \dots\}$. The symbol μ will stand for the sum $\alpha + \beta + 1$ ($\alpha, \beta \in \mathbb{C}$) unless otherwise stated. The key tool used in this paper is the Dirichlet average of a holomorphic function of one variable (real or complex). For the reader's convenience we recall the definition of this average along with some basic properties. Let Ω be a convex set in $\mathbb{C}$ and let f be holomorphic on Ω. For $\alpha, a' \in \mathbb{C}_>$ and $(x, y) \in \Omega^2$ the Dirichlet average of f is defined by [8, (5.1–1)]

$$F(\alpha, a'; x, y) = \int_0^1 f\big[ux + (1-u)y\big] d\mu(u), \tag{2.1}$$

where

$$d\mu(u) = \frac{1}{B(\alpha, \alpha')} u^{\alpha-1}(1-u)^{\alpha'-1} du \tag{2.2}$$

is the Dirichlet measure on $(0, 1)$, and B stands for the beta function. Clearly,

$$F(\alpha, a'; x, y) = F(\alpha', \alpha; y, x).$$

Throughout the sequel the symbol R_n will stand for the Dirichlet average of the monomial $f(t) = t^n$, $n \in \mathbb{N}$. If the parameters α, α' are such that $\alpha + \alpha' \in U$ and if Ω is a circular disk in $\mathbb{C}$ with center c, then the integral average $F(\alpha, \alpha'; x, y)$ has a holomorphic continuation to $\mathbb{C}^2 \times \Omega^2$, where it is represented by

$$F(\alpha, \alpha'; x, y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} R_n(\alpha, \alpha'; x - c, y - c). \tag{2.3}$$

This is a special case of Theorem 6.3–1 in [8]. Also, we will use the generalized Cauchy formula [8, (5.11–2)]

$$F(\alpha, \alpha'; x, y) = \frac{1}{2\pi i} \int_{\varepsilon} f(s) R_{-1}(\alpha, \alpha'; s - x, s - y) ds, \tag{2.4}$$

where ε denotes the rectifiable Jordan curve encircling the convex hull of x and y in the positive direction, f is holomorphic in the inner region of ε and continuous on its closure, R_{-1} stands for the Dirichlet average of $f(t) = t^{-1}$. For the comprehensive discussion of Dirichlet averages the interested reader is referred to [8].

Also, we will use the double Dirichlet average $\mathcal{F}$ of f. Let

$$X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

and let f be a holomorphic function on a domain D in $\mathbb{C}$ containing the convex hull of x, y, z, w. The double Dirichlet average may be defined by

$$\begin{aligned}\mathcal{F}(\alpha, \alpha'; X; \beta, \beta') \\ &= \int_0^1 F\left(\alpha, \alpha'; vx + (1-v)y, vz + (1-v)w\right)\varphi_{(\beta,\beta')}(v)dv \\ &= \int_0^1 F\left(\beta, \beta'; ux + (1-u)z, uy + (1-u)w\right)\varphi_{(\alpha,\alpha')}(u)du\end{aligned}$$

(see [5, (2.8)]). Throughout the sequel the symbol $\mathcal{R}_n$ will stand for the double Dirichlet average of t^n, $n \in \mathbb{N}$, while $\mathcal{R}_{-\nu}$ will denote the double average of $t^{-\nu}$.

The double average $\mathcal{F}$ also has a generalized Cauchy formula [5, (6.11)]

$$\mathcal{F}(\alpha, \alpha'; X; \beta, \beta') = \frac{1}{2\pi i}\int_\varepsilon f(s)\mathcal{R}_{-1}(\alpha, \alpha'; s - X; \beta, \beta')ds, \tag{2.5}$$

where

$$s - X = \begin{bmatrix} s-x & s-y \\ s-z & s-w \end{bmatrix}.$$

All the matrix elements of X are required to lie in the inner region of the positively oriented rectifiable Jordan curve ε, and f is assumed to be holomorphic on ε and its inner region.

3. New Formulas for the Hahn Polynomials and Continuous Dual Hahn Polynomials

The purpose of this section is to derive new representations for the polynomials in question. We shall show that they can be represented either by single averages or by the double Dirichlet averages. Representations involving the contour integrals are also discussed.

For later use let us record a useful formula for the hypergeometric polynomials [8, Ex. 5.7–1]

$${}_{p+1}F_{q+1}\left(\begin{matrix} -n, b_2, \ldots, b_p, b \\ c_1, \ldots, c_q, c \end{matrix} \;\Big|\; x\right) = F(b, c-b; x, 0), \tag{3.1}$$

where F denotes the single Dirichlet average of

$$f(t) = {}_pF_q\left(\begin{matrix} -n, b_2, \ldots, b_p \\ c_1, \ldots, c_q, \end{matrix} \;\Big|\; t\right), \tag{3.2}$$

$p, q, n \in \mathbb{N}$. We assume that the denominator parameters of the ${}_pF_q$ polynomial are such that it is well defined.

It follows from (3.1) – (3.2) and (1.1) that

$$Q_n(x; \alpha, \beta, N) = F(-x, x-N; 1, 0), \tag{3.3}$$

where

$$f(t) = {}_2F_1\left(\begin{matrix} -n, n+\mu \\ \alpha+1 \end{matrix} \;\Big|\; t\right) = P_n^{(\alpha,\beta)}(1-2t)\big/P_n^{(\alpha,\beta)}(1), \tag{3.4}$$

$\mu = \alpha+\beta+1$, and $P_n^{(\alpha,\beta)}$ denotes the nth Jacobi polynomial. Similarly,

$$Q_n(x;\alpha,\beta,N) = F(n+\mu, -n-\beta; 1, 0), \tag{3.5}$$

where

$$f(t) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \;\Big|\; t\right). \tag{3.6}$$

For the continuous dual Hahn polynomials we have the following result

$$S_n(x^2; a, b, c) = (a+b, n)(a+c, n)F(a-ix, c+ix; 1, 0), \tag{3.7}$$

where

$$f(t) = {}_2F_1\left(\begin{matrix} -n, a+ix \\ a+b \end{matrix} \;\Big|\; t\right). \tag{3.8}$$

To obtain the representations in terms of $\mathcal{R}$-polynomials it suffices to use [5, (3.2)]

$$ {}_3F_2\left(\begin{matrix} -n, \beta, \beta' \\ \gamma, \gamma' \end{matrix} \;\Big|\; 1\right) = \mathcal{R}_n(\beta, \gamma-\beta; X; \beta', \gamma'-\beta') \tag{3.9}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

on (1.1) and (1.2). We have

$$Q_n(x;\alpha,\beta,N) = \mathcal{R}_n(n+\mu, -n-\beta; X; -x, x-N) \tag{3.10}$$

and

$$S_n(x^2; a, b, c) = (a+b, n)(a+c, n)\mathcal{R}_n(a+ix, b-ix; X; a-ix, c+ix). \tag{3.11}$$

More representations can be derived by use of the linear transformation

$$(\alpha+\alpha', n)\mathcal{R}_n(\alpha, \alpha'; Y; \beta, \beta') = (\alpha, n)\mathcal{R}_n(1-\alpha-\alpha'-n, \alpha'; Z; \beta, \beta'), \tag{3.12}$$

$$Y = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \qquad Z = \begin{bmatrix} x & y \\ x-z & y-w \end{bmatrix}$$

(see [7, (3.4)]) on (3.10) and (3.11). For instance, use of (3.12) on (3.10) gives

$$Q_n(x;\alpha,\beta,N) = \frac{(n+\mu, n)}{(\alpha+1, n)}\mathcal{R}_n(-x, x-N; Z'; -n-\alpha, -n-\beta), \tag{3.13}$$

$$Z' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Here we have used the transposition symmetry for double averages [5, p. 422]. A second application of (3.12) to (3.13) gives

$$\begin{aligned} Q_n(x;\alpha,\beta,N) &= (-1)^n \frac{(n+\mu,n)(-x,n)}{(\alpha+1,n)(-N,n)} \\ &\qquad \times \mathcal{R}_n(N-n+1, x-N; X; -n-\alpha, -n-\beta) \\ &= (-1)^n \frac{(n+\mu,n)(-x,n)}{(\alpha+1,n)(-N,n)} \\ &\qquad \times {}_3F_2\left(\begin{matrix} -n, N-n+1, -n-\alpha \\ x-n+1, -2n-\alpha-\beta \end{matrix} \;\Big|\, 1\right), \end{aligned}$$

where in the last step we have used (3.9) and the fact that $\mathcal{R}_n$ is homogeneous of order n in its matrix elements.

To obtain a similar representation for S_n we follow the lines introduced above. The result is

$$\begin{aligned} &\frac{S_n(x^2;a,b,c)}{(a+ix,n)(a-ix,n)} \\ &\quad = (-1)^n {}_3F_2\left(\begin{matrix} -n,\, 1-a-b-n,\, 1-a-c-n \\ 1-a+ix-n,\, 1-a-ix-n \end{matrix} \;\Big|\; 1\right). \end{aligned}$$

The last two formulas are contained in Luke's theorem [22, 5.2.1(5)]. I am indebted to Professor Stanislaw Lewanowicz for calling my attention to some formulas contained [21–22].

The contour integrals for the Hahn polynomials and the continuous dual Hahn polynomials can be derived from the following formula for the ${}_3F_2(1)$ functions

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} -n, \beta, \beta' \\ \gamma, \gamma' \end{matrix} \;\Big|\, 1\right) \\ &\quad = \frac{1}{2\pi i} \int_{\varepsilon} {}_2F_1\left(\begin{matrix} -n, \beta \\ \gamma \end{matrix} \;\Big|\; s\right) {}_2F_1\left(\begin{matrix} 1, \beta' \\ \gamma' \end{matrix} \;\Big|\; \frac{1}{s}\right) \frac{ds}{s}, \end{aligned} \tag{3.14}$$

where now ε is the rectifiable Jordan curve encircling the interval $[0,1]$ in the positive direction. To prove (3.14) we put $p=2$, $q=1$, $x=1$, $b_2=\beta$, $b=\beta'$,

$c_1 = \gamma$, $c = \gamma'$ in (3.1) and (3.2). Combining this with (2.4) we obtain

$$
\begin{aligned}
{}_3F_2\begin{pmatrix} -n, \beta, \beta' & \\ & \mid 1 \\ \gamma, \gamma' & \end{pmatrix} &= \frac{1}{2\pi i}\int_\varepsilon {}_2F_1\begin{pmatrix} -n, \beta & \\ & \mid s \\ \gamma & \end{pmatrix} \\
&\qquad \times R_{-1}(\beta', \gamma' - \beta'; s-1, s)ds \\
&= \frac{1}{2\pi i}\int_\varepsilon {}_2F_1\begin{pmatrix} -n, \beta & \\ & \mid s \\ \gamma & \end{pmatrix} \\
&\qquad \times R_{-1}(\beta', \gamma' - \beta'; 1 - \frac{1}{s}, 1)\frac{ds}{s},
\end{aligned}
$$

where in the last step we have used homogeneity of the R_{-1} function. To complete the proof it suffices to apply a formula [8, (5.9 – 11)]

$$
R_{-\nu}(\alpha, a'; z, w) = w^{-\nu}{}_2F_1\begin{pmatrix} \nu, \alpha, & \\ & \mid 1 - z/w \\ \alpha + \alpha' & \end{pmatrix} \tag{3.15}
$$

on the right side of the last identity. The formula (3.15) is valid provided that both z and w belong to the complex plane cut along the non-positive real axis.

Application of (3.14) to (3.3) – (3.4) gives

$$
Q_n(x; \alpha, \beta, N) = \frac{1}{2\pi i}\int_\varepsilon {}_2F_1\begin{pmatrix} -n, n+\mu & \\ & \mid s \\ \alpha + 1 & \end{pmatrix} {}_2F_1\begin{pmatrix} 1, -x & \\ & \mid \frac{1}{s} \\ -N & \end{pmatrix} \frac{ds}{s}. \tag{3.16}
$$

Similarly, use of (3.14) on (3.7) – (3.8) provides

$$
\begin{aligned}
&\frac{S_n(x^2; a, b, c)}{(a+b, n)(a+c, n)} \\
&\quad = \frac{1}{2\pi i}\int_\varepsilon {}_2F_1\begin{pmatrix} -n, a + ix & \\ & \mid s \\ a + b & \end{pmatrix} \\
&\qquad \times {}_2F_1\begin{pmatrix} 1, a - ix & \\ & \mid \frac{1}{s} \\ a + c & \end{pmatrix} \frac{ds}{s}
\end{aligned} \tag{3.17}
$$

More contour integrals for the polynomials under discussion can be obtained from the following representation for ${}_3F_2(1)$

$$
\begin{aligned}
&{}_3F_2\begin{pmatrix} -n, \beta, \beta' & \\ & \mid 1 \\ \gamma, \gamma' & \end{pmatrix} \\
&\quad = \frac{1}{2\pi i}\int_\varepsilon s^n(s-1)^{-1}{}_3F_2\begin{pmatrix} 1, \beta, \beta' & \\ & \mid \frac{1}{1-s} \\ \gamma, \gamma' & \end{pmatrix} ds,
\end{aligned} \tag{3.18}
$$

where the curve ε is the same as in (3.14). This can be established using (2.5), with $f(t) = t^n$, (3.9), and [5, (3.2)]. We omit further details. The desired representations for Q_n and S_n now follow by use of (3.18) on (3.10) and (3.11). We leave it to the reader to derive these formulas.

We close this section with a formula which connects the Hahn polynomials with the Krawtchouk polynomials and the Jacobi polynomials. We have

$$\begin{aligned}Q_n&(x;\alpha,\beta,N)\\&=\sum_{m=0}^{n}p^m\binom{n}{m}\frac{(n+\mu,m)}{(\alpha+1,m)}K_m(x;p,N)\\&\qquad\times P_{n-m}^{(\alpha+m,\beta+m)}(1-2p)\Big/P_{n-m}^{(\alpha+m,\beta+m)}(1)\end{aligned}\tag{3.19}$$

$(0<p<1)$, where K_m stands for the mth Krawtchouk polynomial. The latter polynomials may be obtained as limiting forms of the Hahn polynomials by means of the relation

$$K_m(x;p,N)=\lim_{t\to\infty}Q_m(x;pt,(1-p)t,N)={}_2F_1\left(\begin{matrix}-m,-x\\-N\end{matrix}\;\middle|\,p^{-1}\right)\tag{3.20}$$

(see, e.g., [16, (1.22)]). Similarly, the Jacobi polynomials are the limiting cases of the Hahn polynomials

$$\frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}=\lim_{N\to\infty}Q_n(Nx;\alpha,\beta,N)\tag{3.21}$$

(cf. [16, (1.9)]. Letting $\beta=n+\mu$, $\beta'=-x$, $\gamma=\alpha+1$, $\gamma'=-N$, $t=p$ in

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}-n,\beta,\beta'\\\gamma,\gamma'\end{matrix}\;\middle|\,1\right)\\&=\sum_{m=0}^{n}t^m\binom{n}{m}\frac{(\beta,m)}{(\gamma,m)}{}_2F_1\left(\begin{matrix}-n+m,\beta+m\\\gamma+m\end{matrix}\;\middle|\,t\right)\\&\qquad\times{}_2F_1\left(\begin{matrix}-m,\beta'\\\gamma'\end{matrix}\;\middle|\,\frac{1}{t}\right)\end{aligned}\tag{3.22}$$

we obtain the desired result (3.19). Formula (3.22) follows from [21, 9.1(27)] by letting $p=2$, $q=1$, $r=s=1$, $a_2=(-n,\beta)$, $c_1=\beta'$, $b_1=\gamma$, $d_1=\gamma'$, $u=t=0$, $z=t$, $\omega=1/t$.

4. Generating functions

Most of the results of this section can be derived from a generalization of Meixner's bilinear relation for the hypergeometric polynomials [23]

$$\begin{aligned}
&\sum_{n=0}^{\infty} w^n \binom{\alpha}{n} {}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix} \,\Big|\, u\right) {}_2F_1\left(\begin{matrix}-n, \beta\\ \gamma\end{matrix} \,\Big|\, v\right)\\
&\quad = (1+w)^{\alpha} \sum_{n=0}^{\infty} \frac{w^n}{(1+w)^{2n}}\\
&\qquad \times \binom{\alpha}{n} \frac{(b,n)(\beta,n)}{(c,n)(\gamma,n)} u^n {}_2F_1\left(\begin{matrix}-\alpha+n, b+n\\ c+n\end{matrix} \,\Big|\, \xi u\right)\\
&\qquad \times v^n {}_2F_1\left(\begin{matrix}-\alpha+n, \beta+n\\ \gamma+n\end{matrix} \,\Big|\, \xi v\right),
\end{aligned} \tag{4.1}$$

where $\xi = w/(1+w)$ (see also [10, 2.5(8)]). The last formula is valid provided that u, v, ξu, $\xi v \neq 1, \infty$, and $|w|$ is sufficiently small.

Another bilinear relation can be obtained from (4.1). Substituting $w := w/\alpha$ and then letting $\alpha \to \infty$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{w^n}{n!} {}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix} \,\Big|\, u\right) {}_2F_1\left(\begin{matrix}-n, \beta\\ \gamma\end{matrix} \,\Big|\, v\right)\\
&\quad = e^w \sum_{n=0}^{\infty} \frac{w^n (b,n)(\beta,n)}{n!(c,n)(\gamma,n)} u^n {}_1F_1\left(\begin{matrix}b+n\\ c+n\end{matrix} \,\Big|\, -uw\right)\\
&\qquad \times v^n {}_1F_1\left(\begin{matrix}\beta+n\\ \gamma+n\end{matrix} \,\Big|\, -vw\right)
\end{aligned} \tag{4.2}$$

Generalizations of (4.1) and (4.2) can be obtained by averaging both sides of these formulas. Forming the Dirichlet average $F(b', c'-b'; 1, 0)$ (with respect to the variable u) and repeating the process of averaging by use of $F(\beta', \gamma' - \beta'; 1, 0)$, we arrive at

$$
(4.3)\qquad
\begin{aligned}
&\sum_{n=0}^{\infty} w^n \binom{\alpha}{n} {}_3F_2\left(\begin{matrix}-n, b, b' \\ c, c'\end{matrix} \;\middle|\; 1\right) {}_3F_2\left(\begin{matrix}-n, \beta, \beta' \\ \gamma, \gamma'\end{matrix} \;\middle|\; 1\right) \\
&= (1+w)^{\alpha} \sum_{n=0}^{\infty} \frac{w^n}{(1+w)^{2n}} \binom{\alpha}{n} \frac{(b,n)(b',n)(\beta,n)(\beta',n)}{(c,n)(c',n)(\gamma,n)(\gamma',n)} \\
&\qquad \times {}_3F_2\left(\begin{matrix}-\alpha+n, b+n, b'+n \\ c+n, c'+n\end{matrix} \;\middle|\; \xi\right) \\
&\qquad \times {}_3F_2\left(\begin{matrix}-\alpha+n,\ \beta+n,\ \beta'+n \\ \gamma+n,\ \gamma'+n\end{matrix} \;\middle|\; \xi\right)
\end{aligned}
$$

and

$$
(4.4)\qquad
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{w^n}{n!} {}_3F_2\left(\begin{matrix}-n, b, b' \\ c, c'\end{matrix} \;\middle|\; 1\right) {}_3F_2\left(\begin{matrix}-n, \beta, \beta' \\ \gamma, \gamma'\end{matrix} \;\middle|\; 1\right) \\
&= e^{w} \sum_{n=0}^{\infty} \frac{w^n (b,n)(b',n)(\beta,n)(\beta',n)}{n!(c,n)(c',n)(\gamma,n)(\gamma',n)} \\
&\qquad \times {}_2F_2\left(\begin{matrix}b+n, b'+n \\ c+n, c'+n\end{matrix} \;\middle|\; -w\right) \\
&\qquad \times {}_2F_2\left(\begin{matrix}\beta+n, \beta'+n \\ \gamma+n, \gamma'+n\end{matrix} \;\middle|\; -w\right),
\end{aligned}
$$

respectively. Here we have used (3.1) and two formulas

$$
(4.5)\qquad F(b', c'-b'; 1, 0) = \frac{(b',n)}{(c',n)}\, {}_3F_2\left(\begin{matrix}-\alpha+n,\ b+n,\ b'+n \\ c+n,\ c'+n\end{matrix} \;\middle|\; \xi\right),
$$

where

$$
(4.6)\qquad f(u) = u^n {}_2F_1\left(\begin{matrix}-\alpha+n,\ b+n \\ c+n\end{matrix} \;\middle|\; \xi u\right) \qquad \left(|\xi u| < 1\right)
$$

and

$$
F(b', c'-b'; 1, 0) = \frac{(b',n)}{(c',n)}\, {}_2F_2\left(\begin{matrix}b+n,\ b'+n \\ c+n,\ c'+n\end{matrix} \;\middle|\; -w\right),
$$

where

$$
f(u) = u^n {}_1F_1\left(\begin{matrix}b+n \\ c+n\end{matrix} \;\middle|\; -uw\right).
$$

We shall prove (4.5). The second formula can be established by the same means. It follows from (4.6) that

$$f(u) = \sum_{m=0}^{\infty} \frac{(-\alpha+n,m)(b+n,m)}{(c+n,m)m!}\xi^m u^{n+m}.$$

Hence,

$$\begin{aligned} &F(b', c'-b'; 1, 0) \\ &\quad = \sum_{m=0}^{\infty} \frac{(-\alpha+n,m)(b+n,m)}{(c+n,m)m!}\xi^m R_{n+m}(b', c'-b'; 1, 0), \end{aligned} \tag{4.7}$$

where R_{n+m} stands for the R-hypergeometric polynomial. Making use of [8, (6.2 – 5)] we obtain

$$R_{n+m}(b', c'-b'; 1, 0) = \frac{(b', n+m)}{(c', n+m)}.$$

Since $(z, n+m) = (z+n, m)(z, n)$,

$$R_{n+m}(b', c'-b'; 1, 0) = \frac{(b', n)(b'+n, m)}{(c', n)(c'+n, m)}.$$

This in conjunction with (4.7) gives the assertion.

Two generating functions for the ${}_3F_2(1)$ polynomials can be derived from (4.1) and (4.2). Put $u = 0$ and take the Dirichlet average $F(\beta', \gamma'-\beta'; 1, 0)$ on both sides to obtain

$$(1+w)^{\alpha}{}_3F_2\left(\begin{matrix}-\alpha, \beta, \beta' \\ \gamma, \gamma'\end{matrix} \;\Big|\; \xi\right) = \sum_{n=0}^{\infty} w^n \binom{\alpha}{n} {}_3F_2\left(\begin{matrix}-n, \beta, \beta' \\ \gamma, \gamma'\end{matrix} \;\Big|\; 1\right) \tag{4.8}$$

and

$$e^w {}_2F_2\left(\begin{matrix}\beta, \beta' \\ \gamma, \gamma'\end{matrix} \;\Big|\; -w\right) = \sum_{n=0}^{\infty} \frac{w^n}{n!} {}_3F_2\left(\begin{matrix}-n, \beta, \beta' \\ \gamma, \gamma'\end{matrix} \;\Big|\; 1\right). \tag{4.9}$$

Two bilinear generating functions for the continuous dual Hahn polynomials follow from (4.3), (4.4), and (1.2). We have

$$\begin{aligned} &\sum_{n=0}^{\infty} w^n \binom{\alpha}{n} \frac{S_n(x^2; a, b, c) S_n(y^2; a', b', c')}{(a+b,n)(a+c,n)(a'+b',n)(a'+c',n)} \\ &\quad = (1+w)^{\alpha} \sum_{n=0}^{\infty} \frac{w^n}{(1+w)^{2n}(n!)^2} \binom{\alpha}{n}^{-1} \\ &\qquad \times \left[D_{\xi}^n {}_3F_2\left(\begin{matrix}-\alpha, a+ix, a-ix \\ a+b, a+c\end{matrix} \;\Big|\; \xi\right)\right] \\ &\qquad \times \left[D_{\xi}^n {}_3F_2\left(\begin{matrix}-\alpha, a'+iy, a'-iy \\ a'+b', a'+c'\end{matrix} \;\Big|\; \xi\right)\right] \end{aligned} \tag{4.10}$$

and

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{S_n(x^2; a,b,c) S_n(y^2; a',b',c')}{(a+b,n)(a+c,n)(a'+b',n)(a'+c',n)} \\
&= e^w \sum_{n=0}^{\infty} \frac{w^n}{n!} \left[D_\xi^n \, {}_2F_2 \left(\begin{matrix} a+ix, a-ix \\ a+b, a+c \end{matrix} \;\Big|\; -w \right) \right] \\
&\quad \times \left[D_\xi^n \, {}_2F_2 \left(\begin{matrix} a'+iy, a'-iy \\ a'+b', a'+c' \end{matrix} \;\Big|\; -w \right) \right],
\end{aligned} \tag{4.11}
$$

where $D_t^n = d^n/dt^n$.

The generating functions for the continuous dual Hahn polynomials

$$
\begin{aligned}
&(1+w)^\alpha {}_3F_2 \left(\begin{matrix} -\alpha, a+ix, a-ix \\ a+b, a+c \end{matrix} \;\Big|\; \xi \right) \\
&= \sum_{n=0}^{\infty} w^n \binom{\alpha}{n} \frac{S_n(x^2; a,b,c)}{(a+b,n)(a+c,n)}
\end{aligned} \tag{4.12}
$$

and

$$
e^w {}_2F_2 \left(\begin{matrix} a+ix, a-ix \\ a+b, a+c \end{matrix} \;\Big|\; -w \right) = \sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{S_n(x^2; a,b,c)}{(a+b,n)(a+c,n)} \tag{4.13}
$$

are contained in (4.8) and (4.9), respectively.

To obtain the generating functions for the Hahn polynomials we use (1.1) on (4.8) (with $\alpha = M$, $M \in \mathbb{N}$) and (4.9). The result is

$$
(1+w)^M {}_3F_2 \left(\begin{matrix} -n, n+\mu, -M \\ \alpha+1, -N \end{matrix} \;\Big|\; \xi \right) = \sum_{x=0}^{M} w^x \binom{M}{x} Q_n(x; \alpha, \beta, N) \tag{4.14}
$$

and

$$
e^w {}_2F_2 \left(\begin{matrix} -n, n+\mu \\ \alpha+1, -N \end{matrix} \;\Big|\; -w \right) = \sum_{x=0}^{\infty} \frac{w^x}{x!} Q_n(x; \alpha, \beta, N) \tag{4.15}
$$

$(n = 0, 1, \ldots, N)$. The generating function of Karlin and McGregor [16, (1.11)]

$$
(1+w)^N P_n^{(\alpha,\beta)} \left(\frac{1-w}{1+w} \right) \Big/ P_n^{(\alpha,\beta)}(1) = \sum_{x=0}^{N} w^x \binom{N}{x} Q_n(x; \alpha, \beta, N)
$$

is contained in (4.14). Put $M = N$ and then use

$$
{}_2F_1 \left(\begin{matrix} -n, n+\mu \\ \alpha+1 \end{matrix} \;\Big|\; \xi \right) = P_n^{(\alpha,\beta)} \left(\frac{1-w}{1+w} \right) \Big/ P_n^{(\alpha,\beta)}(1).
$$

5. Summation formulas

This section deals with the summation of a finite series whose terms involve either the Hahn polynomials or the continuous dual Hahn polynomials.

We need the following formula

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}{}_3F_2\left(\begin{matrix}-k,\beta,\beta'\\ \gamma,\gamma'\end{matrix}\;\Big|\;1\right)=\frac{(\beta,m)(\beta',m)}{(\gamma,m)(\gamma',m)}. \tag{5.1}$$

This can be dreived easily from the generating function (4.9). Multiply both sides by e^{-w} and then form the Cauchy product on the right side. By equating coefficients of w^m we obtain the desired results.

Substituting $\beta := n+\mu$, $\beta' := -n$, $\gamma := \alpha+1$, $\gamma' := -N$ into (5.1) we obtain

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}Q_n(k;\alpha,\beta,N)=\frac{(-n,m)(n+\mu,m)}{(\alpha+1,m)(-N,m)}, \tag{5.2}$$

$n=0,1,\dots,N$.

Similarly, letting $\beta := a+ix$, $\beta' := a-ix$, $\gamma := a+b$, $\gamma' := a+c$ in (5.1) we obtain with the aid of (1.2)

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{S_k(x^2;a,b,c)}{(a+b,k)(a+c,k)}=\frac{(a+ix,m)(a-ix,m)}{(a+b,m)(a+c,m)}. \tag{5.3}$$

More summation formulas can be derived from

$$\begin{aligned}{}_{p+1}F_{q+1}\left(\begin{matrix}-n,b_2,\dots,b_p,b\\ c_1,\dots,c_q,c\end{matrix}\;\Big|\;x\right)&=\frac{k!}{(c,k)}\sum_{j=0}^{k}\frac{(b,j)(c-b,k-j)}{j!(k-j)!}\\ &\cdot{}_{p+1}F_{q+1}\left(\begin{matrix}-n,b_2,\dots,b_p,b+j\\ c_1,\dots,c_q,c+k\end{matrix}\;\Big|x\right).\end{aligned} \tag{5.4}$$

We shall prove (5.4) and then give applications to the Hahn polynomials. The following result

$$F(b,c-b;x,0)=\frac{k!}{(c,k)}\sum_{j=0}^{k}\frac{(b,j)(c-b,k-j)}{j!(k-j)!}F(b+j,c+k-b-j;x,0) \tag{5.5}$$

is the special case of the Exercise 5.6–2 in [8]. This formula is valid provided f is continuous on an open interval with endpoints at 0 and x. Let f be defined by (3.2). Application of (3.1) to (5.5) gives the desired result (5.4).

Assume now that $p=2$, $q=1$, $x=1$. Substituting $b_2=-x$, $b=n+\mu$, $c_1=-N$, $c=\alpha+1$ into (5.4) we obtain

$$\begin{aligned}&\frac{k!}{(\alpha+1,k)}\sum_{j=0}^{k}\frac{(n+\mu,j)(-n-\beta,k-j)}{j!(k-j)!}Q_n(x;\alpha+k,\beta-k+j,N)\\ &\qquad=Q_n(x;\alpha,\beta,N).\end{aligned} \tag{5.6}$$

Use of the limit relation (3.21) on (5.6) provides

$$\begin{aligned}(\alpha+n+1,k)&P_n^{(\alpha,\beta)}(x)\\&=\sum_{j=0}^{k}\binom{k}{j}(n+\mu,j)(-n-\beta,k-j)P_n^{(\alpha+k,\beta-k+j)}(x),\end{aligned}$$

$k=0,1,\dots$. Here we have used the formula $P_n^{(\alpha,p)}(1)=(\alpha+1,n)/n!$ (see, e.g., [25, (4.1.1)]).

Similarly, letting $p=2$, $q=1$, $x=1$, $b_2=n+\mu$, $b=-x$, $c_1=\alpha+1$, $c=-N$ in (5.4) we obtain

$$\binom{N}{k}^{-1}\sum_{j=0}^{k}\binom{x}{j}\binom{N-x}{k-j}Q_n(x-j;\alpha,\beta,N-k)=Q_n(x;\alpha,\beta,N), \tag{5.7}$$

$k=0,1,\dots N$. A summation formula of Lee [20, (13)] is a special case of (5.7).

To obtain a summation formula for the continuous dual Hahn polynomials we need a generalization of (5.4)

$$\begin{aligned}{}_{p+1}F_{q+1}\left(\begin{matrix}b_1,\dots,b_p,b\\c_1,\dots,c_q,c\end{matrix}\;\Big|\;x\right)&=\frac{k!}{(c,k)}\sum_{j=0}^{k}\frac{(b,j)(c-b,k-j)}{j!(k-j)!}\\&\cdot{}_{p+1}F_{q+1}\left(\begin{matrix}b_1,\dots,b_p,b+j\\c_1,\dots,c_q,c+k\end{matrix}\;\Big|\;x\right),\end{aligned} \tag{5.8}$$

$|x|<\rho$, where ρ denotes the radius of convergence of the hypergeometric series

$${}_pF_q\left(\begin{matrix}b_1,\dots,b_p\\c_1,\dots,c_q\end{matrix}\;\Big|x\right). \tag{5.9}$$

Formula (5.8) is a special case of [8, Ex. 5.7–1]. When one of the numerator parameters in (5.9) is a non-positive integer, then the latter restriction can be dropped.

Letting $p=2$, $q=1$, $x=1$, $b_1=a+ix$, $b_2=a-ix$, $b=-n$ $(n\in\mathbb{N})$, $c_1=a+c$, $c_2=a+b$ in (5.8), we obtain

$$\begin{aligned}\frac{k!}{(a+b,k)}\sum_{j=0}^{k}\frac{(-n,j)(a+b+n,k-j)}{j!(k-j)!}&\frac{S_{n-j}(x^2;a,b,c,)}{(a+b+k,n-j)(a+c,n-j)}\\&=\frac{S_n(x^2;a,b,c)}{(a+b,n)(a+c,n)}.\end{aligned} \tag{5.10}$$

For more summation formulas for the Hahn polynomials the reader is referred to Bartko [4], Gasper [12–13], and Lee [20].

6. Remarks

The method of Dirichlet averages can be employed to obtain some results for the Hahn polynomials using known results for the Jacobi polynomials. In this section we make some comments concerning the projections formulas discussed in [12] and [13]. Also, we give some results for the continuous dual Hahn polynomials.

A. Gasper's projection formula [12, (1.4)]

$$Q_n(x;\gamma,\delta,N)=\sum_{k=0}^{n} b_{k,n}Q_k(x;\alpha,\beta,N), \tag{6.1}$$

$$\begin{aligned} b_{k,n} &= \binom{n}{k}\frac{(\alpha+1,k)(n+\nu,k)}{(\alpha+1,k)(k+\mu,k)} \\ &\quad\times {}_3F_2\left(\begin{matrix}-n+k,\ k+\alpha+1,\ n+k+\nu\\ k+\gamma,\ 2k+\mu+1\end{matrix}\ \Big|1\right)\end{aligned}$$

($\nu=\gamma+\delta+1$) can be obtained immediately from the formula which connects the Jacobi polynomials of different orders

$$\frac{P_n^{(\gamma,\delta)}(x)}{P_n^{(\gamma,\delta)}(1)}=\sum_{k=0}^{n} b_{k,n}\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} \tag{6.2}$$

(see [1, (7.3), (7.8)]). In (6.2) replace x by $1-2t$ and then average both sides using (3.3) – (3.4).

B. The following summation formula

$$\sum_{k=0}^{n} c_{k,n}Q_k(x;\alpha,\beta,N)=\frac{(-x,n)}{(-N,n)} \tag{6.3}$$

($n=0,1,\dots,N$), where

$$c_{k,n}=(\alpha+1,k)(-1)^k\binom{n}{k}\frac{\mu+2k}{(\mu+k,n+1)},\quad 0\le k\le n$$

plays a key role in Gasper's proof of (6.1). To obtain (6.3) we average both sides of

$$t^n=\sum_{k=0}^{n} c_{k,n}\frac{P_k^{(\alpha,\beta)}(1-2t)}{P_k^{(\alpha,\beta)}(1)}$$

(cf. [24, 136(2)]) using (3.3) and (3.4). The result is

$$R_n(-x,x-N;1,0)=\sum_{k=0}^{n} c_{k,n}Q_k(x;\alpha,\beta,N).$$

Since

$$R_n(-x,x-N;1,0)=\frac{(-x,n)}{(-N,n)},$$

the assertion follows. The last formula is a special case of [8, (6.2–5)].

C. Formula (6.1) contains as a special case a projection formula for the symmetric Hahn polynomials

$$Q_n(x;\beta,\beta,N) = \sum_{k=0}^{[n/2]} a_{k,n} Q_{n-2k}(x;\alpha,\alpha,N), \tag{6.4}$$

where

$$a_{k,n} = \frac{n!\Gamma(\alpha+1/2)(n-2k+\alpha+1/2)(2\alpha+1,n-2k)\Gamma(k+\beta-\alpha)\Gamma(n-k+\beta+1/2)}{k!(n-2k)!(2\beta+1,n)\Gamma(\beta-\alpha)\Gamma(n-k+\alpha+3/2)},$$

$0 \le k \le [n/2]$. This follows easily from Gegenbauer's formula for the symmetric Jacobi polynomials (ultraspherical polynomials)

$$\frac{P_n^{(\beta,\beta)}(x)}{P_n^{(\beta,\beta)}(1)} = \sum_{k=0}^{[n/2]} a_{k,n} \frac{P_{n-2k}^{(\alpha,\alpha)}(x)}{P_{n-2k}^{(\alpha,\alpha)}(1)} \tag{6.5}$$

(see [14]). Replace x by $1-2t$. Use of (3.3) – (3.4) yields the assertion.

D. It follows from (4.13) and the Maclaurin theorem that

$$\frac{S_n(x^2;a,b,c)}{(a+b,n)(a+c,n)} = \left[D_w^n \, {}_2F_2 \left(\begin{matrix} a+ix,\ a-ix \\ a+b,\ a+c \end{matrix} \;\middle|\; w \right) \right]_{w\,=\,0}. \tag{6.6}$$

Use of the Cauchy formula on the right side of (6.6) provides another contour integral for S_n

$$\frac{S_n(x^2;a,b,c)}{(a+b,n)(a+c,n)} = \frac{n!}{2\pi i} \int_\varepsilon s^{-n-1} e^s \, {}_2F_2 \left(\begin{matrix} a+ix,\ a-ix \\ a+b,\ a+c \end{matrix} \;\middle|\, -s \right) ds,$$

where the contour ε encircles the origin of the s-plane in the positive direction. To obtain a real integral for the continuous dual Hahn polynomials one can substitute $s = e^{i\varphi}$, $0 \le \varphi \le 2\pi$. We omit further details.

E. Performing one differentiation in (6.6) we obtain the first order recurrence-difference equation

$$\begin{aligned} S_n(x^2;a,b,c) &= (a+b+n-1)(a+c+n-1)S_{n-1}(x^2;a,b,c) \\ &\quad - (a^2+x^2)S_{n-1}(x^2;a+1,b,c), \end{aligned} \tag{6.7}$$

$n = 1, 2, \ldots, S_0(x^2) = 1$.

F. Assume now that $a,b,c > 0$. Application of [6, (2.8)] to (3.11) gives

$$\left| \frac{S_n(x^2;a,b,c)}{(a+b,n)(a+c,n)} \right| \le \frac{B(a,b)B(a,c)}{|B(a+ix,b-ix)B(a-ix,c+ix)|},$$

$-\infty < x < \infty$, or in terms of the gamma function

$$\left| \frac{S_n(x^2;a,b,c)}{(a+b,n)(a+c,n)} \right| \le \frac{\Gamma^2(a)\Gamma(b)\Gamma(c)}{|\Gamma(a+ix)\Gamma(a-ix)\Gamma(b-ix)\Gamma(c+ix)|}. \tag{6.8}$$

If $a, b, c \geq \frac{1}{2}$ and $x \neq 0$, then one can apply the inequality [8, (3.10–7)]

$$\frac{\Gamma(a)}{|\Gamma(a \pm ix)|} \leq (\operatorname{sech} \pi x)^{-1/2}$$

to the right side of (6.8) to obtain a weaker bound

$$\left|\frac{S_n(x^2; a, b, c)}{(a+b, n)(a+c, n)}\right| \leq (\operatorname{sech} \pi x)^{-2}.$$

7. The Addition Theorems for Krawtchouk Polynomials and Meixner-Pollaczek Polynomials

In this section we shall establish addition theorems for the Krawtchouk polynomials and the Meixner-Pollaczek polynomials. The former family is the limiting case of the Hahn polynomials (see (3.20)) while the latter can be obtained as the limiting case of the continuous dual Hahn polynomials (see [26, p. 698] for more details). The Meixner-Pollaczek polynomials are defined by [2, p. 48]

$$P_n^{(a)}(x; \varphi) = e^{in\varphi} {}_2F_1\left(\begin{matrix} -n, a+ix \\ 2a \end{matrix} \;\middle|\; 1 - e^{-2i\varphi}\right) \tag{7.1}$$

($a > 0$, $0 < \varphi < \pi$). They constitute an orthogonal system on $\mathbb{R}$ with the weight function

$$x \to e^{(2\varphi - \pi)x} |\Gamma(a+ix)|^2$$

(see, e.g., [26, p. 698]).

Application of (3.15) to the right side of (7.1) gives

$$P_n^{(a)}(x; \varphi) = e^{in\varphi} R_n(a+ix, a-ix; e^{-2i\varphi}, 1).$$

Since R_n is homogeneous of order n in its variables,

$$P_n^{(a)}(x; \varphi) = R_n(a+ix, a-ix; e^{-i\varphi}, e^{i\varphi}). \tag{7.2}$$

Similarly, application of (3.15) to the third member of (3.20) gives

$$K_n(x; p, N) = R_n(-x, x-N; 1-1/p, 1), \tag{7.3}$$

$0 < p < 1$, $n = 0, 1, \ldots, N$.

We are now in a position to state and prove the addition theorems for the polynomials discussed in this section. We have

$$\begin{aligned} &K_n(x+y; p, M+N) \\ &= \binom{M+N}{n}^{-1} \sum_{j=0}^{n} \binom{M}{j} \binom{N}{n-j} K_j(x; p, M) K_{n-j}(y; p, N) \end{aligned} \tag{7.4}$$

($M, N \in \mathbb{N}$, $n = 0, 1, \ldots, \min\{M, N\}$) and

$$(7.5)\quad \begin{aligned} &P_n^{(a+b)}(x+y;\varphi) \\ &\quad = \sum_{j=0}^{n} \binom{n}{j} \frac{(2a,j)(2b,n-j)}{(2a+2b,n)} P_j^{(a)}(x;\varphi) P_{n-j}^{(b)}(y;\varphi) \end{aligned}$$

($a > 0$, $b > 0$; $0 < \varphi < \pi$). Both formulas follow easily from

$$(7.6)\quad \begin{aligned} &\frac{(\alpha+\alpha'+\beta+\beta',n)}{n!} R_n(\alpha+\beta,\alpha'+\beta';x,y) \\ &\quad = \sum_{j=0}^{n} \frac{(\alpha+\alpha',j)(\beta+\beta',n-j)}{j!(n-j)!} R_j(\alpha,\alpha';x,y) R_{n-j}(\beta,\beta';x,y) \end{aligned}$$

which is a special case of [8, Ex. 6.6–7].

In order to establish the addition theorem (7.4) we substitute $\alpha = -x$, $\alpha' = x - M$, $\beta = -y$, $\beta' = y - N$, $x = 1 - 1/p$, $y = 1$ into (7.6). This in conjunction with (7.3) completes the proof. For the proof of (7.5) we let $\alpha = a + ix$, $\alpha' = a - ix$, $\beta = b + iy$, $\beta' = b - iy$, $x = \exp(-i\varphi)$, $y = \exp(i\varphi)$ in (7.6). Use of (7.2) on the resulting formula gives the assertion.

Dunkl [9, §4.4] gave a different addition theorem for the Krawtchouk polynomials.

The addition theorem for the Laguerre polynomials

$$L_n^{\alpha+\beta+1}(x+y) = \sum_{j=0}^{n} L_j^{\alpha}(x) L_{n-j}^{\beta}(y)$$

(see, e.g., [8, Ex. 7.9–7]) can be obtained from (7.5) by use of the limit relation

$$\lim_{\varphi\to 0} P_n^{(a)}\left(\frac{x}{2\varphi};\varphi\right) = L_n^{2a-1}(x) \Big/ L_n^{2a-1}(0).$$

The Poisson-Charlier polynomials c_n may be obtained as limiting forms of the Krawtchouk polynomials by means of the relation

$$(7.7)\quad c_n(x;a) = \lim_{N\to\infty} K_n(x;a/N,N) = {}_2F_0\left(\begin{matrix} -n,\ -x \\ - \end{matrix} \;\middle|\; -a^{-1}\right),$$

$a > 0$, $x = 0, 1, \ldots$.

Letting $M = N$ in (7.4) and next using (7.7) we obtain with the aid of

$$\lim_{N\to\infty} \binom{2N}{n}^{-1} \binom{N}{j} \binom{N}{n-j} = 2^{-n} \binom{n}{j}$$

the addition theorem

$$(7.8)\quad c_n(x+y;2a) = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} c_j(x;a) c_{n-j}(y;a).$$

Dunkl's addition theorem for the Poisson-Charlier polynomials [9, (4.6)]

$$c_n(x+y;a) = \sum_{j=0}^{n} (n-j)! \binom{n}{j} \binom{y}{n-j} (-a)^{j-n} c_j(x;a)$$

can be derived from the generating function

$$e^w \left(1 - \frac{w}{a}\right)^x = \sum_{n=0}^{\infty} \frac{w^n}{n!} c_n(x;a), \quad |w| < a$$

(see, e.g., [25, (2.81.3)]).

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.3,249-261,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

ON THE GENERALIZED PICARD AND GAUSS WEIERSTRASS SINGULAR INTEGRALS

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Abstract. In this paper, we give the generalizations of the Picard and the Gauss Weierstrass singular integral operators which are based on the $q-$numbers and depend on $q-$generalization of the Euler gamma integral. Later on, some approximation properties of these two generalized operators are established in $L_p(\mathbb{R})$ and weighted $-L_p(\mathbb{R})$ spaces. We also show that the rates of convergence of these generalized operators to approximating function f in the L_p-norm are at least so faster than that of the classical Picard and Gauss Weierstrass singular integral operators.

1. Introduction

Let f be a real valued function in $\mathbb{R}$. For $\lambda > 0$ and $x \in \mathbb{R}$, the well-known Picard and Gauss Weierstrass singular integral operators are defined as

$$P_\lambda(f;x) := \frac{1}{2\lambda}\int_{-\infty}^{\infty} f(x+t)\, e^{-\frac{|t|}{\lambda}}\,dt$$

and

$$W_\lambda(f;x) := \frac{1}{\sqrt{\pi\lambda}}\int_{-\infty}^{\infty} f(x+t)\, e^{-\frac{t^2}{\lambda}}\,dt,$$

respectively.

For many years scientists have been investigating to develop various aspects of approximation results of above operators. The recent book written by Anastassiou and Gal [2] includes great number of results related to different properties of these type of operators and also includes other references on the subject. For example, in [2, Chapter 16], Jackson type generalization of these operators is one among other generalizations, which satisfy the Global Smoothness Preservation Property (GSPP). It has been shown in [3] that this type of generalization has better rate of convergence and provides better estimates with some modulus of smoothness. Beside, in [4] and [5], Picard and Gauss Weierstrass singular integral operators modified by means of non-isotropic distance and their pointwise approximation

1991 *Mathematics Subject Classification.* 41A17, 41A25, 41A35.

Key words and phrases. $q-$gamma integral, $q-$Picard and $q-$Gauss Weierstrass integral, weighted modulus of continuity.

properties in different normed spaces are analyzed. Furthermore, in [11] and [8], Picard and Gauss Weierstrass singular integrals were considered in exponential weighted spaces for functions of one or two variables.

In this paper, we introduce a new generalization of Picard singular integral operator and Gauss Weierstrass singular integral operator which we call the $q-$Picard singular integral operator and the $q-$Gauss Weierstrass singular integral operator, respectively. As a result, a connection has been constructed between $q-$analysis and approximation theory.

We now give a short background on $q-$analysis that we need throughout the rest of the paper. We use standard notations of $q-$analysis as in [10] and [12].

For $q>0$, $q-$number is

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & q \neq 1 \\ \lambda, & q = 1 \end{cases}$$

for all nonnegative λ. If λ is an integer, i.e. $\lambda = n$ for some n, we write $[n]_q$ and call it $q-$integer. Also, we define a $q-$factorial as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n = 1, 2, ... \\ 1 & n = 0. \end{cases} .$$

For integers $0 \leq k \leq n$, the $q-$binomial coefficients are given by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Furthermore, the $q-$extension of exponential function e^x is

$$E_q(x) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n = (-x;q)_\infty ,$$

where $(a;q)_n = \prod\limits_{k=0}^{n} \left(1 - aq^k\right)$ and $(-x;q)_\infty = \prod\limits_{k=0}^{\infty} \left(1 + xq^k\right)$.

To be able to construct the generalized operators, we need the following $q-$extension of Euler integral representation for the gamma function given in [6] and [1] for $0 < q < 1$

$$c_q(x)\Gamma_q(x) = \frac{1-q}{\ln q^{-1}} q^{\frac{x(x-1)}{2}} \int_0^\infty \frac{t^{x-1}}{E_q((1-q)t)} dt, \quad \Re x > 0 \tag{1.1}$$

where $\Gamma_q(x)$ is the $q-$gamma function defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \ \ 0 < q < 1$$

and $c_q(x)$ satisfies the following conditions:

(1) $c_q(x+1) = c_q(x)$
(2) $c_q(n) = 1$, $n = 0, 1, 2, ...$

(3) $\lim_{q\to 1^-} c_q(x) = 1.$

When $x = n+1$ with n a nonnegative integer, we obtain

$$\Gamma_q(n+1) = [n]_q!. \tag{1.2}$$

In [7], Berg evaluated the following integral

$$\int_{-\infty}^{\infty} \frac{t^{2k}}{E_q(t^2)} dt = \pi \left(q^{1/2};\, q\right)_{1/2} q^{-\frac{k^2}{2}} \left(q^{1/2};\, q\right)_k, \quad k = 0,\, 1,\, 2... \tag{1.3}$$

where

$$(a;\, q)_\alpha = \frac{(a;\, q)_\infty}{(aq^\alpha;\, q)_\infty}$$

for any real number α.

These integrals (1.1) and (1.3) are the starting point of our work. Note that, these definitions are kinds of $q-$deformation of usual ones and are reduced to them in the limit $q \to 1$.

Definition 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $\lambda > 0$ and $0 < q < 1$, the $q-$generalizations of Picard and Gauss-Weierstrass singular integrals of f are*

$$P_\lambda(f; q,\, x) \equiv P_\lambda(f;\, x) := \frac{(1-q)}{2[\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} \frac{f(x+t)}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt \tag{1.4}$$

and

$$W_\lambda(f; q,\, x) \equiv W_\lambda(f;\, x) := \frac{1}{\pi\sqrt{[\lambda]_q}\left(q^{1/2};\, q\right)_{1/2}} \int_{-\infty}^{\infty} \frac{f(x+t)}{E_q\left(\frac{t^2}{[\lambda]_q}\right)} dt, \tag{1.5}$$

respectively.

Note that, this construction is sensitive to the rate of convergence to f. That is, the proposed estimate in Section 2 with rates in terms of L_p-modulus of continuity tells us that, depending on our selection of q, the rates of convergence in L_p-norm of the $q-$Picard and the $q-$Gauss Weierstrass singular integral operators are better than the classical ones.

In the section 3, we give a direct approximation result for these operators using Korovkin type theorem in weighted L_p spaces described in [9]. We give a new type modulus of continuity and in terms of this modulus of continuity, we obtain an inequality for weighted error estimate in section 4. Also we show in section 5 that they posses Global Smoothness Preservation Property.

2. Rate of Convergence in $L_p(\mathbb{R})$

For $f \in L_p(\mathbb{R})$, the modulus of continuity of f is defined by

$$\omega_p(f;\delta) = \sup_{|h|\leq\delta} \|f(\cdot+h) - f(\cdot)\|_p,$$

where $\|f\|_p = \left(\int\limits_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p}$.

Here are some auxiliary lemmas.

Lemma 2.1. *For every $\lambda > 0$,*

a) $\int\limits_{-\infty}^{\infty} P_\lambda(f;x)\, dx = 1,$

b) $\int\limits_{-\infty}^{\infty} W_\lambda(f;x)\, dx = 1.$

Proof. The proof is obvious from (1.1) and (1.3). □

By using Lemma 2.1, for every function $f \in L_p(\mathbb{R})$ with $1 \leq p < \infty$, the operators defined by (1.4) and (1.5) are well defined as expressed in the following lemma.

Lemma 2.2. *Let $f \in L_p(\mathbb{R})$ for some $1 \leq p < \infty$. Then we have*

$$\|P_\lambda(f;\cdot)\|_p \leq \|f\|_p$$

and

$$\|W_\lambda(f;\cdot)\|_p \leq \|f\|_p.$$

Now we give convergence rates for these new operators. A similar approach for classical Picard and Gauss Weierstrass singular integral operators can be found in [13, Th. 1.18]

Theorem 2.3. *If $f \in L_p(\mathbb{R})$ for some $1 \leq p < \infty$ then we have*

$$\|P_\lambda(f;\cdot) - f(\cdot)\|_p \leq \omega_p\left(f;[\lambda]_q\right)\left(1+\frac{1}{q}\right)$$

and

$$\|W_\lambda(f;\cdot) - f(\cdot)\|_p \leq \omega_p\left(f;\sqrt{[\lambda]_q}\right)\left(1+\sqrt{q^{-1/2}\left(1-q^{1/2}\right)}\right).$$

Proof. From Lemma 2.1, we get

$$P_\lambda(f;x) - f(x) = \frac{(1-q)}{2[\lambda]_q \ln q^{-1}} \int\limits_{-\infty}^{\infty} \frac{(f(x+t) - f(x))}{E_q\left(\frac{(1-q)|t|}{[\lambda]_q}\right)} dt.$$

Thus

$$\begin{aligned}
\left\|P_{\lambda}(f;\cdot)-f(\cdot)\right\|_{p} &\leq \frac{(1-q)}{2[\lambda]_{q}\ln q^{-1}}\left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{f(x+t)-f(x)}{E_{q}\left(\frac{(1-q)|t|}{[\lambda]_{q}}\right)}dt\right|^{p}dx\right)^{1/p} \\
&\quad \text{(generalized Minkowski inequality, see [14, pp.271])} \\
&\leq \frac{(1-q)}{2[\lambda]_{q}\ln q^{-1}}\int_{-\infty}^{\infty}\frac{\omega_{p}(f;|t|)}{E_{q}\left(\frac{(1-q)|t|}{[\lambda]_{q}}\right)}dt \\
&\leq \omega_{p}\left(f;[\lambda]_{q}\right)\frac{(1-q)}{2[\lambda]_{q}\ln q^{-1}}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{[\lambda]_{q}}\right)\frac{dt}{E_{q}\left(\frac{(1-q)|t|}{[\lambda]_{q}}\right)} \\
&= \omega_{p}\left(f;[\lambda]_{q}\right)\left(1+\frac{1}{q}\right),
\end{aligned}$$

where we use (1.1), (1.2) and the well known inequality

$$\omega_{p}(f;C\delta)\leq(1+C)\,\omega_{p}(f;\delta)$$

for $C>0$.

Similarly,

$$\begin{aligned}
\left\|W_{\lambda}(f;\cdot)-f(\cdot)\right\|_{p} &\leq \frac{\omega_{p}\left(f;\sqrt{[\lambda]_{q}}\right)}{\pi\sqrt{[\lambda]_{q}}\left(q^{1/2};q\right)_{1/2}}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\sqrt{[\lambda]_{q}}}\right)\frac{dt}{E_{q}\left(\frac{t^{2}}{[\lambda]_{q}}\right)} \\
&\leq \omega_{p}\left(f;\sqrt{[\lambda]_{q}}\right)\left(1+\left(\frac{1}{\pi\sqrt{[\lambda]_{q}}\left(q^{1/2};q\right)_{1/2}}\int_{-\infty}^{\infty}\frac{t^{2}}{[\lambda]_{q}}\frac{dt}{E_{q}\left(\frac{t^{2}}{[\lambda]_{q}}\right)}\right)^{1/2}\right) \\
&\leq \omega_{p}\left(f;\sqrt{[\lambda]_{q}}\right)\left(1+\sqrt{q^{-1/2}\left(1-q^{1/2}\right)}\right),
\end{aligned}$$

where we use (1.3). □

Since for a fixed value of q with $0<q<1$,

$$\lim_{\lambda\to 0}[\lambda]_{q}=\frac{1}{1-q},$$

the above theorem does not give a rate of convergence for $P_{\lambda}(f;\cdot)-f(\cdot)$ in $L_{p}-$norm. However, if we choose q_{λ} such that $0<q_{\lambda}<1$ and $q_{\lambda}\to 1$ as $\lambda\to 0$, then we obtain such a convergence rate.For example, we select q_{λ} as

$$\frac{1}{2}\leq 1-\lambda\leq q_{\lambda}<1$$

for some $\lambda>0$. Then we have

$$[\lambda]_{q_{\lambda}}=\frac{1-q_{\lambda}^{\lambda}}{1-q_{\lambda}}\leq 2\left(1-q_{\lambda}\right)\leq 2\lambda,$$

so that $[\lambda]_{q_{\lambda}}\to 0$ as $\lambda\to 0$. Thus we express Theorem 2.3 as follows.

Theorem 2.3. Let be $q_\lambda \in (0,\,1)$ such that $q_\lambda \to 1$ as $\lambda \to 0$. If $f \in L_p(\mathbb{R})$ for some $1 \le p < \infty$ then we have

$$\|P_\lambda(f; q_\lambda,\,\cdot) - f(\cdot)\|_p \le \omega_p\left(f;\,[\lambda]_{q_\lambda}\right)\left(1 + \frac{1}{q_\lambda}\right)$$

and

$$\|W_\lambda(f; q_\lambda,\,\cdot) - f(\cdot)\|_p \le \omega_p\left(f;\,\sqrt{[\lambda]_{q_\lambda}}\right)\left(1 + \sqrt{q_\lambda^{-1/2}\left(1 - q_\lambda^{1/2}\right)}\right).$$

This theorem tells us that depending on the selection of q_λ, the rate of convergence of $P_\lambda(f;\,\cdot)$ to $f(\cdot)$ in the L_p-norm is $[\lambda]_{q_\lambda}$ that is at least so faster than λ which is the rate of convergence for the classical Picard singular integrals. Similar situation arises when approximating by $W_\lambda(f;\,\cdot)$ to $f(\cdot)$.

3. Convergence in Weighted Space

Now we recall the following Korovkin type theorem in weighted L_p space given in [9].

Let ω be positive continuous function on real axis $\mathbb{R} = (-\infty,\,\infty)$, satisfying the condition

$$\int_{\mathbb{R}} t^{2p}\omega(t)\,dt < \infty. \tag{3.1}$$

We denote by $L_{p,\,\omega}(\mathbb{R})$ the linear space of $p-$absolutely integrable functions on $\mathbb{R}$ with respect to the weight function ω, i.e. for $1 \le p < \infty$

$$L_{p,\,\omega}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R};\ \|f\|_{p,\,\omega} := \left\|f\omega^{\frac{1}{p}}\right\|_p = \left(\int_{\mathbb{R}} |f(t)|^p\,\omega(t)\,dt\right)^{\frac{1}{p}} < \infty \right\}.$$

Theorem A. Let $(L_n)_{n\in N}$ be a uniformly bounded sequence of positive linear operators from $L_{p,\,\omega}(\mathbb{R})$ into $L_{p,\,\omega}(\mathbb{R})$, satisfying the conditions

$$\lim_{n\to\infty} \left\|L_n\left(t^i;\,x\right) - x^i\right\|_{p,\,\omega} = 0, \quad i = 0,\,1,\,2.$$

Then for every $f \in L_{p,\,\omega}(\mathbb{R})$

$$\lim_{n\to\infty} \|L_n f - f\|_{p,\,\omega} = 0.$$

By choosing $\omega(x) = \left(\frac{1}{1+x^{6m}}\right)^p, \quad p \ge 1$,and working on $L_{p,\,\omega}(\mathbb{R})$ space that we denote it by $L_{p,\,m}(\mathbb{R})$, we shall obtain direct approximation result by using Theorem A. Note that this selection of $\omega(x)$ satisfies the condition (3.1). Also note that for $1 \le p < \infty$

$$L_{p,\,m}(\mathbb{R}) = \left\{f :\ f : \mathbb{R} \to \mathbb{R};\ \left(1 + x^{6m}\right)^{-1} f(x) \in L_p(\mathbb{R})\right\},$$

where m is a positive integer.

Lemma 3.1. *If $f \in L_{p,m}(\mathbb{R})$ for some $1 \le p < \infty$ and positive integer m, then*

$$\|P_\lambda(f;\,\cdot)\|_{p,m} \le 2^{6m-1}\left(1 + \frac{[\lambda]_q^{6m}\,[6m]_q!}{q^{3m(6m+1)}}\right)\|f\|_{p,m}$$

and

$$\|W_\lambda(f;\,\cdot)\|_{p,m} \le 2^{6m-1}\left(1 + [\lambda]_q^{3m}\,q^{-\frac{9m^2}{2}}\left(q^{1/2};\,q\right)_{3m}\right)\|f\|_{p,m}$$

for $0 < q < 1$.

Proof. Using $\left(1 + (x+t)^{6m}\right) \le 2^{6m-1}\left(1 + x^{6m}\right)\left(1 + t^{6m}\right)$ for all positive integer m and $x,\, t \in \mathbb{R}$ and (1.1), (1.2) and (1.3), the proof is obvious. □

Theorem 3.2. *Let $q_\lambda \in (0,\,1)$ such that $q_\lambda \to 1$ as $\lambda \to 0$. Then for every $f \in L_{p,m}(\mathbb{R})$*

$$\lim_{\lambda\to 0}\|P_\lambda(f; q_\lambda,\,\cdot) - f\|_{p,m} = 0.$$

Proof. Using Theorem A, it is sufficient to verify that the conditions

$$\lim_{\lambda\to 0}\left\|P_\lambda\left(t^i; q_\lambda,\,\cdot\right) - x^i\right\|_{p,m} = 0, \quad i = 0,\,1,\,2. \tag{3.2}$$

are satisfied. Since $P_\lambda(1; q_\lambda,\,\cdot) = 1$ and $P_\lambda(t; q_\lambda,\,\cdot) = x$, the conditions of (3.2) are fulfilled for $i = 0$ and $i = 1$.

Direct calculation shows that

$$P_\lambda\left(t^2; q_\lambda,\,\cdot\right) = x^2 + \frac{[2]_{q_\lambda}[\lambda]_{q_\lambda}^2}{q_\lambda^3}$$

and then we obtain

$$\left\|P_\lambda\left(t^2; q_\lambda,\,\cdot\right) - x^2\right\|_{p,m} = \frac{[2]_{q_\lambda}[\lambda]_{q_\lambda}^2}{q_\lambda^3}\|1\|_{p,m}.$$

This means that the condition in (3.2) for $i = 2$ also holds and by Theorem A the proof is completed. □

Theorem 3.3. *Let be $q_\lambda \in (0,\,1)$ such that $q_\lambda \to 1$ as $\lambda \to 0$. For every $f \in L_{p,m}(\mathbb{R})$*

$$\lim_{\lambda\to 0}\|W_\lambda(f; q_\lambda,\,\cdot) - f\|_{p,m} = 0.$$

For $f \in L_{p,m}(\mathbb{R})$ with some positive integer m, we define the weighted modulus of continuity $\omega_{p,m}(f;\,\delta)$ as

$$\begin{aligned}\omega_{p,m}(f;\,\delta) &= \sup_{|h|\le\delta}\left(\int_{-\infty}^{\infty}\left|\frac{f(x+h) - f(x)}{\left(1+h^{6m}\right)\left(1+x^{6m}\right)}\right|^p dx\right)^{1/p}\\ &= \sup_{|h|\le\delta}\left\|\frac{f(\cdot+h) - f(\cdot)}{\left(1+h^{6m}\right)}\right\|_{p,m}.\end{aligned}$$

Now , we show that this modulus of continuity satisfies some classical properties of L_p-modulus. For $f \in L_{p,m}$ it is guaranteed that $\omega_{p,m}(f;\delta)$ is bounded as δ tends to ∞ and also, $\omega_{p,m}(f;\delta) \leq 2^{6m}\|f\|_{p,m}$ for any integer m.

4. APPROXIMATION ERROR

The next Lemma 4.1 and Lemma 4.2 will allow us to obtain the approximation error of generalized operators by means of the weighted modulus of continuity $\omega_{p,m}(f;\delta)$ and weighted norm $\|\cdot\|_{p,m}$.

Lemma 4.1. *Given $f \in L_{p,m}(\mathbb{R})$ and $C > 0$,*

$$\omega_{p,m}(f;C\delta) \leq 2^{6m-1}(1+C)^{6m+1}\left(1+\delta^{6m}\right)\omega_{p,m}(f;\delta) \tag{4.1}$$

for $\delta > 0$.

Proof. For positive integer n, we can write

$$\begin{aligned}
\omega_{p,m}(f;n\delta) &= \sup_{|h|\leq\delta}\left\|\frac{f(\cdot+nh)-f(\cdot)}{\left(1+(nh)^{6m}\right)}\right\|_{p,m} \\
&= \sup_{|h|\leq\delta}\left\|\sum_{k=1}^{n}\frac{f(\cdot+kh)-f(\cdot+(k-1)h)}{\left(1+(nh)^{6m}\right)}\right\|_{p,m} \\
&\leq 2^{6m-1}\omega_{p,m}(f;\delta)\sum_{k=1}^{n}\left(1+((k-1)\delta)^{6m}\right) \\
&\leq 2^{6m-1}n\left(1+((n-1)\delta)^{6m}\right)\omega_{p,m}(f;\delta) \\
&\leq 2^{6m-1}n^{6m+1}\left(1+\delta^{6m}\right)\omega_{p,m}(f;\delta).
\end{aligned}$$

Using this estimation

$$\begin{aligned}
\omega_{p,m}(f;C\delta) &\leq 2^{6m-1}(1+[|C|])^{6m+1}\left(1+\delta^{6m}\right)\omega_{p,m}(f;\delta) \\
&\leq 2^{6m-1}(1+C)^{6m+1}\left(1+\delta^{6m}\right)\omega_{p,m}(f;\delta),
\end{aligned}$$

where $[|C|]$ is the greatest integer less than C. □

Lemma 4.2. *If $f \in L_{p,m}(\mathbb{R})$ then $\lim\limits_{\delta\to 0}\omega_{p,m}(f;\delta) = 0$.*

Proof. For a positive real number a, let $\chi_1^a(t)$ be characteristics function of the interval $[a,\infty)$, $\chi_2^a(t) = 1-\chi_1^a(t)$ and $\chi^a(t) = \chi_1^{-a}(t) \cap \chi_2^a(t)$. Since $f \in L_{p,m}$, for each $\varepsilon > 0$ there exists $a \in \mathbb{R}$ large enough such that

$$\left(\int_{-\infty}^{-a}\left|\frac{f(x)}{1+x^{6m}}\right|^p dx\right)^{\frac{1}{p}} + \left(\int_{a}^{\infty}\left|\frac{f(x)}{1+x^{6m}}\right|^p dx\right)^{\frac{1}{p}} < \frac{\varepsilon}{4}.$$

That is,

$$\left\| f\chi_2^{-a} \right\|_{p,\,m} + \left\| f\chi_1^{a} \right\|_{p,\,m} < \frac{\varepsilon}{4}.$$

Similarly, for $\delta > 0$

$$\left\| f\chi_2^{-(a+\delta)} \right\|_{p,\,m} + \left\| f\chi_1^{a+\delta} \right\|_{p,\,m} < \frac{\varepsilon}{2^{6m+1}\left(1+\delta^{6m}\right)}$$

can be written. Hence for $|h| \le \delta$

$$\left\| f\left(\cdot + h\right)\chi_2^{-(a+\delta)}\left(\cdot\right) \right\|_{p,\,m} + \left\| f\left(\cdot + h\right)\chi_1^{a+\delta}\left(\cdot\right) \right\|_{p,\,m} < \frac{\varepsilon}{4}.$$

Thus, we have

$$\omega_{p,\,m}\left(f;\, \delta\right) \le \sup_{|h|\le\delta} \left\| \frac{\left(f\left(\cdot + h\right) - f\left(\cdot\right)\right)\chi^{a+\delta}\left(\cdot\right)}{\left(1+h^{6m}\right)} \right\|_{p,\,m} + \frac{\varepsilon}{2} \tag{4.2}$$

for $\delta > 0$. By the well-known Weierstrass theorem, there exist sequences $\varphi_n\left(x\right) \in C^{\infty}$ (the space of function having continuous derivatives of any order in the interval $[-a-2\delta,\, a+2\delta]$) such that

$$\lim_{n\to\infty} \left\| \left(f\left(\cdot\right) - \varphi_n\left(\cdot\right)\right)\chi^{a+2\delta}\left(\cdot\right) \right\|_{p,\,m} = 0.$$

That is, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\| \left(f\left(\cdot\right) - \varphi_n\left(\cdot\right)\right)\chi^{a+2\delta}\left(\cdot\right) \right\|_{p,\,m} < \frac{\varepsilon}{2^{6m+5}} \tag{4.3}$$

whenever $n \ge n_0$ and $\delta > 0$. Thus we have

$$\begin{aligned} \left\| \left(f\left(\cdot + h\right) - \varphi_n\left(\cdot + h\right)\right)\chi^{a+\delta}\left(\cdot\right) \right\|_{p,\,m} &\le 2^{6m-1}\left\| \left(f\left(\cdot\right) - \varphi_n\left(\cdot\right)\right)\chi^{a+2\delta}\left(\cdot\right) \right\|_{p,\,m} \\ &\le \frac{\varepsilon}{6} \end{aligned} \tag{4.4}$$

for $n \ge n_0$.

Applying the Minkowsky inequality yields

$$\begin{aligned} \left\| \frac{\left(f\left(\cdot + h\right) - f\left(\cdot\right)\right)\chi^{a+\delta}\left(\cdot\right)}{\left(1+h^{6m}\right)} \right\|_{p,\,m} &\le \left\| \left(f\left(\cdot + h\right) - \varphi_n\left(\cdot + h\right)\right)\chi^{a+\delta}\left(\cdot\right) \right\|_{p,\,m} \\ &\quad + \left\| \left(\varphi_n\left(\cdot + h\right) - \varphi_n\left(\cdot\right)\right)\chi^{a+\delta}\left(\cdot\right) \right\|_{p,\,m} \\ &\quad + \left\| \left(\varphi_n\left(\cdot\right) - f\left(\cdot\right)\right)\chi^{a+\delta}\left(\cdot\right) \right\|_{p,\,m}. \end{aligned}$$

From (4.3) and (4.4) it follows that

$$\sup_{|h|\le\delta} \left\| \frac{\left(f\left(\cdot + h\right) - f\left(\cdot\right)\right)\chi^{a+\delta}\left(\cdot\right)}{\left(1+h^{6m}\right)} \right\|_{p,\,m} \le \frac{\varepsilon}{3} + \sup_{|h|\le\delta} \left\| \left(\varphi_n\left(\cdot + h\right) - \varphi_n\left(\cdot\right)\right)\chi^{a+\delta}\left(\cdot\right) \right\|_{p,\,m}, \tag{4.5}$$

for $\delta > 0$. By the properties of $\varphi_n\left(x\right)$, for $|h| \le \delta$ and $n \ge n_0$ we can write

$$\left|\varphi_n\left(x+h\right) - \varphi_n\left(x\right)\right| \le \frac{\varepsilon}{6\left\|\chi^{a+\delta}\right\|_{p,\,m}},$$

where $x \in [-a-2\delta,\, a+2\delta]$ .Thus, we obtain

$$\sup_{|h|\leq\delta}\left\|\left(\varphi_n(\cdot+h)-\varphi_n(\cdot)\right)\chi^{a+\delta}(\cdot)\right\|_{p,\,m}<\frac{\varepsilon}{6}. \tag{4.6}$$

By (4.5) and (4.6) we get

$$\sup_{|h|\leq\delta}\left\|\frac{\left(f(\cdot+h)-f(\cdot)\right)\chi^{a+\delta}(\cdot)}{(1+h^{6m})}\right\|_{p,\,m}<\frac{\varepsilon}{2}, \tag{4.7}$$

for $\delta>0$. From (4.2) and (4.7), we get

$$\omega_{p,\,m}(f;\,\delta)<\varepsilon$$

which shows that $\lim_{\delta\to0}\omega_{p,\,m}(f;\,\delta)=0$. $\square$

Theorem 4.3. *Let $q_\lambda\in(0,\,1)$ such that $q_\lambda\to1$ as $\lambda\to0$. For every $f\in L_{p,\,m}(\mathbb{R})$*

$$\|P_\lambda(f;q_\lambda,\,\cdot)-f(\cdot)\|_{p,m}\leq A\omega_{p,\,m}\left(f;\,[\lambda]_{q_\lambda}\right)$$

and

$$\|W_\lambda(f;q_\lambda,\,\cdot)-f(\cdot)\|_{p,m}\leq B\omega_{p,\,m}\left(f;\,\sqrt{[\lambda]_{q_\lambda}}\right)$$

where

$$A=2^{12m-1}\left(1+\frac{(6m)![\lambda]_{q_\lambda}^{6m}}{q_\lambda^{3m(6m+1)}}+\frac{(6m+1)!}{q_\lambda^{(3m+1)(6m+1)}}+\frac{(12m+1)![\lambda]_{q_\lambda}^{6m}}{q_\lambda^{(12m+1)(6m+1)}}\right)\left(1+[\lambda]_{q_\lambda}^{6m}\right) \tag{4.8}$$

and

$$\begin{aligned}B \;=\; & 2^{12m-1}\left(1+[\lambda]_{q_\lambda}^{6m}\right)\left(1+[\lambda]_{q_\lambda}^{3m}\,q_\lambda^{-\frac{9m^2}{2}}\left(q_\lambda^{1/2};\;q_\lambda\right)_{3m}+\sqrt{q_\lambda^{-\frac{(6m+1)^2}{2}}\left(q_\lambda^{1/2};\;q_\lambda\right)_{6m+1}}\right.\\ & \left.+[\lambda]_{q_\lambda}^{6m}\sqrt{q_\lambda^{-\frac{(12m+1)^2}{2}}\left(q_\lambda^{1/2};\;q_\lambda\right)_{12m+1}}\right).\end{aligned}$$

Proof. Part (a) of Lemma 2.1 implies that,

$$P_\lambda(f;q_\lambda,\,x)-f\;(x)=\frac{(1-q_\lambda)}{2\,[\lambda]_{q_\lambda}\ln q_\lambda^{-1}}\int_{-\infty}^{\infty}\frac{(f\,(x+t)-f\,(x))}{E_{q_\lambda}\left(\frac{(1-q_\lambda)|t|}{[\lambda]_{q_\lambda}}\right)}dt.$$

Then we have

$$\left\|P_\lambda\left(f;q_\lambda,\cdot\right)-f\left(\cdot\right)\right\|_{p,\ m}\leq\frac{(1-q_\lambda)}{2[\lambda]_{q_\lambda}\ln q_\lambda^{-1}}\left(\int\limits_{-\infty}^{\infty}\left|\int\limits_{-\infty}^{\infty}\frac{\left(f\left(x+t\right)-f\left(x\right)\right)}{E_q\left(\frac{(1-q_\lambda)|t|}{[\lambda]_{q_\lambda}}\right)\left(1+x^{6m}\right)}dt\right|^p dx\right)^{1/p}$$

$$\leq\frac{(1-q_\lambda)}{2[\lambda]_{q_\lambda}\ln q_\lambda^{-1}}\int\limits_{-\infty}^{\infty}\left(\int\limits_{-\infty}^{\infty}\left|\frac{f\left(x+t\right)-f\left(x\right)}{\left(1+x^{6m}\right)}\right|dx\right)^{1/p}\frac{dt}{E_q\left(\frac{(1-q_\lambda)|t|}{[\lambda]_{q_\lambda}}\right)}$$

$$\leq\frac{(1-q_\lambda)}{[\lambda]_{q_\lambda}\ln q_\lambda^{-1}}\int\limits_{0}^{\infty}\omega_{p,m}\left(f;\ t\right)\frac{\left(1+t^{6m}\right)}{E_q\left(\frac{(1-q_\lambda)t}{[\lambda]_{q_\lambda}}\right)}dt.$$

By using (4.1) and taking $C=\frac{t}{[\lambda]_{q_\lambda}}$, we have

$$\begin{aligned}\left\|P_\lambda\left(f;q_\lambda,\cdot\right)-f\left(\cdot\right)\right\|_{p,\ m}\ &\leq\ \frac{2^{6m-1}\left(1-q_\lambda\right)}{[\lambda]_{q_\lambda}\ln q_\lambda^{-1}}\left(1+[\lambda]_{q_\lambda}^{6m}\right)\omega_{p,m}\left(f;\ [\lambda]_{q_\lambda}\right)\\&\quad\int\limits_0^\infty\frac{\left(1+\frac{t}{[\lambda]_{q_\lambda}}\right)^{6m+1}\left(1+t^{6m}\right)}{E_q\left(\frac{(1-q_\lambda)t}{[\lambda]_{q_\lambda}}\right)}dt\\&\leq\ \frac{2^{12m-1}\left(1-q_\lambda\right)}{[\lambda]_{q_\lambda}\ln q_\lambda^{-1}}\left(1+[\lambda]_{q_\lambda}^{6m}\right)\omega_{p,m}\left(f;\ [\lambda]_{q_\lambda}\right)\\&\quad\int\limits_0^\infty\frac{1+t^{6m}+\frac{t^{6m+1}}{[\lambda]_{q_\lambda}^{6m+1}}+\frac{t^{12m+1}}{[\lambda]_{q_\lambda}^{6m+1}}}{E_q\left(\frac{(1-q_\lambda)t}{[\lambda]_{q_\lambda}}\right)}dt.\end{aligned}$$

From (1.1) and (1.2) it follows that

$$\left\|P_\lambda\left(f;q_\lambda,\cdot\right)-f\left(\cdot\right)\right\|_{p,\ m}\leq A\omega_{p,m}\left(f;\ [\lambda]_{q_\lambda}\right),$$

where A defined as in (4.8).

For $W_\lambda\left(f;\cdot\right)$, the proof is similar. □

5. Global Smoothness Preservation Property

Further information on G.S.P.P. for different linear positive operators and also singular integral operators can be found in [2].

Theorem 5.1. *Let $q_\lambda\in(0,1)$ such that $q_\lambda\to1$ as $\lambda\to0$. For every $f\in L_{p,m}(\mathbb{R})$ and $\delta>0$*

$$\omega_{p,m}\left(P_\lambda\left(f\right);\ \delta\right)\leq C\omega_{p,m}\left(f;\ \delta\right)$$

and

$$\omega_{p,m}\left(W_{\lambda}\left(f\right);\,\delta\right)\leq D\omega_{p,m}\left(f;\,\delta\right)$$

where

$$C=\left(1+\frac{(6m)!\,[\lambda]_{q_{\lambda}}^{6m}}{q_{\lambda}^{3m(6m+1)}}\right)\qquad \textit{and}\qquad D=q_{\lambda}^{-\frac{9m^{2}}{2}}\left(q_{\lambda}^{1/2};\;q_{\lambda}\right)_{3m}[\lambda]_{q_{\lambda}}^{3m}. \tag{5.1}$$

Proof. Part (a) of Lemma 2.1 implies that,

$$P_{\lambda}\left(f;q_{\lambda},\,x+h\right)-P_{\lambda}\left(f;q_{\lambda},\,x\right)=\frac{(1-q_{\lambda})}{2\,[\lambda]_{q_{\lambda}}\ln q_{\lambda}^{-1}}\int_{-\infty}^{\infty}\frac{\left(f\left(x+t+h\right)-f\left(x+t\right)\right)}{E_{q_{\lambda}}\left(\frac{(1-q_{\lambda})|t|}{[\lambda]_{q_{\lambda}}}\right)}dt.$$

By this equality, we get

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}\left|\frac{P_{\lambda}\left(f;q_{\lambda},\,x+h\right)-P_{\lambda}\left(f;q_{\lambda},\,x\right)}{\left(1+x^{6m}\right)\left(1+h^{6m}\right)}\right|^{p}dx\right)^{1/p}\\
\leq\;&\frac{(1-q_{\lambda})}{2\,[\lambda]_{q_{\lambda}}\ln q_{\lambda}^{-1}}\left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{\left(f\left(x+t+h\right)-f\left(x+t\right)\right)}{E_{q}\left(\frac{(1-q_{\lambda})|t|}{[\lambda]_{q_{\lambda}}}\right)\left(1+x^{6m}\right)\left(1+h^{6m}\right)}dt\right|^{p}dx\right)^{1/p}\\
\leq\;&\frac{(1-q_{\lambda})}{2\,[\lambda]_{q_{\lambda}}\ln q_{\lambda}^{-1}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|\frac{\left(f\left(x+t+h\right)-f\left(x+t\right)\right)}{\left(1+x^{6m}\right)\left(1+h^{6m}\right)}\right|^{p}dx\right)^{1/p}\frac{1}{E_{q_{\lambda}}\left(\frac{(1-q_{\lambda})|t|}{[\lambda]_{q_{\lambda}}}\right)}dt\end{aligned}$$

Using the inequality for $x,\,t\in\mathbb{R}$

$$1+x^{6m}\leq 2^{6m-1}\left(1+(x-t)^{6m}\right)\left(1+t^{6m}\right)$$

we have

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}\left|\frac{P_{\lambda}\left(f\,q_{\lambda},\,x+h\right)-P_{\lambda}\left(f;q_{\lambda},\,x\right)}{\left(1+x^{6m}\right)\left(1+h^{6m}\right)}\right|^{p}dx\right)^{1/p}\\
\leq\;&\frac{2^{6m-1}\left(1-q_{\lambda}\right)}{[\lambda]_{q_{\lambda}}\ln q_{\lambda}^{-1}}\omega_{p,m}\left(f;\,h\right)\int_{0}^{\infty}\frac{\left(1+t^{6m}\right)}{E_{q}\left(\frac{(1-q_{\lambda})|t|}{[\lambda]_{q_{\lambda}}}\right)}dt\end{aligned}$$

Besides, from (1.1) and (1.2) it follows that

$$\omega_{p,m}\left(P_{\lambda}\left(f\right);\,h\right)\leq C\omega_{p,m}\left(f;\,h\right),$$

where C is defined as in (5.1).

For $W_{\lambda}\left(f;\,\cdot\right)$, the proof is similar. □

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.3,263-273,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

The Principal Component Analysis of Continuous Sample Curves with Higher-order B-spline Functions

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ABSTRACT

This article concerns the Principal Component Analysis (PCA) of a vector process with higher-order B-spline functions. The approximated PCA of this well-known process is compared with the classical PCA of the different wavelengths simulated data.
Keywords : Principal components; vector process; B-splines.

1 INTRODUCTION

In many applications, observations are based on a continuous curve rather than a scalar or vector variable. The most common such applications are spectrophotometry, chromatography, absorbances of samples of filter material at wavelengths in the visible spectrum, stochastic processes, kinetic model building, and many others.

Castro et al.(1986) developed the principal components technique based on the concept of a best linear model in the context of continuous sample curves. Aguilera et al.(1996) developed the approximation of estimators in the PCA of a stochastic process using cubic splines.

In the present paper, the PCA technique is used for the reduction of sample curve data to a finite-dimension model, and the principal factors from simulated data using third-degree and fifth-degree B-splines are estimated.

2 PRINCIPAL COMPONENT ANALYSIS

Principal component analysis is a well-known technique for the reduction of vector data to a minimal dimension.Let $Y = \{y_1(x), y_2(x), ..., y_p(x) : x \in [0, 1]\}$ be real valued on the random fields, where $y_1(x), y_2(x), ..., y_p(x)$are scalar variates. The covariance function C(s,t) is defined on the Hilbert space $L^2[0,T]$. Consider the integral equation

$$\int_0^T C(s,t)\phi(t)dt = \lambda\phi(s)\ ,\ 0 \le s \le T \tag{2.1}$$

and the result

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1 \text{ if i = j} \\ 0 \text{ if i} \neq j \end{cases} \tag{2.2}$$

where ϕ is the orthonormal family of eigenfunctions and λ is the decreasing sequence of non-null eigenvalues.Then, the spectral representation of C provides the following orthogonal decomposition of the process, know as the Karhunen-Loeve expansion Adler (1981)

$$Y(x) = \mu(x) + \sum_{i=1}^{k} \phi_i(\mathrm{x})\alpha_{\mathrm{i}} \quad . \tag{2.3}$$

This model was considered by Rice and Silverman (1991). In (2.3), α_i is the family of uncorrelated zero-mean random variables defined by

$$\alpha_i = \int_0^T \phi_i(t)(Y(x) - \mu(x))dt \tag{2.4}$$

The random variable α_i is called the ith principal component and has the maximum variance of all the generalized linear combinations of Y(x) which are uncorrelated with α_j (j =1,..., i-1). The variance E$\{\alpha_i^2\}$=λ_i, for all i = 1,2, ... , is called the ith principal value of the process.

3 PCA OF B-SPLINES

In this section, third-degree and fifth-degree B-splines are used to construct the interpolated process. A detailed escription of B-spline functions generated by subdivision can be found in Schumaker (1981).Suppose that Y(x) is only observed at the knots 0, h, 2h, ... ,(n-1)h = 1. Each sample function Y(x) will be interpolated at the points (x$_i$, Y(x$_i$)) using B-splines.

3.1. Third-degree B-splines

The B-splines are defined as

$$B_0(x) = \frac{1}{6h^3}\begin{cases} x^3 & 0 \le x < h \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3 & h \le x < 2h \\ 3x^3 - 24hx^2 + 60h^2x - 44h^3 & 2h \le x < 3h \\ -x^3 + 12hx^2 - 48h^2x + 64h^3 & 3h \le x < 4h \end{cases} \tag{3.1}$$

$B_i(x) = B_0(x - (i-1)h)$, i = 2, 3, ...

Let

$$Y_I(x_i) = \sum_{i=1}^{n} C_i B_i(x) \tag{3.2}$$

be approximate values of ϕ_i(t) in (2.1) , where the C_i are unknown real coefficients and B_i(x) are cubic splines, I denoting interpolate. The following equation can be written

$$Y_I(x_i) = \sum_{i=1}^{n} C_i B_i((i-1)h) = Y((i-1)h). \tag{3.3}$$

This leads to a system of n linear equations in the n unknowns C_1, C_2, ... , C_n . This system can be written in matrix-vector form as

$$AC = Y , \tag{3.4}$$

where A is the symmetric coefficient matrix given by

$$A = \begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & 0 & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & 0 & & & 1 & 4 & 1 \\ & & & & & 1 & 4 \end{bmatrix} ,$$

$C = [C_1, C_2, \dots , C_n]^T$ and $Y = [Y_1, Y_2, \dots , Y_n]^T$,

T denoting tranpose. Then, the process $Y_I(x)$ in (3.2) can be interpolated upon the n random variables x_i , i = 0, 1, ... , n , by the following matrix expression

$$Y_I = B^T C , \tag{3.5}$$

where $B(x) = [B_1(x), B_2(x), \dots , B_n(x)]^T$ and C can be written in (3.4) as follows

$$C = A^{-1} Y . \tag{3.6}$$

Therefore,

$$Y_I = B^T A^{-1} Y . \tag{3.7}$$

It can be proved that the function of the interpolated process is

$$E [Y_I] = B^T A^{-1} M , \tag{3.8}$$

where M is the random vector. Similarly, the covariance function C(s,t) can be written as

$$C_I(s,t) = B^T(t) A^{-1} C A^{-1} B(s) . \tag{3.9}$$

Thus, the principal factors of Y_I are approximated by the eigenfunctions of the interpolated covariance Kernel $C_I(s,t)$.
Given $x_1, \dots , x_p$, and observations $y_j(x_1), \dots , y_j(x_p)$, j=1, ... , n , form the usual unbiased, consistent estimates

$$\hat{m}_i = \frac{1}{n} \sum_{j=1}^{n} y_j(x_i) \tag{3.10}$$

and

$$\hat{c}_{ij} = \frac{1}{n-1} \sum_{q=1}^{n} [y_q(x_i) - \hat{m}_i][y_q(x_j) - \hat{m}_j] \quad , \tag{3.11}$$

where $\hat{C}$ will be used to indicate quantites estimated from data. Using $\hat{C}$ in equation (3.9), (2.1) can be rewritten as

$$\int_0^1 B^T(t)A^{-1}\hat{C}A^{-1}B(s)\hat{\phi}(s)ds = \hat{\lambda}\hat{\phi}(t) \tag{3.12}$$

where $\hat{\phi}$ is given by

$$\hat{\phi}(t) = \sum_{i=1}^{n} \alpha_i B_i(t). \tag{3.13}$$

That is,

$$\hat{\phi}(t) = B^T(t)\alpha, \tag{3.14}$$

where α is the n-dimensional column vector of coordinates α_i. Substituting (3.14) in (3.12) gives

$$\hat{\lambda}B^T(t)\alpha = \int_0^1 B^T(t)A^{-1}\hat{C}A^{-1}B(s)B^T(s)\alpha ds. \tag{3.15}$$

Then, estimated eigenvectors $\hat{\phi}$(t) are obtained by solving (3.15).

Let $p_{ij} = \langle B_i, B_j\rangle = \int_0^1 B_i(t)B_j(t)dt$, i,j = 1, ... , n, then

$$\mathrm{P} = \begin{bmatrix}
302/35 & 531/70 & 6/7 & 1/140 & & & & & & \\
531/70 & 599/35 & 1191/140 & 6/7 & 1/140 & & & & & \\
6/7 & 1191/140 & 604/35 & 1191/140 & 6/7 & 1/140 & & & & \\
1/140 & 6/7 & 1191/140 & . & . & . & . & & & \\
 & 1/140 & 6/7 & . & . & . & . & . & & \\
 & & 1/140 & . & . & . & . & . & . & \\
 & & & . & . & . & 604/35 & 1191/140 & 6/7 & 1/140 \\
 & & & & . & . & 1191/140 & 604/35 & 1191/140 & 6/7 \\
 & & & & & . & 6/7 & 1191/140 & 599/35 & 531/70 \\
 & & & & & & 1/140 & 6/7 & 531/70 & 302/35
\end{bmatrix}.$$

Therefore, equation (3.15) can be rewritten as

$$\hat{\lambda}B^T(t)\alpha = B^T(t)A^{-1}\hat{C}A^{-1}P\alpha \quad , \tag{3.16}$$

$$\hat{\lambda}\alpha = A^{-1}\hat{C}A^{-1}P\alpha. \tag{3.17}$$

Thus, the problem is reduced to calculating the eigensystem of the nxn matrix $A^{-1}\hat{C}A^{-1}P$ (in general non-symmetric). Since the matrix P is positive definite, it has a unique square root $\mathrm{P}^{1/2}$. Therefore, by defining $\beta=\mathrm{P}^{1/2}\alpha$, (3.17) can be rewritten as

$$\hat{\lambda}\beta = P^{1/2}A^{-1}\hat{C}A^{-1}P^{1/2}\beta, \tag{3.18}$$

where $P^{1/2}A^{-1}\hat{C}A^{-1}P^{1/2}$ is a symmetric matrix. Once the eigenvectors β have been computed, the coefficient vectors α of the approximated principal factors $\hat{\phi}$ in the B-splines basis are obtained as

$$\alpha = \mathrm{P}^{1/2}\beta \ . \tag{3.19}$$

3.2. Fifth-degree B-splines

The following B-splines can be applied to all equations in subsection 3.1.The fifth-degree B-splines are defined as

$$\mathrm{B}_0 = \frac{1}{120h^5}\begin{cases} x^5 & 0 \le x < h \\ -5x^5 + 30hx^4 - 60h^2x^3 + 60h^3x^2 - 30h^4x + 6h^5 & h \le x < 2h \\ x^5 - 120hx^4 + 540h^2x^3 - 1140h^3x^2 + 1170h^4x - 474h^5 & 2h \le x < 3h \\ -10x^5 + 180hx^4 - 1260h^2x^3 + 4260h^3x^2 - 6930h^4x + 4386h^5 & 3h \le x < 4h \\ 5x^5 - 120hx^4 + 1140h^2x^3 - 5340h^3x^2 + 12270h^4x - 10974h^5 & 4h \le x < 5h \\ -x^5 + 30hx^4 - 360h^2x^3 + 2160h^3x^2 - 6480h^4x + 7776h^5 5h & 5h \le x < 6h \end{cases}$$

$\mathrm{B}_i(x) = B_0(x-(i-1)h)$, i = 2, 3, ...

Then, the coefficient matrix in (3.4) is given by

$$
A = \begin{bmatrix}
66 & 26 & 1 & & & & & & & \\
26 & 66 & 26 & 1 & & & & & & \\
1 & 26 & 66 & 26 & 1 & & & & & \\
 & 1 & 26 & 66 & 26 & 1 & & & & \\
 & & . & . & . & . & . & & & \\
 & & & . & . & . & . & . & & \\
 & & & & . & . & . & . & . & \\
 & & & & & 1 & 26 & 66 & 26 & 1 \\
 & & & & & & 1 & 26 & 66 & 26 \\
 & & & & & & & 1 & 26 & 66
\end{bmatrix}
$$

and P in (3.16) is given by

$$
P = \begin{bmatrix}
A_1 & B_1 & C_1 & D_1 & E_1 & F_1 & & & & & \\
B_1 & A_2 & B_2 & C_2 & D_1 & E_1 & F_1 & & & & \\
C_1 & B_2 & A_3 & B_3 & C_2 & D_1 & E_1 & F_1 & & & \\
D_1 & C_2 & B_3 & A_4 & B_3 & C_2 & D_1 & E_1 & F_1 & & \\
E_1 & D_1 & C_2 & B_3 & A_4 & B_3 & C_2 & D_1 & E_1 & F_1 & \\
F_1 & E_1 & D_1 & C_2 & B_3 & A_4 & B_3 & C_2 & D_1 & E_1 & F_1 \\
 & F_1 & E_1 & D_1 & C_2 & B_3 & A_4 & B_3 & C_2 & D_1 & E_1 \\
 & & F_1 & E_1 & D_1 & C_2 & B_3 & A_4 & B_3 & C_2 & D_1 \\
 & & & F_1 & E_1 & D_1 & C_2 & B_3 & A_3 & B_2 & C_1 \\
 & & & & F_1 & E_1 & D_1 & C_2 & B_2 & A_2 & B_1 \\
 & & & & & F_1 & E_1 & D_1 & C_1 & B_1 & A_1
\end{bmatrix},
$$

with the parameters given in Table 3.1. Then, the principal factors $\hat{\phi}$ are obtained from (3.19).

Table 3.1. Parameters of matrix P

Parameter	Value	Parameter	Value
A_1	655177/231	B_3	9738114/2772
A_2	1277215/231	C_1	1086965/1386
A_3	1310333/231	C_2	1101744/1386
A_4	1310354/231	D_1	152637/2772
B_1	8170295/2772	E_1	509/63
B_2	9729001/2772	F_1	1/2772

4 NUMERICAL RESULTS AND CONCLUSIONS

In this section, the methods discussed in section 3 were tested on simulated data from the literature Castro et al. (1986).To examine the effectiveness of the approach on simulated data, sample functions were generated using various observations by means of the formula

$$\mathrm{y(x)} = \mathrm{N}(\mathrm{m}_1, s_1)\exp\left[\frac{-(x-500)^2}{50^2}\right] + N(m_2, s_2)\exp\left[\frac{-(x-600)^2}{25^2}\right], \qquad (4.1)$$

where $\mathrm{N}(\mathrm{m}_i,\mathrm{s}_i)$ is the Gaussian distribution with m_1=7, s_1=0.5, m_2=3, s_2=0.5 .Several typical absorbance curves y(x), for wavelengths of light x between 350 nm and 700 nm are shown in Figure 4.1.

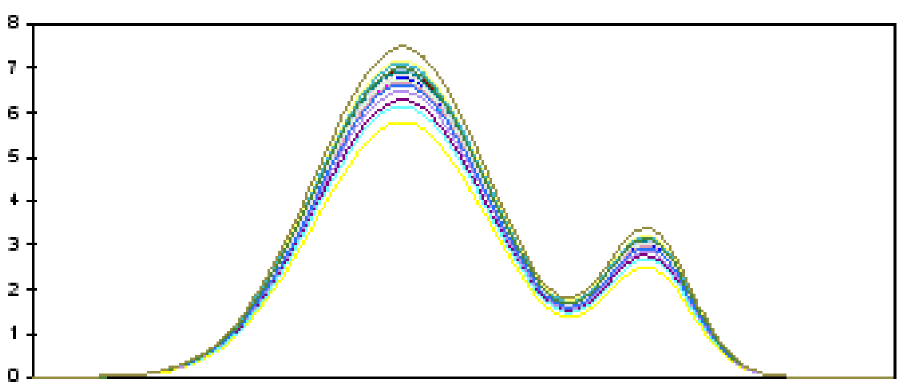

Figure 4.1. Simulated sample curves of filter absorbance

Several typical curves, for p=30 and various values of n, were obtained by using the methods developed in section 3. The results are depicted in Figure 4.2 and Figure 4.3 for third-degree and fifth-degree B-splines. The obtained eigenvalues associated with Kernel C(s,t) are displayed in Table 4.1, based on the method in section 3. Included in these tables are the associated results due to Castro et al.(1986).It is observed from Table 4.1 that, the method of present paper gives better results than the method of Castro et al.(1986).

Table 4.1. Eigenvalues and process variability

Computed results					
Knot number	**Observed number**	Spline order	Eigenvalues λ_1, λ_2		Process variability
12	30	3	0.3558	0.00188	99%
12	30	5	0.3588	0.00191	99%
20	30	3	0.9369	0.0024	99%
20	30	5	0.9377	0.0024	99%

Castro et al. (1986)				
Knot number	**Observed number**	Eigenvalues λ_1, λ_2		Process variability
17	50	0.365	0.244	60%
20	50	0.929	0.096	90%

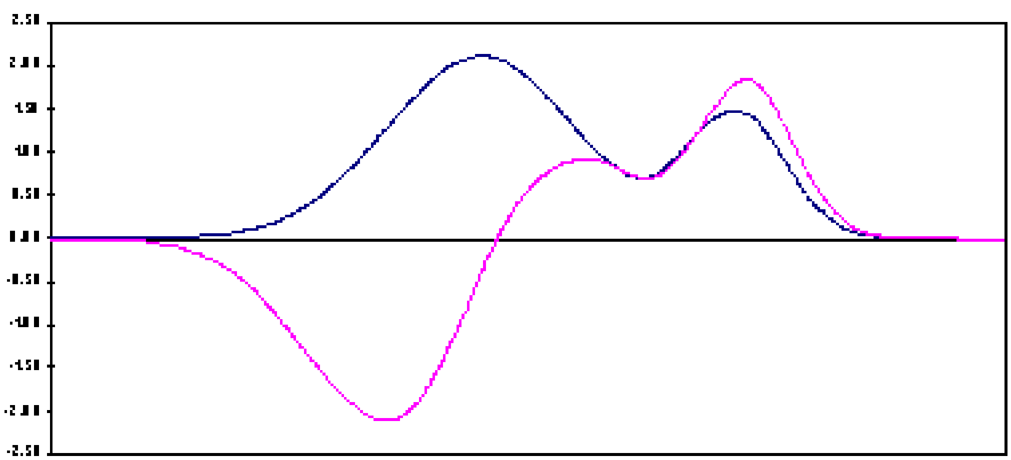

Figure 4.2. First eigenfunction (solid curve) and second eigenfunction (dotted curve) by third-degree B-splines.

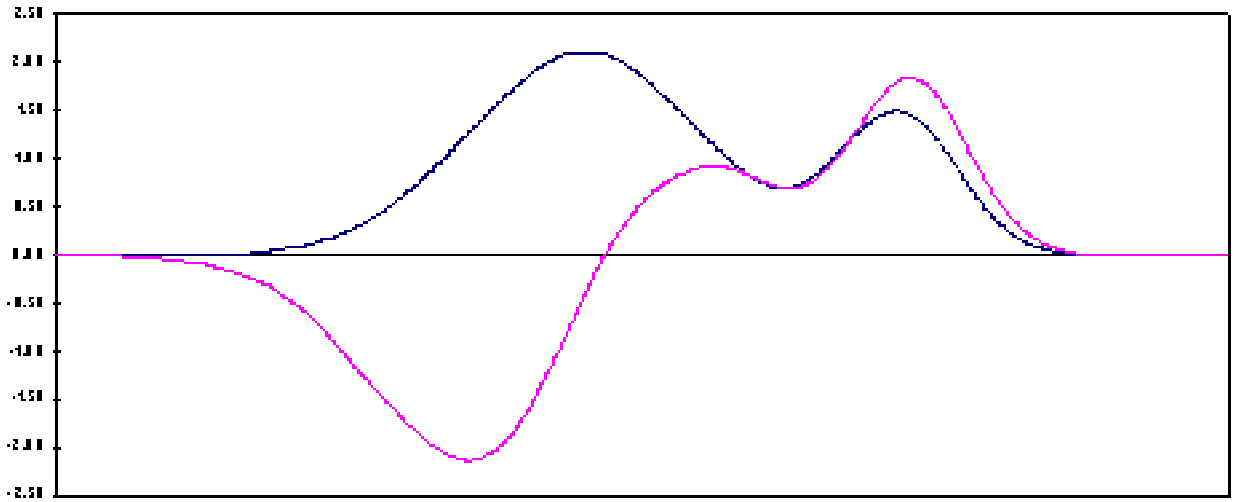

Figure 4.3. First eigenfunction (solid curve) and second eigenfunction (dotted curve) by fifth-degree B-splines.

Refering to Table 4.1, it is seen that, in each case for a one-dimensional model, about 99% of the process variability is explained. The method gave the best results for knot number greater than 12. It shown that the PCA technique using B-spline functions in the continuous sample curves gives the best results. The compute the PCA of high B-spline interpolation of a process a computational algorithm coded in Qbasic has been developed.

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NEW CLASS OF NONLINEAR A-MONOTONE MIXED VARIATIONAL INCLUSION PROBLEMS AND RESOLVENT OPERATOR TECHNIQUE

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1. Introduction and Preliminaries

Just recently, the author [8] announced a new class of mappings - A-monotone mappings - which have a wide range of applications limited not only to variational inclusion problems, but also extend to applications to hemivariational inclusion problems, especially from the fields of engineering science and mechanics On one hand, the class of A-monotone mappings generalizes the well-known class of maximal monotone mappings. Here we first consider the approximation solvability of nonlinear variational inclusion problems involving A- monotone mappings and relaxed cocoercive mappings - a more general notion than the cocoercivity and strong monotonicity - in a Hilbert space setting, and then we consider convergence analysis based on the generalized resolvent operator technique for the solutions. We note that the $A-$*monotonicity* is defined in terms of relaxed monotone mappings - a more general notion than the monotonicity/strong monotonicity. For additional details on inclusion problems and related topics, we recommend [1- 10].

Let X be a real reflexive Banach space and X^* be its dual. Let $T : X \to 2^{X^*}$ be a nonlinear mapping from X into the power set 2^{X^*} of X^* and let $[u^*, u]$ denote the duality pairing between the elements of X^* and X. Next, let the effective domain $D(T)$ of T be defined by

$$D(T) = \{u \in X : T(u) \neq \emptyset\},$$

and the range $R(T)$ of T be defined by

$$R(T) = \bigcup_{u \in X} T(u).$$

The mapping T is maximal monotone if
(i) T is monotone.
(ii) For each $u \in X$ and $u^* \in X^*$,

$$[u^* - v^*, u - v] \geq 0 \,\forall\, v \in D(T), v^* \in T(v),$$

implies $u^* \in T(u)$. The mapping T is said to be pseudomonotone if $\{u_n\}$ is a sequence of X such that $\{u_n\} \rightharpoonup u$ and if $\{u_n^*\} \in T(u_n)$ with

$$\lim_{n\to\infty} sup[u_n^*, u_n - u] \leq 0,$$

imply that for each element $v \in X$ there exists an $u^*(v) \in T(u)$ such that

$$\lim_{n\to\infty} inf[u_n^*, u_n - v] \geq [u^*(v), u - v]\,\forall v \in X.$$

On the top of that, the mapping T is said to be generalized pseudomonotone if $\{u_n\}$ is a sequence of X such that $\{u_n\} \rightharpoonup u$ and if $\{u_n^*\} \in T(u_n)$ with

$$\lim_{n\to\infty} sup[u_n^*, u_n - u] \leq 0,$$

imply that the element $u^* \in T(u)$ and

$$[u_n^*, u_n] \to [u^*, u].$$

Following Rockafellar [5], a monotone mapping $M : X \to 2^{X^*}$ from a real reflexive strictly convex Banach space X into its strictly convex dual X^* is maximal monotone iff

$$R(M + J) = X^*,$$

where $J : X \to X^*$ is the duality mapping on X. This is equivalent to stating that M is $J - monotone$ iff M is monotone and

$$R(M + J) = X^*.$$

This motivated us to extend the notion of the $A-$ monotonicity [8] to a reflexive Banach space setting as follows: a mapping $M : X \to 2^{X^*}$ is $A-$monotone if M is relaxed monotone and $R(M + \lambda A) = X^*$, where $\lambda > 0$, and $A : X \to X^*$ is any nonlinear mapping on X. Also, a variant form of this definition is applied to nonlinear variational inclusion problems in Hilbert space as well as in reflexive Banach space settings.

Definition 1. Let $A : X \to X^*$ and $M : X \to 2^{X^*}$ be any mappings on X. The map M is said to be $A-monotone$ if
$(i)M$ is $(m) - relaxed$ monotone.
$(ii)(A + \rho M)$ is maximal monotone, where $A : X \to X^*$, and $\rho > 0$.

Example 1. [4] Let $A : X \to X^*$ be $(m) - strongly$ monotone and $f : X \to R$ be locally Lipschitz such that ∂f is $(\alpha) - relaxed$ monotone. Then ∂f is $A-monotone$, that is, $A + \partial f$ is maximal monotone for $m - \alpha > 0$, where $m, \alpha > 0$. Clearly, $A + \partial f$ is $(m - \alpha) - strongly$ monotone for $m - \alpha > 0$, that is,

$$[u^* - v^*, u - v] \geq (m - \alpha)\|u - v]\|^2\,\forall u, v \in X,$$

where $u^* \in A(u) + \partial f(u), v^* \in A(v) + \partial f(v)$ and $m - \alpha > 0$. As a matter of fact, $A + \partial f$ is pseudomonotone and hence under the assumptions it is maximal monotone. If $\{u_n\}$ is a sequence of X such that $\{u_n\} \rightharpoonup u$ and if $\{u_n^*\} \in A(u_n) + \partial f(u_n)$ such that

$$\lim_{n\to\infty} sup[u_n^*, u_n - u] \leq 0,$$

then for each element $v \in X$ there exists an $u^*(v) \in A(u) + \partial f(u)$ such that

$$\lim_{n\to\infty} inf[u_n^*, u_n - v] \geq [u^*(v), u - v] \, \forall v \in X.$$

It follows from above inequalities that $\{u_n\} \to u$. Furthermore, the other conditions are fulfilled from the upper semicontinuity of ∂f.

In what follows, H shall denote a real Hilbert space with the norm $\|x\|$ and inner product $\langle x, y\rangle$ for all $x, y \in H$. Let K be a closed convex subset of H.

Lemma 1. Let $A : H \to H$ be $(r) - strongly$ monotone on a real Hilbert space H and $M : H \to 2^H$ be A-monotone. Then the resolvent operator $J_{A,M}^{\rho} := (A + \rho M)^{-1} : H \to H$ is $(\frac{1}{r-\rho m}) - Lipschitz$ continuous for $0 < \rho < \frac{r}{m}$.

Definition 2. A mapping $T : H \to H$ is said to be $(m) - relaxed$ monotone if there exists a positive constant m such that

$$\langle T(x) - T(y), x - y\rangle \geq -m\|x - y\|^2 \forall x, y \in H.$$

Definition 3. A mapping $T : H \to H$ is said to be $(s) - cocoercive$ if there exists a positive constant s such that

$$\langle T(x) - T(y), x - y\rangle \geq s\|T(x) - T(y)\|^2 \, \forall x, y \in H.$$

Definition 4. A mapping $T : H \to H$ is said to be $(m) - relaxed$ cocoercive if there exists a positive constant m such that

$$\langle T(x) - T(y), x - y\rangle \geq -m\|T(x) - T(y)\|^2 \, \forall x, y \in H.$$

Definition 5. A mapping $T : H \to H$ is said to be $(\gamma, m) - relaxed$ cocoercive if there exist positive constants γ, m such that

$$\langle T(x) - T(y), x - y\rangle \geq -m\|T(x) - T(y)\|^2 + \gamma\|x - y\|^2 \, \forall x, y \in H.$$

Example 2. Consider a mapping $T : H \to H$, which is nonexpansive. If we set $A = I - T$, then A is $(\frac{1}{2}) - cocoercive$. For all $u, v \in H$, we have

$$\begin{aligned}
\frac{1}{2}\|A(u) - A(v)\|^2 &= \frac{1}{2}\|u - v\|^2 + \frac{1}{2}\|T(u) - T(v)\|^2 \\
&- \langle u - v, T(u) - T(v)\rangle \\
&\leq \|u - v\|^2 - \langle u - v, T(u) - T(v)\rangle \\
&= \langle u - v, A(u) - A(v)\rangle.
\end{aligned}$$

Clearly, the cocoercivity implies the relaxed cocoercivity, while the converse may not hold in general.

Definition 6. A Hausdorff pseudometric $H^{\wedge} : 2^H \times 2^H \to [0, +\infty) \cup \{+\infty\}$ is defined by

$$H^{\wedge}(A, B) = max\{sup_{u\in A} inf_{v\in B}\|u - v\|, sup_{u\in B} inf_{v\in A}\|u - v\|\} \forall \in A, B \in 2^H.$$

We note that if the domain of $H^\wedge$ is closed bounded subsets, then $H^\wedge$ is the Hausdorff metric.

Definition 7. A mapping $U : H \to 2^H$ is $H^\wedge - (\lambda) - Lipschitz$ continuous if there exists a constant $\lambda > 0$ such that

$$H^\wedge(U(u), U(v)) \leq \lambda\|u - v\| \, \forall u, v \in H.$$

Lemma 2. Let $A : H \to H$ be $(r) - strongly$ monotone on a real Hilbert space H and $M : H \to 2^H$ be A-monotone. Then the resolvent operator $J^\rho_{A,M} := (A + \rho M)^{-1} : H \to H$ is $(r - \rho m) - cocoercive$ for $0 < \rho < \frac{r}{m}$.

Proof. For any $u, v \in H$, it follows from the definition of the resolvent operator that

$$J^\rho_{A,M}(u) = (A + \rho M)^{-1}(u),$$

$$J^\rho_{A,M}(v) = (A + \rho M)^{-1}(v).$$

This further implies that

$$\frac{1}{\rho}[u - A(J^\rho_{A,M}(u))] \in M(J^\rho_{A,M}(u)),$$

$$\frac{1}{\rho}[v - A(J^\rho_{A,M}(v))] \in M(J^\rho_{A,M}(v)).$$

Since M is $A-monotone$ (and hence , it is $(m) - relaxed$ monotone), we have

$$\begin{aligned} & \frac{1}{\rho}\langle u - v - [A(J^\rho_{A,M}(u) - A(J^\rho_{A,M}(v)], J^\rho_{A,M}(u) - J^\rho_{A,M}(v)\rangle \\ \geq \quad & -m\|J^\rho_{A,M}(u) - J^\rho_{A,M}(v)\|^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \langle u - v, J^\rho_{A,M}(u) - J^\rho_{A,M}(v)\rangle \\ \geq \quad & \langle J^\rho_{A,M}(u) - J^\rho_{A,M}(v), A(J^\rho_{A,M}(u)) - A(J^\rho_{A,M}(v))\rangle \\ - \quad & \rho m\|J^\rho_{A,M}(u) - J^\rho_{A,M}(v)\|^2 \\ \geq \quad & r\|J^\rho_{A,M}(u) - J^\rho_{A,M}(v)\|^2 - \rho m\|J^\rho_{A,M}(u) - J^\rho_{A,M}(v)\|^2 \\ = \quad & (r - \rho m)\|J^\rho_{A,M}(u) - J^\rho_{A,M}(v)\|^2. \end{aligned}$$

This completes the proof.

2. Algorithms and Variational Inclusions

Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $A : H \to H$ and $M : H \to 2^H$ be two nonlinear mappings. Let $S : H \times H \to H$ and $U : H \to 2^H$ be any mappings. Then the problem of finding an element $a \in H$, $u \in U(a)$ such that

$$0 \in S(a, u) + M(a) \tag{1}$$

is called a class of nonlinear variational inclusion (abbreviated CNVI) problems.
Next, a special case of the $CNVI\,(1)$ problem is: determine an element $a \in H$ such that

$$0 \in S(a,a) + M(a). \tag{2}$$

When $M(x) = \partial_K(x)$ for all $x \in K$, where K is a nonempty closed convex subset of H and ∂_K denotes the indicator function of K, the $CNVI(1)$ problem reduces to: determine an element $a \in K$ such that

$$\langle S(a,a), x-a\rangle \geq 0 \text{ for all } x \in K, \tag{3}$$

Lemma 3. Let H be a Hilbert space. Let $A : H \to H$ be strictly monotone, $M : H \to 2^H$ be $A-monotone$. Let $S : H \times H \to H$ be any mapping. Then a given element $a \in H$ such that $u \in U(a)$ is a solution to the $CNVI(1)$ problem iff a and u satisfy

$$a = J^{\rho}_{A,M}(A(a) - \rho S(a,u)), \tag{4}$$

where $U : H \to 2^H$ is a multivalued mapping on H.

Algorithm 1.
Step 1. Choose $a^0 \in H$ and $u^0 \in U(a^0)$ such that

$$a^{k+1} = J^{\rho}_{A,M}[A(a^k) - \rho S(a^k,u^k)].$$

Step 2. For an $a^{k+1} \in H$, choose an $u^{k+1} \in U(a^{k+})$ such that

$$\|u^{k+1} - u^k\| \leq (1 + \frac{1}{1+k})H^{\wedge}(U(a^{k+1}), U(a^k)),$$

where $H^{\wedge}(.,.)$ denotes the Hausdorff pseudometric on 2^H. Step 3. If the sequences $\{a^k\}$ and $\{u^k\}$ satisfy to a sufficient degree of accuracy

$$a^{k+1} = J^{\rho}_{A,M}[A(a^k) - \rho S(a^k,u^k)],$$

stop. Otherwise, set $k = k+1$ and return to *Step* 1.

Algorithm 2.
Step 1. Choose $a^0 \in H$ such that

$$a^{k+1} = J^{\rho}_{A,M}[A(a^k) - \rho S(a^k,a^k)].$$

Step 2. If the sequence $\{a^k\}$ satisfies to a sufficient degree of accuracy

$$a^{k+1} = J^{\rho}_{A,M}[A(a^k) - \rho S(a^k,a^k)],$$

stop. Otherwise, set $k = k+1$ and return to *Step* 1.

Theorem 1. Let H be a real Hilbert space. Let $A : H \to H$ be $(r) - strongly$ monotone and $(\alpha) - Lipschitz$ continuous, and let $M : H \to 2^H$ be $A-monotone$. Let $S : H \times H \to H$ be such that $S(.,y)$ is $(\lambda, s) - relaxed$ cocoercive and $(\beta) - Lipschitz$

continuous in the first variable and let $S(x,.)$ be $(\tau)-Lipschitz$ continuous in the second variable for all $(x,y) \in H \times H$. Suppose that $U : H \to C(H)$ is $(p, H^\wedge) - Lipschitz$ continuous, where $C(H)$ denotes the collection of all closed subsets of H. If, in addition, the $CNVI(1)$ admits a solution (a,u), if sequences $\{a^k\}$ and $\{u^k\}$ are generated by $Algorithm\,1$ and if there exists a positive constant ρ such that

$$\sqrt{\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2} + \rho\tau p < (r - \rho m),$$

then sequences $\{a^k\}$ and $\{u^k\}$ converge to a and u, respectively.

Corollary 1. Let H be a real Hilbert space. Let $A : H \to H$ be $(r) - strongly$ monotone and $(\alpha) - Lipschitz$ continuous, and let $M : H \to 2^H$ be $A-monotone$. Let $S : H \times H \to H$ be such that $S(.,y)$ is $(s) - strongly$ monotone and $(\beta) - Lipschitz$ continuous in the first variable and let $S(x,.)$ be $(\tau)-Lipschitz$ continuous in the second variable for all $(x,y) \in H \times H$. Let $a, u \in H$ form a solution to the $CNVI(1)$ problem. Suppose that $U : H \to C(H)$ is $(p, H^\wedge) - Lipschitz$ continuous, where $C(H)$ denotes the collection of all closed subsets of H. If, in addition, sequences $\{a^k\}$ and $\{u^k\}$ are generated by $Algorithm\,1$ and if there exists a positive constant ρ such that

$$\sqrt{\alpha^2 - 2\rho s + \rho^2\mu^2} + \rho\tau p < r - \rho m,$$

then sequences $\{a^k\}$ and $\{u^k\}$ converge, respectively, to a and u,
which form a solution to the $CNVI\,(1)$ problem.

Proof of Theorem 1. Using $Algorithm\,1$ and $Lemma\,1$, we obtain

$$\begin{aligned}
\|a^{k+1} - a^k\| &= \|J^A_{M,\rho}[A(a^k) - \rho S(a^k, u^k)] \\
&- J^A_{M,\rho}[A(a^{k-1}) - \rho S(a^{k-1}, u^{k-1})]\| \\
&\leq \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, u^k) - S(a^{k-1}, u^{k-1}))\|] \\
&\leq \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, u^k] - S(a^{k-1}, u^k))\| \\
&+ \|\rho(S(a^{k-1}, u^k) - S(a^{k-1}, u^{k-1}))\|] \\
&\leq \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, u^k) - S(a^{k-1}, u^k))\| \\
&+ \rho\tau\|u^k - u^{k-1}\|] \\
&\leq \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, u^k) - S(a^{k-1}, u^k))\| \\
&+ \rho\tau p(1 + \frac{1}{k})\|a^k - a^{k-1}\|]
\end{aligned}$$

Since

$$\begin{aligned}
& \|[A(a^k) - A(a^{k-1}) - \rho(S(a^k, u^k] - S(a^{k-1}, u^k))\|^2 \\
= \;& \|A(a^k) - A(a^{k-1})\|^2 - 2\rho\langle A(a^k) - A(a^{k-1}), S(a^k, u^k) - S(a^{k-1}, u^k)\rangle \\
+ \;& \rho^2\|S(a^k, u^k) - S(a^{k-1}, u^k)\|^2 \\
\leq \;& (\alpha^2 + 2\rho\lambda\beta^2 - 2\rho s + \rho^2\beta^2)\|a^k - a^{k-1}\|^2 \\
= \;& (\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2)\|a^k - a^{k-1}\|^2,
\end{aligned}$$

we have

$$\begin{aligned}
& \|a^{k+1} - a^{k-1}\| \\
\leq \;& \frac{1}{r - \rho m}[\theta\|a^k - a^{k-1}\| + \rho\tau p(1 + \frac{1}{k})\|a^k - a^{k-1}\|] \\
= \;& [\frac{1}{r - \rho m}(\theta + \rho\tau p(1 + \frac{1}{k}))]\|a^k - a^{k-1}\|,
\end{aligned}$$

where

$$\theta = \sqrt{\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2}$$

and $\theta + \rho\tau p < r - \rho m$, that is,

$$\sqrt{\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2} + \rho\tau p < r - \rho m.$$

Under the assumptions of the theorem, it follows from the above inequality that $\{a^k\}$ is a Cauchy sequence. As a result, there exists an $a \in H$ such that the sequence $\{a^k\}$ converges to a as $k \to \infty$.

To conclude the proof, we show that the sequence $\{u^k\}$ converges to $u \in U(a)$. Since

$$\begin{aligned}
\|S(a^{k-1}, u^k) - S(a^{k-1}, u^{k-1})\| &\leq \tau\|u^k - u^{k-1}\| \\
&\leq \tau p(1 + \frac{1}{k}))\|a^k - a^{k-1}\|,
\end{aligned}$$

it follows that $\{u^k\}$ is a Cauchy sequence. Thus, there exists an $u \in H$ such that $\{u^k\} \to u \, as \, k \to \infty$.
Next, we show $u \in U(a)$. Since $U(a)$ is closed and

$$\begin{aligned}
& d(u, U(a)) = inf\{\|u - v\| : v \in U(a)\} \\
\leq \;& \|u - u^k\| + d(u^k, U(a)) \\
\leq \;& \|u - u^k\| + H^\wedge(U(a^k), U(a)) \\
\leq \;& \|u - u^k\| + p\|a^k - a\| \to 0,
\end{aligned}$$

it implies that $u \in U(a)$. As a matter of fact, the continuity ensures that a and u satisfy

$$a = J^{\rho}_{A,M}(A(a) - \rho S(a,u)).$$

Finally, it follows from *Lemma* 3 that (a,u) is a solution to the $CNVI\,(1)$ problem. This concludes the proof.

Theorem 2. Let H be a real Hilbert space. Let $A : H \to H$ be $(r) - strongly$ monotone and $(\alpha) - Lipschitz$ continuous, and let $M : H \to 2^H$ be $A-monotone$. Let $S : H \times H \to H$ be such that $S(.,y)$ is $(\lambda, s) - relaxed$ cocoercive and $(\beta) - Lipschitz$ continuous in the first variable and let $S(x,.)$ be $(\tau) - Lipschitz$ continuous in the second variable for all $(x,y) \in H \times H$. If, in addition, the $CNVI(2)$ admits a solution $a \in H$, if the sequence $\{a^k\}$ is generated by *Algorithm* 2 and if there exists a positive constant ρ such that

$$\sqrt{\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2} + \rho\tau p < (r - \rho m),$$

then the sequence $\{a^k\}$ converge to a.

Proof. Using *Algorithm* 1 and *Lemma* 1, we obtain

$$\begin{aligned}
& \|a^{k+1} - a^k\| = \|J^A_{M,\rho}[A(a^k) - \rho S(a^k, a^k)] \\
- \;& J^A_{M,\rho}[A(a^{k-1}) - \rho S(a^{k-1}, a^{k-1})]\| \\
\le \;& \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, a^k) - S(a^{k-1}, a^{k-1}))\|] \\
\le \;& \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, a^k] - S(a^{k-1}, a^k))\| \\
+ \;& \|\rho(S(a^{k-1}, a^k) - S(a^{k-1}, a^{k-1}))\|] \\
\le \;& \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, a^k) - S(a^{k-1}, a^k))\| \\
+ \;& \rho\tau\|a^k - a^{k-1}\|] \\
\le \;& \frac{1}{r - \rho m}[\|A(a^k) - A(a^{k-1}) - \rho(S(a^k, a^k) - S(a^{k-1}, a^k))\| \\
+ \;& \rho\tau p(1 + \frac{1}{k})\|a^k - a^{k-1}\|]
\end{aligned}$$

Since

$$\begin{aligned}
& \|[A(a^k) - A(a^{k-1}) - \rho(S(a^k, a^k] - S(a^{k-1}, a^k))\|^2 \\
= \;& \|A(a^k) - A(a^{k-1})\|^2 \\
- \;& 2\rho\langle A(a^k) - A(a^{k-1}), S(a^k, a^k) - S(a^{k-1}, a^k)\rangle \\
+ \;& \rho^2\|S(a^k, a^k) - S(a^{k-1}, a^k)\|^2 \\
\le \;& (\alpha^2 + 2\rho\lambda\beta^2 - 2\rho s + \rho^2\beta^2)\|a^k - a^{k-1}\|^2 \\
= \;& (\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2)\|a^k - a^{k-1}\|^2,
\end{aligned}$$

we have

$$\|a^{k+1} - a^{k-1}\| \leq \frac{1}{r - \rho m}[\theta\|a^k - a^{k-1}\| + \rho\tau p(1 + \frac{1}{k})\|a^k - a^{k-1}\|]$$
$$= [\frac{1}{r - \rho m}(\theta + \rho\tau p(1 + \frac{1}{k}))]\|a^k - a^{k-1}\|,$$

where

$$\theta = \sqrt{\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2}$$

and $\theta + \rho\tau p < r - \rho m$, that is,

$$\sqrt{\alpha^2 - 2\rho s + \rho^2\beta^2 + 2\rho\lambda\beta^2} + \rho\tau p < r - \rho m.$$

Under the assumptions of the theorem, it follows from the above inequality that $\{a^k\}$ is a Cauchy sequence. As a result, there exists an $a \in H$ such that the sequence $\{a^k\}$ converges to a as $k \to \infty$.

We remark that the obtained results can be extended to the case of a system of nonlinear variational inclusion problems on two Hilbert spaces. Let H_1 and H_2 be two real Hilbert spaces and K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $A : H_1 \to H_1$, $B : H_2 \to H_2$, $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ be nonlinear mappings. Let $S : H_1 \times H_2 \to H_1$ and $T : H_1 \times H_2 \to H_2$ be any two multivalued mappings. Then the problem of finding $(a, b) \in H_1 \times H_2$ such that

$$0 \in S(a, b) + M(a), \tag{5}$$

$$0 \in T(a, b) + N(b), \tag{6}$$

is called the system of nonlinear variational inclusion (abbreviated SNVI) problems.

When $M(x) = \partial_{K_1}(x)$ and $N(y) = \partial_{K_2}(y)$ for all $x \in K_1$ and $y \in K_2$, where K_1 and K_2, respectively, are nonempty closed convex subsets of H_1 and H_2, and ∂_{K_1} and ∂_{K_2} denote indicator functions of K_1 and K_2, respectively, the $SNVI(1) - (2)$ reduces to: determine an element $(a, b) \in K_1 \times K_2$ such that

$$\langle S(a, b), x - a\rangle \geq 0 \text{ for all } x \in K_1, \tag{7}$$

$$\langle T(a, b), y - b\rangle \geq 0 \text{ for all } y \in K_2. \tag{8}$$

Lemma 4. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_1$ and $B : H_2 \to H_2$ be strictly monotone, $M : H_1 \to 2^{H_1}$ be $A-$*monotone* and $N : H_2 \to 2^{H_2}$ be $B-$*monotone*. Let $S : H_1 \times H_2 \to H_1$ and $T : H_1 \times H_2 \to H_2$ be any two multivalued mappings. Then a given element $(a, b) \in H_1 \times H_2$ is a solution to the $SNVI(1) - (2)$ problem iff (a, b) satisfies

$$a = J^{\rho}_{A,M}(A(a) - \rho S(a, b)), \tag{9}$$

$$b = J^{\eta}_{B,N}(B(b) - \eta T(a, b)). \tag{10}$$

Algorithm 3. Choose $(a^0, b^0) \in H_1 \times H_2$, such that

$$a^{k+1} = J^{\rho}_{A,M}[A(a^k) - \rho S(a^k, b^k)]$$

$$b^{k+1} = J^{\eta}_{B,N}[B(a^k) - \rho T(a^k, b^k)].$$

Theorem 3. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_1$ be $(r_1) - strongly$ monotone and $(\alpha_1) - Lipschitz$ continuous, and $B : H_2 \to H_2$ be $(r_2) - strongly$ monotone and $(\alpha_2) - Lipschitz$ continuous. Let $M : H_1 \to 2^{H_1}$ be $A-monotone$ and $N : H_2 \to 2^{H_2}$ be $B-monotone$. Let $S : H_1 \times H_2 \to H_1$ be such that $S(., y)$ is $(\gamma, r) - relaxed$ cocoercive and $(\mu) - Lipschitz$ continuous in the first variable and $S(x, .)$ is $(\nu) - Lipschitz$ continuous in the second variable for all $(x, y) \in H_1 \times H_2$. Let $T : H_1 \times H_2 \to H_2$ be such that $T(u, .)$ is $(\lambda, s) - relaxed$ cocoercive and $(\beta) - Lipschitz$ continuous in the second variable and $T(., v)$ is $(\tau) - Lipschitz$ continuous in the first variable for all $(u, v) \in H_1 \times H_2$. Let $(a, b) \in H_1 \times H_2$ form a solution to the $SNVI$ (4)$-$(5) problem. If, in addition, there exist positive constants ρ, η and sequence $\{(a^k, b^k)\}$ is generated by *Algorithm* 3 such that

$$(r_2 - \eta p)\sqrt{\alpha_1^2 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2} + \eta\tau(r_1 - \rho m) < (r_1 - \rho m)(r_2 - \eta p)$$

$$(r_1 - \rho m)\sqrt{\alpha_2^2 - 2\eta s + 2\eta\lambda\beta^2 + \eta^2\beta^2} + \rho\nu(r_2 - \eta p) < (r_1 - \rho m)(r_2 - \eta p),$$

then the $SNVI$(4)$-$(5) problem has a solution (a, b), where M is $(m)-relaxed$ monotone and N is $(p) - relaxed$ monotone

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Existence and Asymptotic Stability for Viscoelastic Evolution Problems on Compact Manifolds, Part II

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Abstract. The present paper makes a further study on the existence and stabilization in a earlier article (J. Concr. Appl. Math). One considers the nonlinear viscoelastic evolution equation

$$u_{tt} + Au + F(x,t,u,u_t) - g * A\,u = 0 \quad \text{on } \Gamma \times (0,\infty)$$

where Γ is a compact manifold. When $F \neq 0$ and $g \neq 0$ we prove existence of global solutions as well as uniform (exponential and algebraic) decay rates, provided the kernel of the memory decays exponentially and F satisfies suitable growth assumptions.

Key words: Asymptotic Stability, Viscoelastic Evolution Problem
2000 AMS Subject Classification 35G25,37C75

1 Introduction

This manuscript is devoted to the study of the existence and uniform decay rates of solutions $u = u(x,t)$ of the evolution viscoelastic problem

$$(*) \quad \begin{cases} u_{tt} + Au + F(x,t,u,u_t) - \displaystyle\int_0^t g(t-\tau)Au(\tau)\,d\tau = 0 & \text{on } \Gamma \times (0,\infty) \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in \Gamma \end{cases}$$

where Γ is the boundary, assumed compact and smooth, of a domain Ω of $\mathbf{R}^n$, not necessarily bounded.

We will assume that A is the self-adjoint operator, not necessarily bounded, defined by the triple $\left\{H^{1/2}(\Gamma), L^2(\Gamma), ((\cdot,\cdot))_{H^{1/2}(\Gamma)}\right\}$. In this case, A is characterized by

$$\begin{aligned} D(A) \;=\; & \Big\{u \in H^{1/2}(\Gamma);\ \text{there exists } f_u \in L^2(\Gamma) \text{ such that} \\ & (f_u, v)_{L^2(\Gamma)} = ((u,v))_{H^{1/2}(\Gamma)}; \quad \text{for all } v \in H^{1/2}(\Gamma)\Big\}, \quad f_u = Au \end{aligned}$$

$$(Au, v)_{L^2(\Gamma)} = ((u, v))_{H^{1/2}(\Gamma)}\,; \quad \text{for all } u \in D(A) \text{ and for all } v \in H^{1/2}(\Gamma). \quad (1.1)$$

Since the embedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact, we recall that the spectral theorem for self-adjoint operators guarantees the existence of a complete orthonormal system $\{\omega_\nu\}_{\nu\in\mathbf{N}}$ of $L^2(\Gamma)$ given by eigen-functions of A. If $\{\lambda_\nu\}_{\nu\in\mathbf{N}}$ are the corresponding eigenvalues of A, then $\lambda_\nu \to +\infty$ as $\nu \to +\infty$. Besides,

$$D(A) = \{u \in L^2(\Gamma); \sum_{\nu=1}^{+\infty} \lambda_\nu^2 |\,(u, \omega_\nu)_{L^2(\Gamma)}\,|^2 < +\infty\},$$

$$Au = \sum_{\nu=1}^{+\infty} \lambda_\nu\, (u, \omega_\nu)_{L^2(\Gamma)}\, \omega_\nu; \quad \text{for all } u \in D(A).$$

Considering in $D(A)$ the norm $|Au|_{L^2(\Gamma)}$, it turns out that $\{\omega_\nu\}$ is a complete system in $D(A)$. In fact, it is known that $\{\omega_\nu\}$ is also a complete system in $H^{1/2}(\Gamma)$. Now, since A is positive, given $\delta > 0$ one has

$$D(A^\delta) = \{u \in L^2(\Gamma); \sum_{\nu=1}^{+\infty} \lambda_\nu^{2\delta} |\,(u, \omega_\nu)_{L^2(\Gamma)}\,|^2 < +\infty\},$$

$$A^\delta u = \sum_{\nu=1}^{+\infty} \lambda_\nu^\delta\, (u, \omega_\nu)_{L^2(\Gamma)}\, \omega_\nu; \quad \text{for all } u \in D(A^\delta).$$

In $D(A^\delta)$ we consider the topology given by $\left|A^\delta u\right|_{L^2(\Gamma)}$. We observe that from the spectral theory, such operators are also self-adjoint, that is,

$$(A^\delta u, v)_{L^2(\Gamma)} = (u, A^\delta v)_{L^2(\Gamma)}; \quad \text{for all } u, v \in D(A^\delta)$$

and, in particular,

$$D(A^{1/2}) = H^{1/2}(\Gamma). \quad (1.2)$$

At this point it is convenient to observe that, according to J. L. Lions and E. Magenes [[11], Remark 7.5] one has

$$H^{1/2}(\Gamma) = D[(-\Delta_\Gamma)^{1/2}], \quad (1.3)$$

where Δ_Γ is the Laplace-Beltrami operator on Γ. Then, from (1.1), (1.2) and (1.3) we deduce that

$$(Au, v)_{L^2(\Gamma)} = (-\Delta_\Gamma u, v)_{L^2(\Gamma)}\,; \quad \text{for all } u \in D(A), \text{ for all } v \in H^{1/2}(\Gamma), \quad (1.4)$$

that is, $Au = -\Delta_\Gamma u$ for all $u \in D(A)$ which implies that $A \leq -\Delta_\Gamma$. This means that when A is the operator defined by the above triple, problem $(*)$ can also be viewed like the wave operator on the compact manifold Γ.

Now, if one considers the extension $\tilde{A} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ of A defined by

$$< \tilde{A}u, v >_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = ((u, v))_{H^{1/2}(\Gamma)}\,; \quad \text{for all } u, v \in H^{1/2}(\Gamma) \quad (1.5)$$

it is well known that $\tilde{A}$ is bijective, self-adjoint, coercive and continuous (indeed isometry).

In the present manuscript we derive exponential and algebraic decay rates (as in the earlier article) assuming that the kernel decays exponentially, and, moreover, that $\int_0^\infty g(s)\,ds$ is sufficiently small. For this purpose we make use of the perturbed energy method due to A. Haraux and E. Zuazua [7].

It is worth mentioning the papers in connection with viscoelastic effects on the boundary Γ of a domain Ω of $\mathbf{R}^n$. This was considered by M. Aassila, M. M. Cavalcanti and J. Soriano [4] whom considered the linear wave equation in Ω subject to nonlinear feedback and viscoelastic effects on the boundary and proved uniform (exponential and algebraic) decay rates. Also, we can cite the article of D. Andrade and J. E. Muñoz Rivera [2] whom considered a one-dimension nonlinear wave equation in $\Omega = (0,1)$ subject to nonlocal and nonlinear boundary memory effect. They showed that the dissipation introduced by the memory term is strong enough to secure global estimates, which allow them to prove existence of global smooth solution for small data and to derive exponential (or polynomial) decay provided the kernel decays exponentially (or polynomially).

A natural question in this context is about the non-existence results for the nonlinear wave equation in Ω when we have viscoelastic effects on the boundary. In this context we can mention the work of M. Kirane and N. Tatar [9] who derive non-existence results.

Our paper is organized as follows: In section 2 we present some notations, the assumptions on g and F and state our main result. In section 3 we prove existence and uniqueness for regular and weak solutions and in section 4 we give the proof of the uniform decay.

2 Assumptions and Main Result

Define

$$(u,v) = \int_\Gamma u(x)v(x)\,dx; \quad |u|^2 = (u,u)\,, \quad ||u||_p^p = \int_\Gamma |u(x)|^p\,dx.$$

The precise assumptions on the function $F(x,t,u,u_t)$ and on the memory term g of $(*)$ are given in the sequel.

(A.1) Assumptions on $F(x,t,u,u_t)$

We represent by (x,t,ξ,η) a point of $\Gamma \times [0,\infty) \times \mathbf{R}^2$. Let

$$F : \Gamma \times [0,\infty) \times \mathbf{R}^2 \to \mathbf{R}$$

satisfying the conditions

$$F \in C^1\left(\Gamma \times [0,\infty) \times \mathbf{R}^2\right). \tag{$H.1$}$$

There exist positive constants C, D and $\beta > 0$ such that

$$|F(x,t,\xi,\eta)| \le C\left(1 + |\xi|^{\gamma+1} + |\eta|^{\rho+1}\right), \tag{$H.2$}$$

where $0 < \xi, \rho \leq \frac{1}{n-2}$ if $n \geq 3$ and $\xi, \rho > 0$ if $n = 1, 2$;

$$F(x,t,\xi,\eta)\zeta \geq |\xi|^{\gamma}\xi\zeta + \beta|\eta|^{\rho+1}|\zeta|; \quad \text{for all } \zeta \in \mathbf{R}; \tag{$H.3$}$$

$$|F_t(x,t,\xi,\eta)| \leq C\left(1 + |\eta|^{\rho+1} + |\xi|^{\gamma+1}\right); \tag{$H.4$}$$

$$|F_\xi(x,t,\xi,\eta)| \leq C\left(1 + |\eta|^{\rho} + |\xi|^{\gamma}\right); \tag{$H.5$}$$

$$F_\eta(x,t,\xi,\eta) \geq \beta|\eta|^{\rho}; \tag{$H.6$}$$

$$\left(F(x,t,\xi,\eta) - F(x,t,\hat{\xi},\hat{\eta})\right)\left(\zeta - \hat{\zeta}\right) \tag{$H.7$}$$

$$\geq -D\left(|\xi|^{\gamma} + \left|\hat{\xi}\right|\right)\left|\xi - \hat{\xi}\right|\left|\zeta - \hat{\zeta}\right| \quad \text{for all } \zeta, \hat{\zeta} \in \mathbf{R}.$$

A simple variant of the above function is given by the following example

$$F(x,t,\xi,\eta) = \beta|\eta|^{\rho}\eta + |\xi|^{\gamma}\xi.$$

(A.2) Assumptions on the Kernel

We assume that $g : \mathbf{R}_+ \to \mathbf{R}_+$ is a bounded C^2 function satisfying

$$1 - \int_0^\infty g(s)\,ds = l > 0 \tag{$H.8$}$$

and such that there exist positive constants ξ_1, ξ_2 and ξ_3 verifying

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t); \quad \text{for all } t \geq 0, \tag{$H.9$}$$

$$0 \leq g''(t) \leq \xi_3 g(t); \quad \text{for all } t \geq 0, \tag{$H.10$}$$

$$0 \geq g'''(t) \geq \xi_4 g'(t); \quad \text{for all } t \geq 0. \tag{$H.11$}$$

Now we are in a position to state our main result.

Theorem 2.4. *Let the initial data $\{u^0, u^1\}$ belong to $D(A) \times H^{1/2}(\Gamma)$ and assume that the assumptions in (A.1) and (A.2) hold. Then, problem $(*)$ possesses a unique regular solution u in the class*

$$u \in L^\infty(0,\infty; H^{1/2}(\Gamma)),\ u' \in L^\infty(0,\infty; H^{1/2}(\Gamma),\ u'' \in L^\infty(0,\infty; L^2(\Gamma)). \tag{2.6}$$

Moreover, assuming that the kernel $||g||_{L^1(0,\infty)}$ is sufficient small, the energy

$$E(t) = \frac{1}{2}\left\{|u'(t)|^2 + |A^{1/2}u(t)|^2 + \frac{2}{\gamma+2}||u(t)||_{\gamma+2}^{\gamma+2}\right\}, \tag{2.7}$$

has the following decay rates

$$E(t) \leq (\varepsilon\theta t + [E(0)]^{-\rho/2})^{-2/\rho},\ \text{for all } t \geq 0,\ \text{for all } \varepsilon \in (0,\varepsilon_0], (\text{ if } \rho > 0) \tag{2.8}$$

where θ and ε_0 are positive constants, and

$$E(t) \leq CE(0)e^{-\varepsilon\omega t} \text{ for all } , t \geq 0 \text{ for all } \varepsilon \in (0, \varepsilon_0], (\text{ if } \rho = 0), \tag{2.9}$$

where C, ω and ε_0 are positive constants.

Theorem 2.5. *Let the initial data belong to $H^{1/2}(\Gamma) \times L^2(\Gamma)$ and assume the same hypotheses of theorem 2.4 hold. Then, problem $(*)$ possesses a unique weak solution u in the class*

$$u \in C^0([0, \infty), H^{1/2}(\Gamma)) \cap C^1([0, \infty); L^2(\Gamma)). \tag{2.10}$$

Besides, the weak solution has the same decay rates given in (2.8) and (2.9).

3 Existence and Uniqueness of Solutions

In this section we first prove existence and uniqueness of regular solutions to problem $(*)$ making use of Faedo-Galerkin method. Then, we extend the same result to weak solutions using a density argument.

3.1 Regular Solutions

Now, let us consider the existence of regular solutions. For this end, let us consider the operator $A : D(A) \subset L^2(\Gamma) \to L^2(\Gamma)$ defined by the triple $\left\{H^{1/2}(\Gamma), L^2(\Gamma), ((\cdot,\cdot))_{H^{1/2}(\Gamma)}\right\}$. Let $\{\omega_\nu\}$ be a basis in $D(A)$, consider $V_m = [\omega_1, \cdots, \omega_m]$ and $u_m = \sum_{j=1}^{m} \delta_{jm}(t)\omega_j$ verifying

$$\begin{aligned} &(u_m''(t), w) + (Au_m(t), w) + (F(x, t, u_m(t), u_m'(t)), w) \\ &- \int_0^t g(t-\tau)\,(Au_m(\tau), w)\,d\tau = 0; \quad \text{for all } w \in V_m, \end{aligned} \tag{3.1}$$

$$u_m(0) = u_{0m} \to u^0 \text{ in } D(A); \; u_m'(0) = u_{1m} \to u^1 \text{ in } H^{1/2}(\Gamma). \tag{3.2}$$

3.1.4 - A Priori Estimates

The First Estimate: Considering $w = u_m'(t)$ in (3.1), we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{|u_m'(t)|^2 + |A^{1/2}u_m(t)|^2 + \frac{2}{\gamma+2}||u_m(t)||_{\gamma+2}^{\gamma+2}\right\} \\ &+\beta||u_m'(t)||_{\rho+2}^{\rho+2} + g(0)|A^{1/2}u_m(t)|^2 \\ &= \frac{d}{dt}\left\{\int_0^t g(t-\tau)(A^{1/2}u_m(\tau), A^{1/2}u_m(t))\,d\tau\right\} \\ &- \int_0^t g'(t-\tau)(A^{1/2}u_m(\tau), A^{1/2}u_m(t))\,d\tau. \end{aligned} \tag{3.3}$$

We observe that in view of assumption $(H.9)$, making use of Cauchy-Schwarz and $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$ inequalities, we have

$$\begin{aligned}
&\int_0^t g'(t-\tau)(A^{1/2}u_m(\tau), A^{1/2}u_m(t))\, d\tau \qquad (3.4)\\
&\leq \frac{\xi_1^2}{4\eta}\left(\int_0^t g(t-\tau)|A^{1/2}u_m(\tau)|d\tau\right)^2 + \eta|A^{1/2}u_m(t)|^2\\
&\leq \frac{\xi_1^2}{4\eta}\,||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)|A^{1/2}u_m(\tau)|^2 d\tau + \eta|A^{1/2}u_m(t)|^2.
\end{aligned}$$

Combining (3.3) and (3.4) we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{|u_m'(t)|^2 + |A^{1/2}u_m(t)|^2 + \frac{2}{\gamma+2}\,||u_m(t)||_{\gamma+2}^{\gamma+2}\right\} \qquad (3.5)\\
&+\beta\,||u_m'(t)||_{\rho+2}^{\rho+2} + (g(0)-\eta)|A^{1/2}u_m(t)|^2\\
&\leq \frac{d}{dt}\left\{\int_0^t g(t-\tau)(A^{1/2}u_m(\tau), A^{1/2}u_m(t))d\tau\right\}\\
&+\frac{\xi_1^2}{4\eta}\,||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)|A^{1/2}u_m(\tau)|^2 d\tau.
\end{aligned}$$

Integrating (3.5) over (0,t), we obtain

$$\begin{aligned}
&\frac{1}{2}\left\{|u_m'(t)|^2 + |A^{1/2}u_m(t)|^2 + \frac{2}{\gamma+2}\,||u_m(t)||_{\gamma+2}^{\gamma+2}\right\} \qquad (3.6)\\
&+\beta\int_0^t ||u_m'(s)||_{\rho+2}^{\rho+2}\, ds + (g(0)-\eta)\int_0^t |A^{1/2}u_m(s)|^2 ds\\
&\leq \frac{1}{2}\left\{|u_{1m}|^2 + |A^{1/2}u_{0m}|^2 + \frac{2}{\gamma+2}\,||u_{0m}||_{\gamma+2}^{\gamma+2}\right\}\\
&+\int_0^t g(t-\tau)(A^{1/2}u_m(\tau), A^{1/2}u_m(t))d\tau\\
&+\frac{\xi_1^2}{4\eta}\,||g||_{L^1(0,\infty)}^2\int_0^t |A^{1/2}u_m(s)|^2\, ds.
\end{aligned}$$

Finally, we observe that for an arbitrary $\eta > 0$, we infer

$$\begin{aligned}
&\int_0^t g(t-\tau)(A^{1/2}u_m(\tau), A^{1/2}u_m(t))d\tau \qquad (3.7)\\
&\leq \eta|A^{1/2}u_m(t)|^2 + \frac{1}{4\eta}\,||g||_{L^1(0,\infty)}\,||g||_{L^\infty(0,\infty)}\int_0^t |A^{1/2}u_m(s)|^2\, ds.
\end{aligned}$$

From (3.6) and (3.7) we have

$$\frac{1}{2}\left|u_m'(t)\right|^2+(\frac{1}{2}-\eta)|A^{1/2}u_m(t)|^2+\frac{1}{\gamma+2}\left|\left|u_m(t)\right|\right|_{\gamma+2}^{\gamma+2} \tag{3.8}$$
$$+\beta\int_0^t\left|\left|u_m'(s)\right|\right|_{\rho+2}^{\rho+2}ds+(g(0)-\eta)\int_0^t|A^{1/2}u_m(s)|^2ds$$
$$\leq\frac{1}{2}\left\{\left|u_{1m}\right|^2+|A^{1/2}u_{0m}|^2+\frac{2}{\gamma+2}\left|\left|u_{0m}\right|\right|_{\gamma+2}^{\gamma+2}\right\}$$
$$+\left(\frac{\xi_1^2}{4\eta}\left|\left|g\right|\right|_{L^1(0,\infty)}^2+\frac{1}{4\eta}\left|\left|g\right|\right|_{L^1(0,\infty)}\left|\left|g\right|\right|_{L^\infty(0,\infty)}\right)\int_0^t|A^{1/2}u_m(s)|^2\,ds.$$

From (3.2), (3.8), choosing $\eta>0$ sufficiently small and employing Gronwall's lemma, we obtain the first estimate

$$\left|u_m'(t)\right|^2+|A^{1/2}u_m(t)|^2+\left|\left|u_m(t)\right|\right|_{\gamma+2}^{\gamma+2}+\int_0^t\left|\left|u_m'(s)\right|\right|_{\rho+2}^{\rho+2}ds\leq L_4 \tag{3.9}$$

where L_4 is a positive constant independent of $m\in\mathbf{N}$ and $t\in[0,T]$.
The Second Estimate: Considering $w=u_m''(0)$ and $t=0$ in (3.1),

$$\left|u_m''(0)\right|^2\leq\left[\left|Au_{0m}\right|+\left|F(x,t,u_m(t),u_m'(t)\right|\right]\left|u_m''(0)\right|.$$

From the last inequality and making use of the assumption $(H.2)$ on F, we obtain

$$\left|u_m''(0)\right|\leq L_5 \tag{3.10}$$

where L_5 is a positive constant independent of $m\in\mathbf{N}$.

Now, getting the derivative of (3.1) with respect to t and substituting $w=u_m''(t)$ in the obtained expression, it results that

$$\frac{1}{2}\frac{d}{dt}\left\{\left|u_m''(t)\right|^2+\left|A^{1/2}u_m'(t)\right|^2\right\}+\beta\int_\Gamma\left|u_m'\right|^\rho\left|u_m''\right|^2d\Gamma+g(0)|A^{1/2}u_m'(t)|^2$$
$$\leq C\int_\Gamma\left(1+\left|u_m'\right|^{\rho+1}+\left|u_m\right|^{\gamma+1}\right)\left|u_m''\right|d\Gamma$$
$$+C\int_\Gamma\left(1+\left|u_m'\right|^\rho+\left|u_m\right|^\gamma\right)\left|u_m'\right|\left|u_m''\right|d\Gamma$$
$$-g'(0)(A^{1/2}u_m(t),A^{1/2}u_m'(t))-\int_0^t g''(t-\tau)(A^{1/2}u_m(\tau),A^{1/2}u_m'(t))d\tau$$
$$+g(0)\frac{d}{dt}(A^{1/2}u_m(t),A^{1/2}u_m'(t)) \tag{3.11}$$
$$+\frac{d}{dt}\left\{\int_0^t g'(t-\tau)(A^{1/2}u_m(\tau),A^{1/2}u_m'(t))d\tau\right\}.$$

We observe that from $(H.10)$ it holds, from an arbitrary $\eta > 0$, that

$$\int_0^t g''(t-\tau)(A^{1/2}u_m(\tau), A^{1/2}u'_m(t))d\tau \tag{3.12}$$
$$\leq \frac{\xi_3^2}{4\eta}\,||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)|A^{1/2}u_m(\tau)|^2 d\tau + \eta|A^{1/2}u'_m(t)|^2.$$

We also have

$$g'(0)\left(A^{1/2}u_m(t), A^{1/2}u'_m(t)\right) \leq \frac{(g'(0))^2}{4\eta}\left|A^{1/2}u_m(t)\right|^2 + \eta\left|A^{1/2}u'_m(t)\right|^2 \tag{3.13}$$

Integrating (3.11) over (0,t) taking the generalized Hölder inequality and (3.12) and (3.13) into account, it holds that

$$\frac{1}{2}\left|u''_m(t)\right|^2 + \frac{1}{2}\left|A^{1/2}u'_m(t)\right|^2 \tag{3.14}$$
$$+(\beta - 2C\eta)\int_0^t\int_\Gamma |u'_m|^\rho\left|u''_m\right|^2 d\Gamma ds + (g(0)-2\eta)\int_0^t\left|A^{1/2}u'_m(s)\right|^2 ds$$
$$\leq \frac{1}{2}\left|u''_m(0)\right|^2 + \frac{1}{2}\left|A^{1/2}u_{1m}\right|^2 + CT\,meas(\Gamma)$$
$$+C_1\int_0^t\left(||u_m(s)||^{2(\gamma+1)}_{H^{1/2}(\Gamma)} + |u'_m(s)|^2\right)ds + \frac{C}{2\eta}\int_0^t ||u'_m(s)||^{\rho+2}_{\rho+2}ds$$
$$+C\int_0^t\left|u''_m(s)\right|^2 ds + C_2\int_0^t ||u_m(s)||^\gamma_{H^{1/2}(\Gamma)}\,||u'_m(s)||_{H^{1/2}(\Gamma)}\left|u''_m(t)\right| ds$$
$$+\frac{(g'(0))^2}{4\eta}\int_0^t\left|A^{1/2}u_m(s)\right|^2 ds + \frac{\xi_3^2}{4\eta}\,||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)\left|A^{1/2}u_m(\tau)\right|^2 d\tau$$
$$+g(0)\left(A^{1/2}u_m(t), A^{1/2}u'_m(t)\right) + \int_0^t g'(t-\tau)\left(A^{1/2}u_m(\tau), A^{1/2}u'_m(t)\right)d\tau.$$

But, as in (3.7) taking the assumption $(H.9)$ into account, we have

$$\int_0^t g'(t-\tau)\left(A^{1/2}u_m(\tau), A^{1/2}u'_m(t)\right)d\tau \tag{3.15}$$
$$\leq \eta\left|A^{1/2}u'_m(t)\right|^2 + \frac{\xi_1^2}{4\eta}\,||g||_{L^1(0,\infty)}\,||g||_{L^\infty(0,\infty)}\int_0^t\left|A^{1/2}u_m(s)\right|^2 ds.$$

We also infer

$$g(0)\left(A^{1/2}u_m(t), A^{1/2}u'_m(t)\right) \leq \frac{(g(0)^2}{4\eta}\left|A^{1/2}u_m(t)\right|^2 + \eta\left|A^{1/2}u'_m(t)\right|^2. \tag{3.16}$$

Combining (3.14)-(3.16), choosing $\eta > 0$ sufficiently small, making use of the first estimate (3.9), considering (3.10) and employing Gronwall's lemma, we obtain the second estimate

$$\left|u''_m(t)\right|^2 + \frac{1}{2}\left|A^{1/2}u'_m(t)\right|^2 + \int_0^t\int_\Gamma |u'_m|^\rho\left|u''_m\right|^2 d\Gamma ds \leq L_6 \tag{3.17}$$

where L_6 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0, T]$.

3.1.5 - Passage to the Limit.

Having in mind that $\left|A^{1/2}u\right| = ||u||_{H^{1/2}(\Gamma)}$; for all $u \in H^{1/2}(\Gamma)$, and using compactness arguments then we can pass to the limit in (3.1) to obtain

$$u'' + \tilde{A}u + F(x,t,u,u') - g * \tilde{A}u = 0 \text{ in } \mathcal{D}'(0,T;H^{-1/2}(\Gamma)) \tag{3.18}$$

where $\tilde{A} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the isometric and self-adjoint extension of A defined as in (1.5). Since, $u'', F(x,t,u,u') \in L^2(\Gamma)$, from (3.18) it follows that

$$\tilde{A}(u - g * u) \in L^2_{loc}(0,\infty;L^2(\Gamma)). \tag{3.19}$$

Therefore,

$$u - g * u \in L^2_{loc}(0,\infty;D(A)), \tag{3.20}$$

$$u'' + A\left(u - g * u\right) + F(x,t,u,u') = 0 \text{ in } L^2_{loc}(0,\infty;L^2(\Gamma)). \tag{3.21}$$

3.1.6 - Uniqueness.

Let u and $\hat{u}$ be two regular solutions of $(**)$ satisfying theorem 2.4. Defining $z = u - \hat{u}$, from (3.21) we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{|z'(t)|^2 + \left|A^{1/2}z(t)\right|^2\right\} + g(0)\left|A^{1/2}z(t)\right| \\ &\le C(\gamma)\left(||u(t)||^{\gamma}_{2(\gamma+1)} + ||\hat{u}||^{\gamma}_{2(\gamma+1)}\right)||z(t)||_{2(\gamma+1)}\,|z'(t)| \\ &- \int_0^t g'(t-\tau)\left(A^{1/2}z(\tau), A^{1/2}z(t)\right)d\tau \\ &+ \frac{d}{dt}\left\{\int_0^t g(t-\tau)\left(A^{1/2}z(\tau), A^{1/2}z(t)\right)d\tau\right\}. \end{aligned} \tag{3.22}$$

Note that

$$\begin{aligned} &\int_0^t g'(t-\tau)\left(A^{1/2}z(\tau), A^{1/2}z(t)\right)d\tau \\ &\le \frac{\xi_1^2}{4\eta}||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)\left|A^{1/2}z(\tau)\right|^2 d\tau + \eta\left|A^{1/2}z(t)\right|^2. \end{aligned} \tag{3.23}$$

Integrating (3.22) over (0,t) taking (3.23) into account, and having in mind that $\left|A^{1/2}z(t)\right| = ||z(t)||_{H^{1/2}(\Gamma)}$, we infer

$$\begin{aligned} &\frac{1}{2}|z'(t)|^2 + \frac{1}{2}\left|A^{1/2}z(t)\right|^2 + (g(0)-\eta)\int_0^t\left|A^{1/2}z(s)\right|^2 ds \\ &\le C(\gamma)\int_0^t\left\{\frac{1}{2}\left|A^{1/2}z(s)\right|^2 + \frac{1}{2}|z'(s)|^2\right\}ds \\ &+ \frac{\xi_1^2}{4\eta}||g||^2_{L^1(0,\infty)}\int_0^t\left|A^{1/2}z(s)\right|^2 ds + \int_0^t g(t-\tau)\left(A^{1/2}z(\tau), A^{1/2}z(t)\right)d\tau. \end{aligned} \tag{3.24}$$

Finally, observing that

$$\int_0^t g(t-\tau)\left(A^{1/2}z(\tau), A^{1/2}z(t)\right) d\tau \tag{3.25}$$
$$\leq \eta \left|A^{1/2}z(t)\right|^2 + \frac{1}{4\eta}\,||g||_{L^1(0,\infty)}\,||g||_{L^\infty(0,\infty)} \int_0^t \left|A^{1/2}z(s)\right|^2 ds,$$

then, from (3.24), (3.25), choosing η sufficiently small and employing Gronwall's lemma we conclude that $|z'(t)|= ||z(t)||_{H^{1/2}(\Gamma)}= 0$, which finishes the proof of uniqueness for regular solutions of $(*)$. $\diamondsuit$

3.2 Weak Solutions

Given $\{u^0, u^1\} \in H^{1/2}(\Gamma)\times L^2(\Gamma)$, since $D(A)\times H^{1/2}(\Gamma)$ is dense in $H^{1/2}(\Gamma)\times L^2(\Gamma)$ the procedure used in the earlier article is similar. Since $g \neq 0$, the unique difference is due to the memory term which we have already handled, see (3.3)-(3.9) and the section 3.1.6. For this reason the proof will be omitted. Analogously we deduce there exists a unique function u verifying

$$\left\{ \begin{array}{ll} u'' + \tilde{A}(u - g*u) + F(x,t,u,u') = 0 & \text{in } L^2_{loc}(0,\infty; H^{-1/2}(\Gamma)) \\ u(0) = u^0, \quad u'(0) = u^1. & \end{array} \right. \tag{3.26}$$

4 Asymptotic Stability

In this section we obtain the uniform decay of the energy for regular solutions, since the same occurs for weak solutions using standard density arguments.

Let us consider, now, the case $F \neq 0$ and $g \neq 0$. From assumption $(H.3)$ and taking (3.21) into account, we deduce

$$E'(t) \leq -\beta\,||u'(t)||^{\rho+2}_{\rho+2} + \int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u'(t)\right) d\tau, \tag{4.1}$$

where $E(t)$ is defined in (2.7).

A direct computation shows that

$$\int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u'(t)\right) d\tau \tag{4.2}$$
$$= \frac{1}{2}\left(g' \diamond A^{1/2}u\right)(t) - \frac{1}{2}\left(g \diamond A^{1/2}u\right)'(t)$$
$$+ \frac{d}{dt}\left\{\frac{1}{2}\left(\int_0^t g(s)\,ds\right)\left|A^{1/2}u(t)\right|^2\right\} - \frac{1}{2}g(t)\left|A^{1/2}u(t)\right|^2,$$

where

$$(g \diamond y)(t) = \int_0^t g(t-\tau)\,|y(t) - y(s)|^2\,ds.$$

Defining the modified energy by

$$\begin{aligned} e(t) &= \frac{1}{2}\,|u'(t)|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left|A^{1/2}u(t)\right|^2 \\ &+ \frac{1}{\gamma+2}\,||u(t)||^{\gamma+2}_{\gamma+2} + \frac{1}{2}\left(g \diamond A^{1/2}u\right)(t) \end{aligned} \tag{4.3}$$

we obtain from (4.1) and (4.2) that

$$e'(t) = -\beta \, ||u'(t)||_{\rho+2}^{\rho+2} + \frac{1}{2}\left(g' \diamond A^{1/2}u\right)(t) - \frac{1}{2}g(t)\left|A^{1/2}u(t)\right|^2. \tag{4.4}$$

We observe that taking the assumption (H.8) into account, we deduce that $e(t) \geq 0$. Now, from (4.4) and considering the hypothesis (H.9) on the kernel g, we have $e'(t) \leq 0$. Furthermore, since

$$E(t) \leq \left(l^{-1}+1\right)e(t) \tag{4.5}$$

the decay of $E(t)$ is a consequence of the $e(t)$ decay.

Let us define, as in the previous case

$$\psi(t) = [e(t)]^{\rho/2}\left(u'(t), u(t)\right). \tag{4.6}$$

Taking the derivative of $\psi(t)$ with respect to t, substituting

$$u'' = -A(u - g * u) - F(x,t,u,u')$$

in the obtained expression, it holds that

$$\begin{aligned} \psi'(t) &= \frac{\rho}{2}[e(t)]^{\frac{\rho-2}{2}} e'(t)\left(u'(t), u(t)\right) \qquad (4.7)\\ &+ [e(t)]^{\rho/2}\left\{-\left|A^{1/2}u(t)\right|^2 + \int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right)d\tau\right.\\ &- \left.\left(F(x,t,u,u'), u(t)\right) + |u'(t)|^2\right\}. \end{aligned}$$

Since $-e'(t) > 0$, we deduce, from (4.5) and (4.7) that

$$\begin{aligned} \psi'(t) &\leq -C_1 e'(t) \qquad (4.8)\\ &+ [e(t)]^{\rho/2}\left\{-\left|A^{1/2}u(t)\right|^2 - ||u(t)||_{\gamma+2}^{\gamma+2} - \beta\int_\Gamma |u'|^{\rho+1}|u|\,d\Gamma\right.\\ &+ \left.\int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right)d\tau + |u'(t)|^2\right\}, \end{aligned}$$

where $C_1 = C_1\left(l^{-1}, e(0)\right)$.

Repeating the same procedure we have done in the previous paper, we deduce, from (4.8) that

$$\begin{aligned} \psi'(t) &\leq -C_1 e'(t) \qquad (4.9)\\ &+ [e(t)]^{\rho/2}\left\{-\left(1 - \eta 2^{\rho/2}\left(l^{-1}+1\right)^{\rho/2}[e(0)]^{\rho/2}\right)\left|A^{1/2}u(t)\right|^2 + k(\eta)\,||u'(t)||_{\rho+2}^{\rho+2}\right.\\ &- \left.||u(t)||_{\gamma+2}^{\gamma+2} + \int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right)d\tau + |u'(t)|^2\right\}, \end{aligned}$$

where η is an arbitrary positive number and $k = k(\eta)$ is a positive constant which depends on η.

Choosing $\eta > 0$ such that $1 - \eta 2^{\rho/2}\left(l^{-1}+1\right)^{\rho/2}[e(0)]^{\rho/2} = \frac{1}{2}$, from (4.9) we have

$$\begin{aligned} \psi'(t) \;&\leq\; -C_1 e'(t) + k\,[e(0)]^{\rho/2}\,||u'(t)||_{\rho+2}^{\rho+2} \\ &+\; [e(t)]^{\rho/2}\left\{-\frac{1}{2}\left|A^{1/2}u(t)\right|^2 - \frac{1}{\gamma+2}\,||u(t)||_{\gamma+2}^{\gamma+2}\right\} \\ &+\; [e(t)]^{\rho/2}\left\{\int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right)d\tau + |u'(t)|^2\right\}. \end{aligned} \tag{4.10}$$

From (4.4) and (4.10) we deduce

$$\begin{aligned} \psi'(t) \;&\leq\; -(C_1+C_2)e'(t) \\ &+\; [e(t)]^{\rho/2}\left\{-\frac{1}{2}\left|A^{1/2}u(t)\right|^2 - \frac{1}{\gamma+2}\,||u(t)||_{\gamma+2}^{\gamma+2}\right\} \\ &+\; [e(t)]^{\rho/2}\left\{\int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right)d\tau + |u'(t)|^2\right\}, \end{aligned} \tag{4.11}$$

where $C_2 = \beta k\,[e(0)]^{\rho/2}$.

Defining the perturbed energy by

$$e_\varepsilon(t) = \left(1 + \varepsilon\left(C_1 + C_2\right)\right)e(t) + \varepsilon\psi(t) \tag{4.12}$$

we also deduce that there exists $L = L\left(l^{-1}, e(0)\right)$ such that

$$|e_\varepsilon(t) - e(t)| \leq \varepsilon L\, e(t), \quad \text{for all } \varepsilon > 0. \tag{4.13}$$

Considering $\varepsilon \in (0, 1/2L]$, from (4.13) we obtain

$$\frac{1}{2}e(t) \leq e_\varepsilon(t) \leq 2e(t) \tag{4.14}$$

and

$$2^{-\frac{\rho+2}{2}}\,[e(t)]^{\frac{\rho+2}{2}} \leq [e_\varepsilon(t)]^{\frac{\rho+2}{2}} \leq 2^{\frac{\rho+2}{2}}\,[e(t)]^{\frac{\rho+2}{2}}\,; \quad \varepsilon \in (0, 1/2L]. \tag{4.15}$$

Taking the derivative of (4.12) with respect to t, taking (4.28), $(H.9)$ and (4.11) into account, we conclude

$$\begin{aligned} e'_\varepsilon(t) \;&\leq\; -\beta\,||u'(t)||_{\rho+2}^{\rho+2} - \frac{\xi_2}{2}\left(g \diamond A^{1/2}u\right)(t) - \frac{1}{2}g(t)\left|A^{1/2}u(t)\right|^2 \\ &+\; \varepsilon\,[e(t)]^{\rho/2}\left\{-\frac{1}{2}\left|A^{1/2}u(t)\right|^2 - \frac{1}{\gamma+2}\,||u(t)||_{\gamma+2}^{\gamma+2}\right\} \\ &+\; \varepsilon\,[e(t)]^{\rho/2}\left\{\int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right)d\tau + |u'(t)|^2\right\}. \end{aligned} \tag{4.16}$$

Having in mind that

$$\begin{aligned} &-\frac{1}{2}\left|A^{1/2}u(t)\right|^2 - \frac{1}{\gamma+2}\,||u(t)||_{\gamma+2}^{\gamma+2} = -e(t) + \frac{1}{2}\,|u'(t)|^2 \\ &-\frac{1}{2}\left(\int_0^t g(s)\,ds\right)\left|A^{1/2}u(t)\right|^2 + \frac{1}{2}\left(g \diamond A^{1/2}u\right)(t), \end{aligned} \tag{4.17}$$

and since $L^{\rho+2}(\Gamma) \hookrightarrow L^2(\Gamma)$, from (4.16) it holds that

$$\begin{aligned} e'_\varepsilon(t) &\leq -\beta C_0 \left|u'(t)\right|^{\rho+2} - \frac{\xi_2}{2}\left(g \diamond A^{1/2}u\right)(t) \qquad (4.18)\\ &+\frac{3}{2}\varepsilon \left[e(t)\right]^{\rho/2} \left|u'(t)\right|^2 - \varepsilon \left[e(t)\right]^{\frac{\rho+2}{2}} \\ &+\frac{\varepsilon}{2}\left[e(t)\right]^{\rho/2}\left(g \diamond A^{1/2}u\right)(t) + \varepsilon \left[e(t)\right]^{\rho/2} \int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right) d\tau. \end{aligned}$$

But, since $\frac{\rho}{\rho+2} + \frac{2}{\rho+2} = 1$ the Hölder inequality yields

$$\begin{aligned} \left[e(t)\right]^{\rho/2}\left|u'(t)\right|^2 &\leq \frac{\rho}{\rho+2}\left(\mu\left[e(t)\right]^{\rho/2}\right)^{\frac{\rho+2}{\rho}} + \frac{2}{\rho+2}\left(\frac{1}{\mu}\left|u'(t)\right|^2\right)^{\frac{\rho+2}{2}} \qquad (4.19)\\ &\leq \mu^{\frac{\rho+2}{\rho}}\left[e(t)\right]^{\frac{\rho+2}{2}} + \frac{1}{\mu^{\frac{\rho+2}{2}}}\left|u'(t)\right|^{\rho+2}, \end{aligned}$$

where μ is an arbitrary positive constant. Then, combining (4.18)-(4.19) we deduce

$$\begin{aligned} e'_\varepsilon(t) &\leq -\left(\beta C_0 - \frac{3}{2}\varepsilon\frac{1}{\mu^{\frac{\rho+2}{2}}}\right)\left|u'(t)\right|^{\rho+2} - \varepsilon\left(1 - \frac{3}{2}\mu^{\frac{\rho+2}{\rho}}\right)\left[e(t)\right]^{\frac{\rho+2}{2}} \qquad (4.20)\\ &- \left(\frac{\xi_2}{2} - \frac{\varepsilon}{2}\left[e(0)\right]^{\rho/2}\right)\left(g \diamond A^{1/2}u\right)(t) \\ &+ \varepsilon\left[e(t)\right]^{\rho/2}\int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right) d\tau. \end{aligned}$$

Estimate for $J_2 := \int_0^t g(t-\tau)\left(A^{1/2}u(\tau), A^{1/2}u(t)\right) d\tau$.

We have

$$\begin{aligned} |J_2| &\leq \int_0^t g(t-\tau)\left|A^{1/2}u(t)\right|\left\{\left|A^{1/2}u(\tau) - A^{1/2}u(t)\right| + \left|A^{1/2}u(t)\right|\right\} d\tau \\ &\leq \eta\left|A^{1/2}u(t)\right|^2 + \frac{1}{4\eta}\left(\int_0^t g(t-\tau)\left|A^{1/2}u(\tau) - A^{1/2}u(t)\right| d\tau\right)^2 \\ &+\left(\int_0^t g(s)\,ds\right)\left|A^{1/2}u(t)\right|^2 \qquad (4.21)\\ &\leq 2\eta l^{-1}e(t) + \frac{1}{4\eta}\left|\left|g\right|\right|_{L^1(0,\infty)}\left(g \diamond A^{1/2}u\right)(t) + 2\left|\left|g\right|\right|_{L^1(0,\infty)} l^{-1}e(t). \end{aligned}$$

From (4.20) and (4.21) we infer

$$\begin{aligned} e'_\varepsilon(t) &\leq -\left(\beta C_0 - \frac{3}{2}\varepsilon\frac{1}{\mu^{\frac{\rho+2}{2}}}\right)\left|u'(t)\right|^{\rho+2} \qquad (4.22)\\ &- \left[1 - \left(\frac{3}{2}\mu^{\frac{\rho+2}{\rho}} + 2\eta l^{-1} + 2l^{-1}\left|\left|g\right|\right|_{L^1(0,\infty)}\right)\right]\varepsilon\left[e(t)\right]^{\frac{\rho+2}{2}} \\ &- \left\{\frac{\xi_2}{2} - \varepsilon\left[e(0)\right]^{\rho/2}\left(\frac{1}{2} + \frac{1}{4\eta}\left|\left|g\right|\right|_{L^1(0,\infty)}\right)\right\}\left(g \diamond A^{1/2}u\right)(t). \end{aligned}$$

Choosing μ, η and $||g||_{L^1(0,\infty)}$ sufficiently small so that

$$\theta = 1 - \left(\frac{3}{2} \mu^{\frac{\rho+2}{\rho}} + 2\eta l^{-1} + 2 l^{-1} \, ||g||_{L^1(0,\infty)} \right) > 0$$

and choosing ε small enough in order to have

$$\beta C_0 - \frac{3}{2} \varepsilon \frac{1}{\mu^{\frac{\rho+2}{2}}} \geq 0 \quad \text{and} \quad \frac{\xi_2}{2} - \varepsilon \, [e(0)]^{\rho/2} \left(\frac{1}{2} + \frac{1}{4\eta} \, ||g||_{L^1(0,\infty)} \right) \geq 0$$

from (4.22) we conclude

$$e'_\varepsilon(t) \leq -\varepsilon \theta \, [e(t)]^{\frac{\rho+2}{2}}$$

as we obtained earlier. From this inequality we conclude the desired estimate as in the previous case.

We observe that when $\rho = 0$, then, combining (4.14) and (4.22) the exponential decay holds and, in this case, we are able to deduce directly from the proof that is not necessary to consider $\int_0^\infty g(s) \, ds$ sufficiently small. So, the proof of theorem 2.4 and (by density) theorem 2.5 is completed. $\diamondsuit$

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Volume 8,Number 4 October 2006
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

A quarterly international publication of Eudoxus Press, LLC

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Basic Convergence with Rates of Smooth Picard Singular Integral Operators

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AMS 2000 Mathematics Subject Classification:
Primary: 26A15, 41A17, 41A35
Secondary: 26D15, 41A44

Key Words and Phrases: Best constant, Picard singular integral, modulus of smoothness, sharp inequality.

Abstract. In this article we introduce and study the smooth Picard singular integral operators on the line of very general kind. We establish their convergence to the unit operator with rates. The estimates are mostly sharp and they are pointwise or uniform. The established inequalities involve the higher order modulus of smoothness. To prove optimality we use mainly the geometric moment theory method.

1 Introduction

The rate of convergence of singular integrals has been studied earlier in [8], [9], [11], [3], [5], [6] and these motivate this work. Here we consider some very general operators, the *smooth Picard singular integral operators over* $\mathbb{R}$ and we study the degree of approximation to the unit operator with rates over smooth functions. We establish related inequalities involving the higher modulus of smoothness with respect to $\|\cdot\|_\infty$. The estimates are pointwise or uniform. Most of the times these are optimal in sense that the inequalities are attained by basic functions. We use the geometric moment theory method to give best upper bounds in the main theorems and also we give handy estimates there. The discussed operators are not in general positive.

Other motivation comes from [1], [2].

2 Results

In the next we deal with the following *smooth Picard singular integral operators* $P_{r,\xi}(f;x)$ defined as follows.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$P_{r,\xi}(f;x) := \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) e^{-|t|/\xi} dt. \tag{2}$$

We assume that $P_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$P_{r,\xi}(f;x) = \frac{1}{2\xi} \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} f(x+jt) e^{-|t|/\xi} dt \right). \tag{3}$$

We notice by $\frac{1}{2\xi} \int_{-\infty}^{\infty} e^{-|t|/\xi} dt = 1$ that $P_{r,\xi}(c,x) = c$, c constant and

$$P_{r,\xi}(f;x) - f(x) = \frac{1}{2\xi} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x+jt) - f(x)) \right) e^{-|t|/\xi} dt. \tag{4}$$

Since

$$\int_{-\infty}^{\infty} x^k e^{-|x|} dx = \begin{cases} 0, & k \text{ odd}, \\ 2k!, & k \text{ even}, \end{cases} \tag{5}$$

we get the useful here formula

$$\int_{-\infty}^{\infty} t^k e^{-|t|/\xi} dt = \begin{cases} 0, & k \text{ odd}, \\ 2k!\xi^{k+1}, & k \text{ even}. \end{cases} \tag{6}$$

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$ with the rth modulus of smoothness finite, i.e.

$$\omega_r(f^{(n)}, h) := \sup_{|t| \le h} \|\Delta_t^r f^{(n)}(x)\|_{\infty,x} < \infty, \quad h > 0, \tag{7}$$

where

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x+jt), \tag{8}$$

see [7], p. 44.

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \dots, n \in \mathbb{N}, \tag{9}$$

and the even function

$$G_n(t) := \int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!} \omega_r(f^{(n)}, w) dw, \quad n \in \mathbb{N} \tag{10}$$

with

$$G_0(t) := \omega_r(f, |t|), \quad t \in \mathbb{R}. \tag{11}$$

Denote by $\lfloor \cdot \rfloor$ the integral part.

We present our first result

Theorem 1. *It holds that*

$$\left| P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x) \delta_{2m} \xi^{2m} \right| \leq \frac{1}{\xi} \int_0^{\infty} G_n(t) e^{-t/\xi} dt, \quad n \in \mathbb{N}. \tag{12}$$

In L.H.S.(12) *the sum collapses when* $n = 1$.

Proof. By Taylor's formula we obtain

$$\begin{aligned} f(x+jt) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(jt)^k + \int_0^{jt} \frac{(jt-z)^{n-1}}{(n-1)!} f^{(n)}(x+z) dz \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(jt)^k + j^n \int_0^{t} \frac{(t-w)^{n-1}}{(n-1)!} f^{(n)}(x+jw) dw. \end{aligned} \tag{13}$$

Multiplying both sides of (13) by α_j and summing up we get

$$\sum_{j=0}^{r} \alpha_j (f(x+jt) - f(x)) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k t^k + \mathcal{R}_n(0,t), \tag{14}$$

where

$$\mathcal{R}_n(0,t) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!} \tau(w) dw, \tag{15}$$

with

$$\tau(w) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x+jw) - \delta_n f^{(n)}(x).$$

Notice also that

$$-\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}=(-1)^r\binom{r}{0}. \tag{16}$$

According to [3], p. 306, [1], we get

$$\tau(w)=\Delta_w^r f^{(n)}(x). \tag{17}$$

Therefore

$$|\tau(w)|\le\omega_r(f^{(n)},|w|), \tag{18}$$

all $w\in\mathbb{R}$ independently of x. We do have after integration, see also (4), that

$$\begin{aligned}
P_{r,\xi}(f;x)-f(x)&=\frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j(f(x+jt)-f(x))\right)e^{-|t|/\xi}dt\\
&=\frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k t^k+\mathcal{R}_n(0,t)\right)e^{-|t|/\xi}dt\\
&=\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}t^k e^{-|t|/\xi}dt\right)+\mathcal{R}_n^*,
\end{aligned} \tag{19}$$

where

$$\mathcal{R}_n^*:=\frac{1}{2\xi}\int_{-\infty}^{\infty}\mathcal{R}_n(0,t)e^{-|t|/\xi}dt. \tag{20}$$

Here by (10) and (15) we get

$$|\mathcal{R}_n(0,t)|\le\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}|\tau(w)|dw\le G_n(t). \tag{21}$$

Hence by (20) we have

$$\begin{aligned}
|\mathcal{R}_n^*|&\le\frac{1}{2\xi}\int_{-\infty}^{\infty}G_n(t)e^{-|t|/\xi}dt\\
&=\frac{1}{\xi}\int_0^{\infty}G_n(t)e^{-t/\xi}dt.
\end{aligned} \tag{22}$$

Using (6) we obtain

$$P_{r,\xi}(f;x)-f(x)-\sum_{m=1}^{\lfloor\frac{n}{2}\rfloor}f^{(2m)}(x)\delta_{2m}\xi^{2m}=\mathcal{R}_n^*. \tag{23}$$

Inequality (12) is now clear via (23) and (22).

Finally we would like to prove (21) with the use of (18). We have that for $t > 0$ it is obvious. Let $t < 0$, then

$$\begin{aligned}
|\mathcal{R}_n(0,t)| &= \left|\int_t^0 \frac{(t-w)^{n-1}}{(n-1)!}\tau(w)dw\right| \\
&\leq \int_t^0 \frac{(w-t)^{n-1}}{(n-1)!}|\tau(w)|dw \leq \int_t^0 \frac{(-t-(-w))^{n-1}}{(n-1)!}\omega_r(f^{(n)},|w|)dw \\
&= -\left(\int_t^0 \frac{(-t-(-w))^{n-1}}{(n-1)!}\omega_r(f^{(n)},|-w|)d(-w)\right) \\
&= -\left(\int_{-t}^0 \frac{(-t-\theta)^{n-1}}{(n-1)!}\omega_r(f^{(n)},|\theta|)d\theta\right) \\
&= \int_0^{-t} \frac{(-t-\theta)^{n-1}}{(n-1)!}\omega_r(f^{(n)},|\theta|)d\theta \\
&= \int_0^{|t|} \frac{(|t|-\theta)^{n-1}}{(n-1)!}\omega_r(f^{(n)},\theta)d\theta = G_n(t).
\end{aligned}$$

The last completes the proof of Theorem 1. ∎

Corollary 1. *Assume $\omega_r(f,\xi) < \infty$, $\xi > 0$. Then it holds for $n = 0$ that*

$$|P_{r,\xi}(f;x) - f(x)| \leq \frac{1}{\xi}\int_0^\infty \omega_r(f,t)e^{-t/\xi}dt. \tag{24}$$

Proof. We notice that

$$\begin{aligned}
P_{r,\xi}(f;x) - f(x) &= \frac{1}{2\xi}\left(\int_{-\infty}^\infty \left(\sum_{j=1}^r \alpha_j(f(x+jt)-f(x))\right)e^{-|t|/\xi}dt\right) \\
&= \frac{1}{2\xi}\left(\int_{-\infty}^\infty \left(\sum_{j=1}^r (-1)^{r-j}\binom{r}{j}(f(x+jt)-f(x))\right)e^{-|t|/\xi}dt\right. \\
&= \frac{1}{2\xi}\left(\int_{-\infty}^\infty \left(\sum_{j=1}^r (-1)^{r-j}\binom{r}{j}f(x+jt)\right.\right. \\
&\quad \left.\left. - \left(\sum_{j=1}^r (-1)^{r-j}\binom{r}{j}\right)f(x)\right)e^{-|t|/\xi}dt\right) \\
&\overset{(16)}{=} \frac{1}{2\xi}\left(\int_{-\infty}^\infty \left(\sum_{j=1}^r (-1)^{r-j}\binom{r}{j}f(x+jt)\right.\right. \\
&\quad \left. + (-1)^r\binom{r}{0}f(x)\right)e^{-|t|/\xi}dt \\
&= \frac{1}{2\xi}\left(\int_{-\infty}^\infty \left(\sum_{j=0}^r (-1)^{r-j}\binom{r}{j}f(x+jt)\right)e^{-|t|/\xi}dt\right. \\
&\overset{(8)}{=} \frac{1}{2\xi}\left(\int_{-\infty}^\infty ((\Delta_t^r f)(x)\right)e^{-|t|/\xi}dt.
\end{aligned}$$

I.e. we have proved

$$P_{r,\xi}(f;x)-f(x)=\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}(\Delta_t^r f(x))e^{-|t|/\xi}dt\right). \tag{25}$$

Hence by (25) we find

$$\begin{aligned}|P_{r,\xi}(f;x)-f(x)|&\le\frac{1}{2\xi}\int_{-\infty}^{\infty}|\Delta_t^r f(x)|e^{-|t|/\xi}dt\\&\le\frac{1}{2\xi}\int_{-\infty}^{\infty}\omega_r(f,|t|)e^{-|t|/\xi}dt\\&=\frac{1}{\xi}\int_0^{\infty}\omega_r(f,t)e^{-|t|/\xi}dt.\end{aligned}$$

That is proving (24). ∎

Inequality (12) is sharp.

Theorem 2. *Inequality (12) at $x=0$ is attained by $f(x)=x^{r+n}$, $r,n\in\mathbb{N}$ with $r+n$ even.*

Proof. As in [3], p. 307, [1], [12], p. 54 and (7), (8) we get

$$\omega_r(f^{(n)},t)=(r+n)(r+n-1)\cdots(r+1)r!t^r,$$

$t>0$. And

$$G_n(t)=r!|t|^{r+n},\quad t\in\mathbb{R}.$$

Also we have $f^{(k)}(0)=0$, $k=0,1,\dots,n$. Thus the right hand side of (12) equals

$$\frac{r!}{\xi}\int_0^{\infty}t^{r+n}e^{-t/\xi}dt=r!(r+n)!\xi^{r+n}. \tag{26}$$

The left hand side of (12) equals

$$\begin{aligned}|P_{r,\xi}(f;0)|&=\frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(jt)\right)e^{-|t|/\xi}dt\right|\\&=\frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(jt)\right)e^{-|t|/\xi}dt\right|\\&=\frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^{-n}(jt)^{r+n}\right)e^{-|t|/\xi}dt\right|\\&=\frac{1}{2\xi}\left|\left(\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}j^{r}\right)\left(\int_{-\infty}^{\infty}t^{r+n}e^{-|t|/\xi}dt\right)\right|\\&=\frac{1}{2\xi}\left|(\Delta_1^r x^r)(0)\int_{-\infty}^{\infty}t^{r+n}e^{-|t|/\xi}dt\right|\end{aligned}$$

$$= \frac{1}{2\xi}\left|r!\int_{-\infty}^{\infty} t^{r+n}e^{-|t|/\xi}dt\right|$$

$$\overset{(6)}{=} \frac{1}{2\xi}\left|r!2(r+n)!\xi^{r+n+1}\right| = r!(r+n)!\xi^{r+n}.$$

I.e. we have proved

$$|P_{r,\xi}(f;0)| = r!(r+n)!\xi^{r+n}. \tag{27}$$

Thus by (26) and (27) we have established the claim of the theorem.

Inequality (24) is sharp. ■

Corollary 2. *Inequality (24) is attained at $x = 0$ by $f(x) = x^r$, r even.*

Proof. Notice that $\Delta_t^r x^r = r!t^r$ and $\omega_r(f^{(n)}, t) = r!t^r$, $t > 0$. Thus

$$\text{R.H.S.}(24) = \frac{r!}{\xi}\int_0^{\infty} t^r e^{-t/\xi}dt = (r!)^2\xi^r.$$

Also $f(0) = 0$. Therefore

$$\begin{aligned}
\text{L.H.S.}(24) &= |P_{r,\xi}(f;0)| = \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j j^r t^r\right)e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}j^r\right)t^r e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|(\Delta_1^r x^r)(0)\int_{-\infty}^{\infty} t^r e^{-|t|/\xi}dt\right| \\
&= \frac{1}{2\xi}\left|r!\int_{-\infty}^{\infty} t^r e^{-|t|/\xi}dt\right| \\
&\overset{(6)}{=} \frac{1}{2\xi}|r!2r!\xi^{r+1}| = (r!)^2\xi^r.
\end{aligned}$$

That is (24) is attained. ■

Remark 1. On inequalities (12) and (24). We have the uniform estimates

$$\left\|P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x)\delta_{2m}\xi^{2m}\right\|_{\infty,x} \le \frac{1}{\xi}\int_0^{\infty} G_n(t)e^{-t/\xi}dt,\ n \in \mathbb{N}, \tag{28}$$

and

$$\|P_{r,\xi}(f) - f\|_{\infty} \le \frac{1}{\xi}\int_0^{\infty}\omega_r(f,t)e^{-t/\xi}dt, \quad n = 0. \tag{29}$$

Remark 2. The next regards the convergence of operators $P_{r,\xi}$. From (10) we have

$$G_n(t) \le \omega_r(f^{(n)}, |t|)\int_0^{|t|}\frac{(|t|-w)^{n-1}}{(n-1)!}dw,$$

i.e.

$$G_n(t) \le \frac{|t|^n}{n!}\omega_r(f^{(n)}, |t|). \tag{30}$$

Furthermore from (28) and (30) we get

$$\frac{1}{\xi}\int_0^\infty G_n(t)e^{-t/\xi}dt \le \frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)}, t)e^{-t/\xi}dt. \tag{31}$$

That is from (28) we get

$$\begin{aligned} K_1 &:= \left\| P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x)\delta_{2m}\xi^{2m} \right\|_{\infty,x} \\ &\le \frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)}, t)e^{-t/\xi}dt, \quad n \in \mathbb{N}. \end{aligned} \tag{32}$$

Using $\omega_r(f^{(n)}, t) \le t^r \|f^{(r+n)}\|_\infty$, $t > 0$ we get

$$\begin{aligned} \frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)}, t)e^{-t/\xi}dt &\le \frac{\|f^{(r+n)}\|_\infty}{\xi n!}\int_0^\infty t^{n+r}e^{-t/\xi}dt \\ &= \frac{\|f^{(r+n)}\|_\infty}{n!}\xi^{n+r}(n+r)! = \left(\prod_{i=1}^r (n+i)\right)\|f^{(r+n)}\|_\infty \xi^{n+r}. \end{aligned}$$

I.e.

$$\frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)}, t)e^{-t/\xi}dt \le \left(\prod_{i=1}^r (n+i)\right)\|f^{(r+n)}\|_\infty \xi^{n+r}. \tag{33}$$

That is for $f \in C^{n+r}(\mathbb{R})$ we have

$$K_1 \le \prod_{i=1}^r (n+i)\|f^{(r+n)}\|_\infty \xi^{n+r}, \quad n \in \mathbb{N}. \tag{34}$$

Here is assumed that $\|f^{(r+n)}\|_\infty$ is finite.

One may use also that

$$\omega_r(f^{(n)}, t) \le 2^r \|f^{(n)}\|_\infty.$$

Then

$$\begin{aligned} \frac{1}{\xi n!}\int_0^\infty t^n \omega_r(f^{(n)}, t)e^{-t/\xi}dt &\le \frac{2^r\|f^{(n)}\|_\infty}{\xi n!}\int_0^\infty t^n e^{-t/\xi}dt \\ &= 2^r \|f^{(n)}\|_\infty \xi^n. \end{aligned} \tag{35}$$

I.e.

$$K_1 \le 2^r \|f^{(n)}\|_\infty \xi^n, \quad n \in \mathbb{N}. \tag{36}$$

Here is assumed that $\|f^{(n)}\|_\infty < \infty$. Clearly from (34) or (36) as $\xi \to 0$ we obtain that $P_{r,\xi} \to$ unit operator I pointwise as $\xi \to 0$ with rates, $n \in \mathbb{N}$.

Next using $\omega_r(f, \lambda t) \leq (\lambda + 1)^r \omega_r(f, t)$, $\lambda, t > 0$, we get from (29) that

$$\begin{aligned}\frac{1}{\xi}\int_0^\infty \omega_r(f,t)e^{-t/\xi}dt &= \frac{1}{\xi}\int_0^\infty \omega_r\left(f, \xi\left(\frac{t}{\xi}\right)\right)e^{-t/\xi}dt \\ &\leq \omega_r(f,\xi)\int_0^\infty \left(1+\frac{t}{\xi}\right)^r e^{-t/\xi}dt/\xi \\ &= \omega_r(f,\xi)\int_0^\infty (1+u)^r e^{-u}du \\ &= \omega_r(f,\xi)\left(\sum_{k=0}^r \binom{r}{k} k!\right).\end{aligned}$$

I.e. we find for the case $n = 0$, see (29), that

$$\|P_{r,\xi}(f) - f\|_\infty \leq \left(\sum_{k=0}^r \binom{r}{k} k!\right)\omega_r(f,\xi). \tag{37}$$

Here is assumed that $\omega_r(f, \xi) < \infty$. Now as $\xi \to 0$ we obtain

$$P_{r,\xi} \xrightarrow{u} I \text{ with rates}, n = 0.$$

Note 1. The operators $P_{r,\xi}$ *are not in general positive* and they are *of convolution type.*

Let $r = 2$, $n = 3$. Then $\alpha_0 = \frac{23}{8}$, $\alpha_1 = -2$, $\alpha_2 = \frac{1}{8}$. Consider $f(t) = t^2 \geq 0$ and $x = 0$. Then

$$P_{r,\xi}(t^2; 0) = -3\xi^2 < 0.$$

Next using Geometric Moment theory methods [10], [3] we find best upper bounds for the right hand side of (12) and (24).

Theorem 3. *Let ψ be a continuous and strictly increasing function on $\mathbb{R}_+$ such that $\psi(0) = 0$, and let*

$$\psi^{-1}\left(\frac{1}{\xi}\int_{\mathbb{R}_+} \psi(t)e^{-t/\xi}dt\right) =: d_\xi > 0, \quad \xi > 0. \tag{38}$$

Assume $H_n := G_n \circ \psi^{-1}$ is concave on $\mathbb{R}_+$, $n \in \mathbb{Z}^+$. Then we obtain the best upper bound

$$\frac{1}{\xi}\int_{\mathbb{R}_+} G_n(t)e^{-t/\xi}dt \leq G_n(d_\xi). \tag{39}$$

Corollary 3. *Consider the upper concave envelope $H_n^*(u)$ of $H_n(u)$. We find the best upper bound*

$$\frac{1}{\xi}\int_{\mathbb{R}_+} G_n(t)e^{-t/\xi}dt \leq H_n^*(\psi(d_\xi)), \quad n \in \mathbb{Z}_+. \tag{40}$$

Note 2. When H_n, $n \in \mathbb{Z}_+$ is concave, then $H_n^*(\psi(d\xi)) = G_n(d_\xi)$.

Proof of Theorem 3. Here H_n is concave by assumption. It follows from the moment method of optimal distance [10], [3] that

$$\sup_{\mu \in \{\text{probability measures as in (38)}\}} \int_{\mathbb{R}_+} G_n(t)\mu(dt) = G_n(d_\xi).$$

Here is assumed that the last integrals are finite. Since by concavity of H_n the set

$$\Gamma_1 := \{(u, H_n(u)) \colon 0 \leq u < \infty\}$$

describes the upper boundary of the convex hull conv Γ_0 of the curve

$$\Gamma_0 := \{(\psi(t), G_n(t)) \colon 0 \leq t < \infty\}.$$

Notice here that $\frac{1}{\xi} e^{-t/\xi} dt$ is a probability measure on $\mathbb{R}_+$. ■

The fact that H_n can be a concave function is not strange at all, see [3], p. 310, Lemma 9.2.1(i) which we adjust here. Let g be a general modulus of smoothness function and consider

$$\tilde{G}_n(y) := \int_0^{|y|} \frac{(|y| - t)^{n-1}}{(n-1)!} g(t)dt, \tag{41}$$

all $y \in \mathbb{R}$, $n \in \mathbb{N}$.

Then we have

Lemma 1. *Let $\psi \in C^n((0,\infty))$ such that $\psi^{(k)}(0) \leq 0$, for $k = 1, \dots, n-1$ and $g(y)/\psi^{(n)}(y)$ is non-increasing, whenever $\psi^{(n)}(y) > 0$. Then $\tilde{H}_n := \tilde{G}_n \circ \psi^{-1}$ is a concave function, $n \in \mathbb{N}$.*

For the right hand side of inequality (12) we find the following simple upper bound without any special assumptions.

Theorem 4. *Call*

$$\tau_\xi := \xi((n+1)!)^{1/n+1}, \quad n \in \mathbb{N},\ \xi > 0, \tag{42}$$

which the same as

$$\left(\frac{1}{\xi} \int_{\mathbb{R}_+} y^{n+1} e^{-y/\xi} dy\right)^{1/n+1} = \tau_\xi. \tag{43}$$

Let

$$G_n^*(y) := \int_0^{|y|} \frac{(|y| - t)^{n-1}}{(n-1)!} \omega_1(f^{(n)}, t)dt, \tag{44}$$

all $y \in \mathbb{R}$, where $\omega_1(f^{(n)}, t)$ is the first modulus of continuity of $f^{(n)}$ and is finite, $f \in C^n(\mathbb{R})$. Assume also that

$$\int_{\mathbb{R}_+} G_n^*(y) e^{-y/\xi} dy < \infty.$$

Then

$$\frac{1}{\xi}\int_{\mathbb{R}_+} G_n(y)e^{-y/\xi}dy \le 2^r G_n^*(\tau_\xi), \quad r \in \mathbb{N}. \tag{45}$$

Proof. We have $\omega_r(f^{(n)}, |y|) \le 2^{r-1}\omega_1(f^{(n)}, |y|)$, for all $y \in \mathbb{R}$, see [7], p. 45. Furthermore by [7], p. 43 we get

$$\omega_1(f^{(n)}, |y|) \le \overline{\omega}_1(|y|) \le 2\omega_1(f^{(n)}, |y|),$$

for all $y \in \mathbb{R}$, where $\overline{\omega}_1$ is the least concave majorant of ω_1.

Thus

$$\omega_r(f^{(n)}, |y|) \le 2^{r-1}\overline{\omega}_1(|y|) \le 2^r\omega_1(f^{(n)}, |y|),$$

for all $y \in \mathbb{R}$. Set

$$\overline{G}_n(y) := \int_0^{|y|} \frac{(|y|-t)^{n-1}}{(n-1)!}\overline{\omega}_1(t)dt,$$

for all $y \in \mathbb{R}$. Hence

$$\begin{aligned} G_n(y) = \int_0^{|y|} \frac{(|y|-t)^{n-1}}{(n-1)!}\omega_r(f^{(n)}, t)dt &\le 2^{r-1}G_n^*(y) \\ \le 2^{r-1}\overline{G}_n(y) \le 2^r G_n^*(y), &\quad \text{for all } y \in \mathbb{R}. \end{aligned}$$

The function $\psi(y) = y^{n+1}$ on $\mathbb{R}_+$ is continuous, strictly increasing and $\psi(0) = 0$. And $\psi^{(n)}(y) = (n+1)!y > 0$, for all $y \in \mathbb{R}_+ - \{0\}$, along with $\psi^{(k)}(0) = 0$, $k = 1, \ldots, n-1$. Since $\overline{\omega}_1(y)$ is concave on $\mathbb{R}_+$, this implies $\overline{\omega}_1(y)/y$ is decreasing in $y > 0$, so that $\overline{\omega}_1(y)/\psi^{(n)}(y)$ is decreasing on $(0, \infty)$.

Thus by Lemma 1 we get that $\overline{H}_n := \overline{G}_n \circ \psi^{-1}$ is a concave function on $\mathbb{R}_+$; and by Theorem 3 we obtain

$$\frac{1}{\xi}\int_0^\infty \overline{G}_n(y)e^{-y/\xi}dy \le \overline{G}_n(\tau_\xi)$$

giving us

$$\begin{aligned} \frac{1}{\xi}\int_{\mathbb{R}_+} G_n(y)e^{-y/\xi}dy &\le 2^{r-1}\frac{1}{\xi}\int_{\mathbb{R}_+} \overline{G}_n(y)e^{-y/\xi}dy \\ &\le 2^{r-1}\overline{G}_n(\tau_\xi) \le 2^r G_n^*(\tau_\xi). \end{aligned}$$

The proof of the claim is now finished. ∎

A related convergence theorem follows.

Theorem 5. *Let $f \in C(\mathbb{R})$ with $\omega_1(f, y)$ finite, $y > 0$. Then*

$$\|P_{r,\xi}(f) - f\|_\infty \le 2^r\omega_1(f, \xi). \tag{46}$$

I.e. as $\xi \to 0$ we get again $P_{r,\xi} \xrightarrow{u} I$, $n = 0$.

Proof. Notice
$$\frac{1}{\xi}\int_{\mathbb{R}_+} ye^{-y/\xi}dy = \xi. \tag{47}$$

We have again
$$\omega_r(f,|y|) \le 2^{r-1}\omega_1(f,|y|), \quad \forall y \in \mathbb{R},$$

see [7], p. 45. Furthermore
$$\omega_1(f,|y|) \le \overline{\omega}_1(|y|) \le 2\omega_1(f,|y|) \quad \forall y \in \mathbb{R},$$

where $\overline{\omega}_1$ is the least concave majorant of ω_1, see [7], p. 43. Thus
$$\omega_r(f,|y|) \le 2^{r-1}\overline{\omega}_1(|y|) \le 2^r\omega_1(f,|y|), \quad \forall y \in \mathbb{R}.$$

Notice that for $n = 0$ we get
$$\begin{aligned}
|P_{r,\xi}(f;x) - f(x)| &= \frac{1}{2\xi}\left|\int_{\mathbb{R}}\left(\sum_{j=0}^{r}\alpha_j(f(x+jt) - f(x))\right)e^{-|t|/\xi}dt\right| \\
&\overset{(24)}{\le} \frac{1}{\xi}\int_0^\infty \omega_r(f,y)e^{-y/\xi}dy \\
&\le \frac{2^{r-1}}{\xi}\int_0^\infty \overline{\omega}_1(y)e^{-y/\xi}dy.
\end{aligned}$$

The probability measure $\frac{1}{\xi}e^{-y/\xi}dy$ fulfills (47). By moment theory [10], [3] we get
$$\sup_{\mu\in\{\text{probability measures as in (47)}\}}\int_{\mathbb{R}_+}\overline{\omega}_1(y)\mu(dy) = \overline{\omega}_1(\xi) \le 2\omega_1(f,\xi).$$

Hence
$$|P_{r,\xi}(f;x) - f(y)| \le 2^{r-1}\cdot 2\omega_1(f,\xi) = 2^r\omega_1(f,\xi).$$

∎

In the next we consider $f \in C^n(\mathbb{R})$, $n \ge 2$ even and the simple *smooth singular operator of symmetric convolution type*
$$P_\xi(f,x_0) := \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x_0+y)e^{-|y|/\xi}dy, \quad \text{for all } x_0 \in \mathbb{R},\ \xi > 0. \tag{48}$$

That is
$$P_\xi(f;x_0) = \frac{1}{2\xi}\int_0^\infty \big(f(x_0+y)+f(x_0-y)\big)e^{-y/\xi}dy, \quad \text{for all } x_0 \in \mathbb{R},\ \xi > 0. \tag{48*}$$

We assume that f is such that
$$P_\xi(f;x_0) \in \mathbb{R}, \quad \forall x_0 \in \mathbb{R}, \xi > 0 \quad \text{and} \quad \omega_2(f^{(n)},h) < \infty,\ h > 0.$$

Note that $P_{1,\xi} = P_\xi$ and if $P_\xi(f;x_0) \in \mathbb{R}$ then $P_{r,\xi}(f;x_0) \in \mathbb{R}$. Let the central second order difference

$$(\tilde{\Delta}_y^2 f)(x_0) := f(x_0+y) + f(x_0-y) - 2f(x_0). \tag{49}$$

Notice that

$$(\tilde{\Delta}_{-y}^2 f)(x_0) = (\tilde{\Delta}_y^2 f)(x_0).$$

Using Taylor's formula with Cauchy remainder we eventually obtain

$$(\tilde{\Delta}_y^2 f)(x_0) = 2\sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} y^{2\rho} + \mathcal{R}_1, \tag{50}$$

where

$$\mathcal{R}_1 := \int_0^y (\tilde{\Delta}_t^2 f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt. \tag{51}$$

Notice that

$$P_\xi(f;x_0) - f(x_0) = \frac{1}{2\xi}\int_0^\infty (\tilde{\Delta}_y^2 f(x_0)) e^{-y/\xi} dy. \tag{52}$$

So immediately we get

Proposition 1. *Assume $\omega_2(f,h) < \infty$, $h > 0$. Then it holds*

$$|P_\xi(f;x_0) - f(x_0)| \le \frac{1}{2\xi}\int_0^\infty w_2(f,y) e^{-y/\xi} dy. \tag{53}$$

Hence

$$\|P_\xi(f) - f\|_\infty \le \frac{1}{2\xi}\int_0^\infty w_2(f,y) e^{-y/\xi} dy. \tag{54}$$

Furthermore we observe by (50) and (52) that

$$\begin{aligned} P_\xi(f;x_0) - f(x_0) &= \frac{1}{2\xi}\int_0^\infty \Bigg(2\sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} y^{2\rho} \\ &\quad + \int_0^y (\tilde{\Delta}_t^2 f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt \Bigg) e^{-y/\xi} dy \\ &= \sum_{\rho=1}^{n/2} f^{(2\rho)}(x_0) \xi^{2\rho} \\ &\quad + \frac{1}{2\xi}\int_0^\infty \left(\int_0^y (\tilde{\Delta}_t^2 f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy. \end{aligned}$$

Clearly we got the representation

$$\begin{aligned} K_2(x_0) &= P_\xi(f;x_0) - f(x_0) - \sum_{\rho=1}^{n/2} f^{(2\rho)}(x_0)\xi^{2\rho} \\ &= \frac{1}{2\xi}\int_0^\infty \left(\int_0^y (\tilde{\Delta}_t^2 f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!} dt \right) e^{-y/\xi} dy. \end{aligned} \tag{55}$$

Therefore

$$|K_2(x_0)| \le \frac{1}{2\xi}\int_0^{\infty}\left(\int_0^y |\tilde{\Delta}_t^2 f^{(n)}(x_0)|\frac{(y-t)^{n-1}}{(n-1)!}dt\right)e^{-y/\xi}dy$$
$$\le \frac{1}{2\xi}\int_0^{\infty}\left(\int_0^y \omega_2(f^{(n)},t)\frac{(y-t)^{n-1}}{(n-1)!}dt\right)e^{-y/\xi}dy.$$

We have proved that

Theorem 6. *Let $f \in C^n(\mathbb{R})$, n even, $P_\xi(f)$ real valued. Then*

$$|K_2(x_0)| \le \frac{1}{2\xi}\int_0^{\infty}\left(\int_0^y w_2(f^{(n)},t)\frac{(y-t)^{n-1}}{(n-1)!}dt\right)e^{-y/\xi}dy$$
$$\le \frac{1}{2\xi n!}\int_0^{\infty}\omega_2(f^{(n)},y)y^n e^{-y/\xi}dy. \tag{56}$$

Remark 3. The operators P_ξ are positive operators. From (54) we obtain

$$\frac{1}{2\xi}\int_0^{\infty}\omega_2(f,y)e^{-y/\xi}dy = \frac{1}{2\xi}\int_0^{\infty}\omega_2\left(f,\xi\left(\frac{y}{\xi}\right)\right)e^{-y/\xi}dy$$
$$\le \frac{1}{2\xi}\omega_2(f,\xi)\int_0^{\infty}\left(1+\frac{y}{\xi}\right)^2 e^{-y/\xi}dy = \frac{5}{2}\omega_2(f,\xi).$$

I.e.

$$\|P_\xi(f)-f\|_\infty \le \frac{5}{2}\omega_2(f,\xi), \quad \xi > 0. \tag{57}$$

Acting similarly on the last part of inequality (56) it leads us to obtain

$$\|K_2\|_\infty \le \left(\frac{n^2+5n+5}{2}\right)\omega_2(f^{(n)},\xi)\xi^n, \quad \xi > 0. \tag{58}$$

Then from the inequality (57) as $\xi \to 0$ we obtain $P_\xi \xrightarrow{u} I$ with rates. And we get the pointwise convergence of $P_\xi \to I$ with rates from inequality (58). Call here for $n \ge 2$ even

$$T_n(y) := \int_0^y \omega_2(f^{(n)},t)\frac{(y-t)^{n-1}}{(n-1)!}dt, \quad y \in \mathbb{R}_+. \tag{59}$$

Then by (56) and (59) we have

$$|K_2(x_0)| \le \frac{1}{2\xi}\int_0^{\infty}T_n(y)e^{-y/\xi}dy, \tag{60}$$

and

$$\|K_2\|_\infty \le \frac{1}{2\xi}\int_0^{\infty}T_n(y)e^{-y/\xi}dy. \tag{61}$$

We set also

$$T_0(y) := \omega_2(y), \quad y > 0.$$

Optimality of Theorem 6 follows.

Proposition 2. *The first inequality of (56) is sharp, namely attained at $x_0 = 0$ by*

$$f_*(y) := \frac{|y|^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)}, \quad 0<\alpha\le 2,\ y\in\mathbb{R},\ n \text{ even.} \tag{62}$$

Proof. See that $f_*^{(n)}(y) = |y|^\alpha$ and by Proposition 9.1.1, p. 298 of [3], [2] we get $\omega_2(f_*^{(n)}, |y|) = 2|y|^\alpha$. Also $f_*^{(k)}(0) = 0$, $k = 0,\dots,n$. Then

$$\begin{aligned}
K_2(0) = P_\xi(f_*;0) &= \frac{1}{\xi}\int_0^\infty \frac{y^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)} e^{-y/\xi}dy \\
&= \frac{\xi^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)}\int_0^\infty x^{\alpha+n}e^{-x}dx = \frac{\xi^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)}\Gamma(\alpha+n+1) \\
&= \frac{\xi^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)}\left(\prod_{i=1}^{n}(\alpha+i)\right)\Gamma(\alpha+1) = \Gamma(\alpha+1)\xi^{\alpha+n}.
\end{aligned}$$

I.e.

$$K_2(0) = \Gamma(\alpha+1)\xi^{\alpha+n} > 0.$$

On the other hand we see that

$$\begin{aligned}
&\frac{1}{2\xi}\int_0^\infty \left(\int_0^y \omega_2(f_*^{(n)}, t)\frac{(y-t)^{n-1}}{(n-1)!}dt\right)e^{-y/\xi}dy \\
&= \frac{1}{2\xi(n-1)!}\int_0^\infty\left(\int_0^y (y-t)^{n-1}2t^\alpha dt\right)e^{-y/\xi}dy \\
&= \frac{1}{\xi(n-1)!}\int_0^\infty\left(\int_0^y (y-t)^{n-1}(t-0)^{(\alpha+1)-1}dt\right)e^{-y/\xi}dy \\
&= \frac{\xi^{n+\alpha}}{(n-1)!}\int_0^\infty\left(\frac{\Gamma(n)\Gamma(\alpha+1)}{(\Gamma(n+\alpha+1)}\left(\frac{y}{\xi}\right)^{n+\alpha}\right)e^{-y/\xi}\frac{dy}{\xi} \\
&= \frac{\xi^{n+\alpha}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}\int_0^\infty x^{n+\alpha}e^{-x}dx = \xi^{n+\alpha}\Gamma(\alpha+1).
\end{aligned}$$

That is proving equality in the first part of inequality (56). ■

It follows the optimality of inequality (53).

Proposition 3. *Inequality (53) is attained by $f^*(y) = |y|^\alpha$, $y\in\mathbb{R}$, $0<\alpha\le 2$ at $x_0 = 0$.*

Proof. We notice that

$$P_\xi(f^*;0) = \frac{1}{\xi}\int_0^\infty y^\alpha e^{-y/\xi}dy = \xi^\alpha\Gamma(\alpha+1) > 0.$$

Also we see again by Proposition 9.1.1, p. 298, [3], [2] that

$$\frac{1}{2\xi}\int_0^{\infty}\omega_2(f^*,y)e^{-y/\xi}dy=\frac{1}{\xi}\int_0^{\infty}y^{\alpha}e^{-y/\xi}dy.$$

That is proving equality to (53). ■

Next we present a Lipschitz type of related optimal result.

Theorem 7. *Let $n\geq 2$ even and $f\in C^n(\mathbb{R})$ such that*

$$\omega_2(f^{(n)},|y|)\leq 2A|y|^{\alpha},\quad 0<\alpha\leq 2,\quad A>0.$$

Then for $x_0\in\mathbb{R}$ we have

$$\left|P_{\xi}(f;x_0)-f(x_0)-\sum_{\rho=1}^{n/2}f^{(2\rho)}(x_0)\xi^{2\rho}\right|\leq\Gamma(\alpha+1)A\xi^{n+\alpha}. \tag{63}$$

Inequality (63) is sharp, namely it is attained at $x_0=0$ by

$$f_*(y)=\frac{A|y|^{\alpha+n}}{\prod_{i=1}^{n}(\alpha+i)}.$$

Proof. For $y>0$ we see that

$$\begin{aligned}T_n(y)&=\int_0^y\omega_2(f^{(n)},t)\frac{(y-t)^{n-1}}{(n-1)!}dt\\&\leq\int_0^y 2At^{\alpha}\frac{(y-t)^{n-1}}{(n-1)!}dt=\frac{2Ay^{n+\alpha}}{\prod_{i=1}^{n}(\alpha+i)}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2\xi}\int_0^{\infty}T_n(y)e^{-y/\xi}dy&\leq\frac{A}{\xi\prod_{i=1}^{n}(\alpha+i)}\int_0^{\infty}y^{n+\alpha}e^{-y/\xi}dy\\&=\frac{A\xi^{n+\alpha}}{\prod_{i=1}^{n}(\alpha+i)}\Gamma(n+\alpha+1)=\Gamma(\alpha+1)A\xi^{n+\alpha}.\end{aligned}$$

Using (60) we have proved (63).

Notice that $f_*^{(n)}(y)=A|y|^{\alpha}$, and by Proposition 9.1.1, p. 298, [3], [2] we get that

$$\omega_2(f_*^{(n)},|y|)=2A|y|^{\alpha}.$$

Also $f_*^{(k)}(0)=0$, $k=0,\dots,n$. Then $K_2(0)=\Gamma(\alpha+1)A\xi^{\alpha+n}>0$. That is proving equality to (63). ■

Let $f \in C^n(\mathbb{R})$ be such that $\omega_2(f^{(n)}, |t|) \le g(t)$, where g is given arbitrary, bounded, even, positive function and Borel measurable. We consider the even function

$$\hat{T}_n(y) := \int_0^y g(t)\frac{(y-t)^{n-1}}{(n-1)!}dt, \quad y \in \mathbb{R}. \tag{64}$$

Theorem 8. *Let ψ be a function on $\mathbb{R}_+$ such that $\psi(0) = 0$, which is continuous and strictly increasing. Assume that*

$$\psi^{-1}\left(\frac{1}{\xi}\int_0^\infty \psi(y)e^{-y/\xi}dy\right) = d_\xi > 0. \tag{65}$$

Suppose ($n \ge 2$ even) that $M_n(u) := \hat{T}_n(\psi^{-1}(u))$ is concave on $\mathbb{R}_+$. Then for any $x_0 \in \mathbb{R}$ we get

$$|K_2(x_0)| \le \frac{1}{2}\hat{T}_n(d_\xi). \tag{66}$$

Proof. Here we are applying geometric moment theory, see [10], [3]. Notice that

$$\sup_{\mu \in (\mu \text{ be probability measures as in (65))}} \int_0^\infty \hat{T}_n(y)\mu(dy) = \hat{T}_n(d_\xi).$$

Since by the concavity of M_n, the set

$$\Gamma_1 := \{(u, M_n(u)) \colon 0 \le u < \infty\}$$

is the upper boundary of the convex hull of the curve

$$\Gamma_0 := \{(\psi(y), \hat{T}_n(y)) \colon 0 \le y < \infty\}.$$

Now theorem follows from (59) and (60). ∎

A more general result follows.

Theorem 9. *All here as in Theorem 8, but we consider now M_n^*, the upper concave envelope of the not necessarily concave M_n. Then*

$$|K_2(x_0)| \le \frac{1}{2}M_n^*(\psi(d_\xi)), \quad \forall x_0 \in \mathbb{R}. \tag{67}$$

If M_n is concave then

$$\text{R.H.S.}(67) = \frac{1}{2}\hat{T}_n(d_\xi).$$

Let g be an arbitrary, continuous, even, positive function on $\mathbb{R}$ such that $g(0) = 0$. Let ψ be continuous, strictly increasing function on $\mathbb{R}_+$ with $\psi(0) = 0$ and $\hat{T}_n$ be as above, see (64).

Next we give sufficient conditions for $M_n = \hat{T}_n \circ \psi^{-1}$ to be concave on $\mathbb{R}_+$, $n \ge 2$ even. The result is similar to Theorem 9.1.3(ii), p. 302, [3], [2].

Theorem 10. *Assume $\psi \in C^n((0,\infty))$, $n \geq 2$ even, that satisfies*

$$\psi^{(k)}(0) \leq 0, \quad \text{for } k = 0, \dots, n-1.$$

Suppose, further that $g(y)/\psi^{(n)}(y)$ is non-increasing on each interval where $\psi^{(n)}$ is positive. Then $M_n = \hat{T}_n \circ \psi^{-1}$ is concave. In particular $\hat{T}_n(y)/\psi(y)$ is non-increasing.

Finally we give to both operators $P_{r,\xi}$, P_ξ some alternative kind of estimates.

Theorem 11. *Assuming $f \in C^n(\mathbb{R})$ and $\omega_r(f^{(n)}, \xi) < \infty$, $\xi > 0$, $n \in \mathbb{N}$ and G_n as in (10). Then*

$$\frac{1}{\xi}\int_0^\infty G_n(t)e^{-t/\xi}dt \leq \delta(\xi), \tag{68}$$

where

$$\delta(\xi) := \omega_r(f^{(n)}, \xi)\xi^n \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)} \left[\lfloor e(n+r)! \rfloor - \lfloor e(n-k-1)! \rfloor \right] \right\}. \tag{69}$$

I.e. from (32) we have

$$K_1 \leq \delta(\xi). \tag{70}$$

That is as $\xi \to 0$ we get again $P_{r,\xi} \to I$, pointwise with rates.

Proof. We observe that for $\xi > 0$

$$\omega_r(f^{(n)}, |w|) = \omega_r\left(f^{(n)}, \xi\left(\frac{|w|}{\xi}\right)\right) \leq \left(1 + \frac{|w|}{\xi}\right)^r \omega_r(f^{(n)}, \xi), \tag{71}$$

see [7], p. 45. Hence by (10) and (71) we see

$$\begin{aligned}
G_n(t) &\leq \frac{\omega_r(f^{(n)}, \xi)}{(n-1)!} \int_0^{|t|} (|t| - w)^{n-1} \left(1 + \frac{w}{\xi}\right)^r dw \\
&= \frac{\omega_r(f^{(n)}, \xi)}{\xi^r (n-1)!} \int_0^{|t|} (|t| - w)^{n-1} (w + \xi)^r dw \\
&= \frac{\omega_r(f^{(n)}, \xi)}{\xi^r (n-1)!} \int_\xi^{(\xi + |t|)} \big((\xi + |t|) - z\big)^{n-1} z^r dz \\
&= \frac{\omega_r(f^{(n)}, \xi)}{\xi^r (n-1)!} \left\{ \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (\xi + |t|)^{n-k-1} \int_\xi^{\xi + |t|} z^{k+r} dz \right\} \\
&= \frac{\omega_r(f^{(n)}, \xi)}{\xi^r} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(k+r+1)} \big[(\xi + |t|)^{n+r} \right. \\
&\quad \left. - \xi^{r+k+1} (\xi + |t|)^{n-k-1} \big] \right\}.
\end{aligned} \tag{72}$$

I.e. we get

$$G_n(t) \le \frac{\omega_r(f^{(n)},\xi)}{\xi^r}\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(k+r+1)}\left[(\xi+|t|)^{n+r}-\xi^{r+k+1}(\xi+|t|)^{n-k-1}\right]\right\}. \tag{73}$$

Hence

$$\begin{aligned}
\frac{1}{\xi}\int_0^\infty G_n(t)e^{-t/\xi}dt &\le \frac{\omega_r(f^{(n)},\xi)}{\xi^r} \\
&\cdot\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\int_0^\infty\big((\xi+t)^{n+r}\right. \\
&\left.-\,\xi^{r+k+1}(\xi+t)^{n-k-1}\big)e^{-t/\xi}d(t/\xi)\right\} \\
&= \frac{\omega_r(f^{(n)},\xi)}{\xi^r}\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\right. \\
&\left.\cdot\left[\xi^{n+r}\int_0^\infty(1+x)^{n+r}e^{-x}dx-\xi^{r+n}\int_0^\infty(1+x)^{n-k-1}e^{-x}dx\right]\right\} \\
&= \omega_r(f^{(n)},\xi)\xi^n\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\right. \\
&\left.\cdot\left[\int_0^\infty(1+x)^{n+r}e^{-x}dx-\int_0^\infty(1+x)^{n-k-1}e^{-x}dx\right]\right\} \\
&= \omega_r(f^{(n)},\xi)\xi^n\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\left[\sum_{j=0}^{n+r}\binom{n+r}{j}\int_0^\infty x^je^{-x}dx\right.\right. \\
&\left.-\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}\int_0^\infty x^je^{-x}dx\right] \\
&= \omega_r(f^{(n)},\xi)\xi^n\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\right. \\
&\cdot\left[\sum_{j=0}^{n+r}\binom{n+r}{j}j!-\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}j!\right] \\
&= \omega_r(f^{(n)},\xi)\xi^n\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(r+k+1)}\right.
\end{aligned}$$

$$\cdot \left[\sum_{j=0}^{n+r} \frac{(n+r)!}{(n+r-j)!} - \sum_{j=0}^{n-k-1} \frac{(n-k-1)!}{(n-k-1-j)!} \right]$$

$$= \omega_r(f^{(n)},\xi)\xi^n \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(r+k+1)} \right.$$

$$\left. \cdot \left[(n+r)! \sum_{j=0}^{n+r} \frac{1}{j!} - (n-k-1)! \sum_{j=0}^{n-k-1} \frac{1}{j!} \right] \right\} = \delta(\xi). \tag{74}$$

Use now

$$m! \sum_{j=0}^{m} \frac{1}{j!} = \lfloor em! \rfloor, \quad m \in \mathbb{N}. \tag{75}$$

That is proving (68). ■

The counterpart of the last theorem follows.

Theorem 12. *Assuming $f \in C^n(\mathbb{R})$, n even and $\omega_2(f^{(n)},\xi) < \infty$, $\xi > 0$, and T_n as in (59). Then*

$$\frac{1}{2\xi} \int_0^\infty T_n(y) e^{-y/\xi} dy \le \tau(\xi), \tag{76}$$

where

$$\tau(\xi) := \frac{1}{2}\omega_2(f^{(n)},\xi)\xi^n \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(k+3)} \right.$$

$$\left. \left[\lfloor e(n+2)! \rfloor - \lfloor e(n-k-1)! \rfloor \right] \right\}. \tag{77}$$

I.e. from (61) we find

$$\|K_2\|_\infty \le \tau(\xi). \tag{78}$$

That is as $\xi \to 0$ we obtain again $P_\xi \to I$, pointwise with rates.

Proof. We observe for $\xi > 0$ that

$$\omega_2(f^{(n)},t) \le \left(1 + \frac{t}{\xi}\right)^2 \omega_2(f^{(n)},\xi), \quad t > 0, \tag{79}$$

see [7], p. 45. And by (59) and (79), we have, $y > 0$, that

$$T_n(y) \le \frac{\omega_2(f^{(n)},\xi)}{\xi^2(n-1)!} \int_0^y (y-t)^{n-1}(t+\xi)^2 dt. \tag{80}$$

That is for $y > 0$ we obtain

$$T_n(y) \le \frac{\omega_2(f^{(n)},\xi)}{\xi^2} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-k-1)!(k+3)} \left[(\xi+y)^{n+2} - \xi^{k+3}(\xi+y)^{n-k-1} \right] \right\}. \tag{81}$$

Hence

$$\frac{1}{2\xi}\int_0^\infty T_n(y)e^{-y/\xi}dy \le \frac{1}{2}\omega_2(f^{(n)},\xi)\xi^n\left\{\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!(k+3)}\right.$$
$$\left.\cdot\left[(n+2)!\sum_{j=0}^{n+2}\frac{1}{j!}-(n-k-1)!\sum_{j=0}^{n-k-1}\frac{1}{j!}\right]\right\}=\tau(\xi). \tag{82}$$

Use at last (75). That is proving (76). ■

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Construction of Lévy Drivers for Financial Models

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Abstract

We extend the Lévy-Khintchine representation for an infinitely divisible distribution to define a driving process in the context of the bond price framework developed earlier. We describe a methodology using subordination to construct such processes and we develop some examples in detail.

Keywords semimartingales, finance, processes with independent increments

2000 AMS Subject Classification 60G51, 91B28, 91B70

1 Introduction

In our previous work [9] we have described the bond price process in terms of semimartingales where we used the characterization in terms of its set of characteristics. One advantage of this approach is that we can impose conditions needed for our results explicitly on the drift, diffusion, or jump components of the model. When the price dynamics is described by a diffusion with jumps driven by a Lévy process then the price itself is represented by a Lévy process. In this case the representation in terms of characteristics (for a fixed t) coincides with its Lévy-Khintchine representation.

In this paper, we first define a Lévy process to be used as driver for our financial model. To this end, we first construct an infinitely divisible distribution to describe the behavior of the increments. We then use a result that allows us to extend its Lévy-Khintchine representation to define the distribution of a Lévy process at each point in time. This extension is a special case of the set of characteristics which describes the process in terms of a semimartingale. Once this set is obtained then it may be used in our financial model since the price process (specified by its characteristics) was defined in terms of the characteristics of the driving process.

2 Summary of the General Model

This section is a summary of the framework for bond price dynamics in the context of a diffusion with jumps described in [9].

2.1 Introduction

We assume the canonical setting. Let $\mathsf{P}(t,T)$ be the price at time t of a bond which matures at time T. It is assumed that for each $T>0$, $(\{\mathsf{P}(t,T)\}_{0\le t\le T}$ is an optional, $\{\mathscr{F}_t\}$-adapted process, and for each t, $\mathsf{P}(t,T)$ is P-a.s. continuously differentiable in the T variable. Let $f(t,T)$ denote the T-*forward rate* at time t, defined by $f(t,T)=-\frac{\partial}{\partial T}\mathsf{P}(t,T)$. The *short rate* r is defined by $r_t=f(t,t)$, and the *money account process* B is defined by

$$\mathsf{B}_t=\exp\left(\int_0^t r_s ds\right).$$

In order to model the bond price dynamics we could start with a description of the forward rate or short rate dynamics. Alternatively, we could follow a direct approach, obtaining $\mathsf{P}(t,T)$ as the solution of a stochastic differential equation. Therefore, we are interested in studying dynamics of the following forms:

$$dr_t=a_t dt+b_t dW_t+\int_E q(t,x)\mu(dt,dx), \tag{1}$$

$$d\mathsf{P}(t,T)=\mathsf{P}(t-,T)\left\{m(t,T)dt+v(t,T)dW_t+\int_E n(t,x,T)\mu(dt,dx)\right\}, \tag{2}$$

$$df(t,T)=\alpha(t,T)dt+\sigma(t,T)dW_t+\int_E \delta(t,x,T)\mu(dt,dx). \tag{3}$$

The coefficients $b(t,T)$, $v(t,T)$, and $\sigma(t,T)$ are assumed to be m-dimensional row vector processes. The following technical assumptions will be needed:

ASSUMPTION

1. For any fixed $T > 0$, $n(t,x,T)$ and $\delta(t,x,T)$ are uniformly bounded. Furthermore, for each t,

$$\int_0^t \int_E h'(n(s,x,T))F(dx)ds < \infty,$$

where $h'(z) = |z|^2 \wedge |z|$ for $z \in \mathbb{R}$.

2. For each fixed ω, t, and (where appropriate) x, all the objects $m(t,T)$, $v(t,T)$, $n(t,x,T)$, $\alpha(t,T)$, $\sigma(t,T)$ and $\delta(t,x,T)$ are assumed to be continuously differentiable in the T-variable.

3. All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.

4. For any t the price curves $\mathsf{P}(\omega,t,T)$ are bounded functions for almost every ω.

Proposition 1. *If $f(t,T)$ satisfies (3), then $\mathsf{P}(t,T)$ satisfies*

$$d\mathsf{P}(t,T) = \mathsf{P}(t-,T)\Bigg[\left(r_t + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right)dt + S(t,T)dW_t + \int_E \left(e^{D(t,x,T)} - 1\right)\mu(dt,dx)\Bigg],$$

where

$$\begin{aligned} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds, \\ D(t,x,T) &= -\int_t^T \delta(t,x,s)ds. \end{aligned} \tag{4}$$

2.2 Bond Markets, Arbitrage

We now present the framework (Björk, Kabanov and Runggaldier [4]) in which we will state results concerning the absence of arbitrage in a model of bond prices. It will be assumed throughout that the filtration $\mathbf{F}$ is the natural filtration generated by W and μ.

A *portfolio* in the bond market is a pair (g,h), where

1. g is a predictable process.
2. For each ω, t, $h_t(\omega,\cdot)$ is a signed finite Borel measure on $[t,\infty)$.
3. For each Borel set A the process $h_t(A)$ is predictable.

The *discounted bond prices* $\overline{\mathsf{P}}(t,T)$ are defined by

$$\overline{\mathsf{P}}(t,T) = \frac{\mathsf{P}(t,T)}{\mathsf{B}_t}.$$

A portfolio (g,h) is said to be *feasible* if the following conditions hold for every t:

$$\int_0^t |g_s| ds < \infty, \qquad \int_0^t \int_s^\infty |m(s,T)||h_s(dT)| ds < \infty,$$

$$\int_0^t \int_s^\infty \int_E |n(s,x,T)||h_s(dT)|\nu(ds,dx) < \infty,$$

$$\text{and} \quad \int_0^t \left\{ \int_s^\infty |v(s,T)||h_s(dT)| \right\}^2 ds < \infty.$$

The *value process* corresponding to a feasible portfolio $\pi = (g,h)$ is defined by

$$V_t^\pi = g_t \mathsf{B}_t + \int_t^\infty \mathsf{P}(t,T) h_t(dT).$$

The *discounted value process* is

$$\overline{V_t^\pi} = \mathsf{B}_t^{-1} V_t^\pi.$$

A feasible portfolio is said to be *admissible* if there is a number $a \geq 0$ such that $V_t^\pi \geq -a$ P-a.s. for all t.

A feasible portfolio is said to be *self-financing* if the corresponding value process satisfies

$$\begin{aligned} V_t^\pi = V_0^\pi &+ \int_0^t g_s d\mathsf{B}_s + \int_0^t \int_s^\infty m(s,t)\mathsf{P}(s,t) h_s(dT) ds \\ &+ \int_0^t \int_s^\infty v(s,t)\mathsf{P}(s,t) h_s(dT) dW_s \\ &+ \int_0^t \int_s^\infty \int_E n(s,x,T)\mathsf{P}(s-,t) h_s(dT)\mu(ds,dx). \end{aligned}$$

The preceding relation can be interpreted formally as follows:

$$dV_t^\pi = g_t d\mathsf{B}_t + \int_t^\infty h_t(dT) d\mathsf{P}(t,T).$$

A *contingent T-claim* is a random variable $X \in L_+^0(\mathscr{F}_T, P)$. An *arbitrage portfolio* is an admissible self-financing portfolio $\pi = (g,h)$ such that the corresponding value process satisfies

1. $V_0^\pi = 0$
2. $V_T^\pi \in L_+^0(\mathscr{F}_T, P)$ with $P(V_T^\pi > 0) > 0$.

If no arbitrage portfolios exist for any $T > 0$ we say that the model is *arbitrage-free*.

Take the measure P as given. We say that a positive martingale $M = \{M_t\}_{t\geq 0}$ with $E^P(M_t) = 1$ for each t is a *martingale density* if for every $T > 0$ the process $\{\overline{\mathsf{P}}(t,T)M_t\}_{0\leq t\leq T}$ ia a P-local martingale. If, moreover, $M_t > 0$ for all $t > 0$ we say that M is a *strict martingale density*.

We say that that a probability measure Q on $(\Omega, \mathscr{F})$ is a *martingale measure* if $Q_t \sim P_t$ and the process $\{\overline{\mathsf{P}}(t,T)\}_{0\leq t\leq T}$ is a Q-local martingale for every $T > 0$. Here Q_t, P_t are the restrictions $Q_{|\mathscr{F}_t}$ and $P_{|\mathscr{F}_t}$, respectively.

Proposition 2. *Suppose that there exists a strict martingale density. Then the bond market model is arbitrage-free.*

We will make the following simplifying assumption:

Assumption For any positive martingale $N = \{N_t\}$ with $E^P(N_t) = 1$ there exists a probability measure Q on $\bigcup_{t\geq 0}\mathscr{F}_t$ such that $N_t = dQ_t/dP_t$.

The following results relate the coefficients in (2) and (3) with a model free of arbitrage.

Theorem 1. *Let the bond price dynamics be given by (2). There exists a martingale measure if and only if the following conditions hold:*

(i) There exists a predictable process ϕ and a $\tilde{\mathscr{P}}$-measurable function $Y(\omega,t,x)$ with $Y > 0$ satisfying

$$\int_0^t \|\phi_s\|^2 ds < \infty, \qquad \int_0^t \int_E |Y(s,x) - 1| F(dx) ds < \infty.$$

and such that $E^P(\mathscr{E}(L)_t) = 1$ for all finite t, where the process L is defined by

$$L = \phi \cdot W + (Y-1) * (\mu - \nu).$$

(ii) For all $T > 0$, and $t \in [0,T]$ we have

$$m(t,T) + \phi_t v(t,T)^T + \int_E Y(t,x)n(t,x,T)F(dx) = r_t. \tag{5}$$

The following theorem gives a similar result when we consider the forward rate dynamics.

Theorem 2. *Let the forward rate dynamics be given by (3). There exists a martingale measure if and only if the following conditions hold:*

(i) There exists a predictable process ϕ and a $\tilde{\mathscr{P}}$-measurable function $Y(\omega,t,x)$ with $Y > 0$ satisfying

$$\int_0^t \|\phi_s\|^2 ds < \infty, \qquad \int_0^t \int_E |Y(s,x) - 1| F(dx) ds < \infty.$$

and such that $E^P(\mathscr{E}(L)_t) = 1$ for all finite t, where the process L is defined by

$$L = \phi \cdot W + (Y-1) * (\mu - \nu).$$

(ii) For all $T > 0$, and $t \in [0,T]$ we have

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2 + \phi_t S(t,T)^T + \int_E Y(t,x)\left(e^{D(t,x,T)} - 1\right)F(dx) = 0,$$

where A, S and D are defined in (4).

3 Semimartingales with Independent Increments

In this short section we state a characterization of semimartingales with independent increments. These results will be used in the following section to establish the connection with Lévy processes.

Theorem 3. *Let X be a d-dimensional process with independent increments. Then X is also a semimartingale if and only if, for each $u \in \mathbb{R}^d$, the function $t \mapsto g(u)_t := E(\exp iu \cdot X_t)$ has finite variation over finite intervals.*

Theorem 4. *Let X be a d-dimensional semimartingale with $X_0 = 0$. Then it is a process with independent increments if and only if there is a version (B, C, ν) of its characteristics that is deterministic. Furthermore, in this case, with $J = \{t : \nu(\{t\} \times \mathbb{R}^d) > 0\}$ and for all $s \le t$, $u \in \mathbb{R}^d$ we have:*

$$\begin{aligned} E(e^{iu\cdot(X_t - X_s)}) = \exp\Bigg[& iu \cdot (B_t - B_s) - \frac{1}{2} u \cdot (C_t - C_s) \cdot u \\ & + \int_s^t \int_{\mathbb{R}^d} (e^{iu\cdot x} - 1 - iu \cdot h(x)) 1_{J^c}(r) \nu(dr, dx) \Bigg] \\ & \times \prod_{s<r\le t} \left\{ e^{-iu\cdot \Delta B_r} \left[1 + \int (e^{iu\cdot x} - 1) \nu(\{r\} \times dx) \right] \right\}. \end{aligned} \tag{6}$$

Corollary 1. *A d-dimensional semimartingale X is a process with stationary independent increments if and only if it is a semimartingale admitting a version (B, C, ν) of its characteristics that has the form*

$$B_t(\omega) = bt, \quad C_t(\omega) = ct, \quad \nu(\omega; dt, dx) = dt\, K(dx)$$

where $b \in \mathbb{R}^d$, c is a symmetric nonnegative $d \times d$ matrix, K is a positive measure on $\mathbb{R}^d$ that integrates $(|x|^2 \wedge 1)$ and satisfies $K(\{0\}) = 0$.

4 Construction of a Driving Process

In this section we develop a description of the type of processes we propose for financial applications. The approach is from specific to general. Infinitely divisible distributions extend quite naturally to additive and Lévy processes in law. Once a cádlág modification is chosen, this is seen to be a special case of our general approach in terms of semimartingales. We adopt the results and notation of K. Sato's beautiful book [18].

4.1 Infinitely Divisible Distributions

The class of infinitely divisible distributions arise naturally in a financial context. Below we define the class membership. Roughly speaking, a random variable follows an infinitely divisible distributions if it can be considered to be the sum of independent innovations. Asset returns, for example, are the accumulation of the returns accrued in non-overlapping time intervals. This class generalizes the Gaussian distribution to allow heavy tails and skewness (Shiryaev [21], Nolan

[15]), and is the only class that contains the limit distributions of sums of iid random variables.

A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^{n*}$, where μ^{n*} denotes the n-fold convolution of μ with itself.

We begin our discussion with the *Lévy-Khintchine representation* of the characteristic function $\widehat{\mu}(z) = \int e^{i\langle z,x\rangle}\mu(dx)$, $z \in \mathbb{R}^d$ of μ.

Theorem 5. *(i) Let $D = \{x \in \mathbb{R}^d : |x| \leq 1\}$. If μ is an infinitely divisible distribution on $\mathbb{R}^d$ then*

$$\begin{aligned}\widehat{\mu}(z) = \exp\Big[&-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle \\ &+ \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_D(x))\nu(dx)\Big], \quad z \in \mathbb{R}^d\end{aligned} \tag{7}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty. \tag{8}$$

(ii) The representation of $\hat{\mu}(z)$ in (i) by $\gamma \in \mathbb{R}^d$, A and ν is unique.

(iii) Conversely, if $\gamma \in \mathbb{R}^d$, A is a symmetric nonnegative-definite $d \times d$ matrix, and ν is a measure satisfying (8), then there is an infinitely divisible distribution μ whose characteristic function is given by (7).

As stated earlier, the motivation for describing a semimartingale in terms of characteristics was to generalize the *generating triplet* (γ, A, ν) for the infinitely divisible distribution μ. Here we start with an infinitely divisible distribution and develop a process and its characteristics in parallel, in order to adapt it to our general framework as a driving process.

The representation (7) can be rewritten in terms of another truncation function $c(x)$ in place of $1_D(x)$. Given a particular Lévy measure, we may be able to simplify the integrand in (7) by choosing an appropriate $c(x)$ while still ensuring that the integral is finite. In fact, if $c : \mathbb{R}^d \to \mathbb{R}$ is a measurable function such that

$$\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle c(x))\nu(dx) < \infty \tag{9}$$

for every $z \in \mathbb{R}^d$ then rearranging terms in (7) we obtain

$$\widehat{\mu}(z) = \exp\Big[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_c, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle c(x))\nu(dx)\Big], \tag{10}$$

with $\gamma_c \in \mathbb{R}^d$ defined by

$$\gamma_c = \gamma + \int_{\mathbb{R}^d} x(c(x) - 1_D(x))\nu(dx).$$

The representation (γ_c, A, ν) implied by (10) will also be called the generating triplet for μ. Note that the components A and ν are independent of the choice of c.

If $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ then (9) is satisfied with $c \equiv 0$ and we obtain the representation (γ_0, A, ν):

$$\widehat{\mu}(z) = \exp\Big[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\Big], \tag{11}$$

Likewise, if $\int_{|x|>1} |x|\nu(dx) < \infty$ we obtain the representation (γ_1, A, ν) from (10) with $c \equiv 1$.

4.2 Lévy Processes

An $\mathbb{R}^d$-valued stochastic process $\{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ is said to be an *additive process in law* if each of the following conditions hold.

1. X has the independent increments property.
2. $X_0 = 0$ a.s.
3. X is stochastically continuous.

An additive process in law with the stationary increments property is said to be a *Lévy process in law.* An additive (Lévy) process in law which is cádlág is called an *additive* (*Lévy*) *process.* An $\mathbb{R}$-valued increasing Lévy process is said to be a *subordinator.*

The following two results establish the correspondence between a family of infinitely divisible distributions and additive processes in law. Then the associated family of generating triplets offers a natural representation for the corresponding process. Later this will be seen to be a special case of the characteristics described in the context of semimartingales. However, we will need a restriction to ensure that an additive process is a semimartingale. In the case of a Lévy process, no restriction is needed.

Theorem 6. *(i) Let $\{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued additive process in law and, for $0 \leq s \leq t < \infty$, let $\mu_{s,t}$ be the distribution of $X_t - X_s$. Then $\mu_{s,t}$ is infinitely divisible and*

$$\mu_{s,t} * \mu_{t,u} = \mu_{s,u} \;\; for\ 0 \leq s \leq t \leq u < \infty,$$

$$\mu_{s,s} = \delta_0 \;\; for\ 0 \leq s < \infty,$$

$$\mu_{s,t} \to \delta_0 \;\; as\ s \uparrow t,$$

$$\mu_{s,t} \to \delta_0 \;\; as\ t \downarrow s.$$

(ii) Conversely, if $\{\mu_{s,t}\}_{0\leq s\leq t<\infty}$ is a system of probability measures on $\mathbb{R}^d$ satisfying the properties in (i), then there is an additive process in law $\{X_t\}_{t\geq 0}$ such that for $0 \leq s \leq t < \infty$, $X_t - X_s$ has the distribution $\mu_{s,t}$.

(iii) If $\{X_t\}$ and $\{X'_t\}$ are $\mathbb{R}^d$-valued additive processes in law such that $X_t \stackrel{\mathrm{d}}{=} X'_t$ for any $t \geq 0$, then $\{X_t\}$ and $\{X'_t\}$ are identical in law.

Theorem 7. *(i) Suppose that $\{X_t\}_{t\geq 0}$ is an $\mathbb{R}^d$-valued additive process in law. Let $(\gamma(t), A_t, \nu_t)$ be the generating triplet of the infinitely divisible distribution $\mu_t = P_{X_t}$ for $t \geq 0$. Then the following conditions are satisfied.*

(a) $\gamma(0) = 0$, $A_0 = 0$, $\nu_0 = 0$.

(b) If $0 \leq s \leq t < \infty$, then $\langle z, A_s z\rangle \leq \langle z, A_t z\rangle$ for $z \in \mathbb{R}^d$ and $\nu_s(B) \leq \nu_t(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$.

(c) As $s \to t$ in $[0,\infty)$, $\gamma(s) \to \gamma(t)$, $\langle z, A_s z\rangle \to \langle z, A_t z\rangle$ for $z \in \mathbb{R}^d$, and $\nu_s(B) \to \nu_t(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$ with $B \subset \{x : |x| > \epsilon\}$, $\epsilon > 0$.

(ii) Let $\{\mu_t\}_{t\geq 0}$ be a system of infinitely divisible probability measures on $\mathbb{R}^d$ with generating triplets $(\gamma(t), A_t, \nu_t)$ satisfying (1)-(3) Then there exists, uniquely up to identity in law, an $\mathbb{R}^d$-valued additive process in law such that $P_{X_t} = \mu_t$ for $t \geq 0$.

Let $\{X_t\}$ be an $\mathbb{R}^d$-valued additive process in law. Let (γ_t, A_t, ν_t) be its system of generating triplets. Construct the measure $\tilde{\nu}$ on $[0,\infty) \times \mathbb{R}^d$ such that

$$\tilde{\nu}([0,t] \times B) = \nu_t(B), \quad \text{for } t \geq 0 \text{ and } B \in \mathcal{B}(\mathbb{R}^d) \tag{12}$$

by defining a set function as in (12) on the field of sets $[0,t]\times B$ with $t\geq 0$ and $B\in\mathcal{B}(\mathbb{R}^d)$, and then extending to the σ-field which is equivalent to the Borel σ-field of $[0,\infty)\times\mathbb{R}^d$. By Theorem 7*(i)* and (12) it follows that the following statements hold.

$$\tilde{\nu}(\{t\}\times\mathbb{R}^d)=0 \quad \text{for } t\geq 0, \tag{13}$$

$$\int_{[0,t]\times\mathbb{R}^d}(1\wedge|x|^2)\tilde{\nu}(ds,dx)<\infty \quad \text{for } t\geq 0. \tag{14}$$

Conversely, if a measure $\tilde{\nu}$ satisfies (13) and (14) then for each $t\geq 0$, the Lévy measure ν_t defined by (12) satisfies the conditions in Theorem 7*(i)*.

The following result implies that we can choose a modification $\{X'_t\}$ of $\{X_t\}$ that is an additive process.

Theorem 8. *Let $\{X_t\}$ be an an $\mathbb{R}^d$-valued additive or Lévy process in law. Then it has a cádlág modification.*

Since our interest is in semimartingales, by virtue of Theorem 3 we require $\{X'_t\}$ to be such that the function $t\mapsto\widehat{P}_{X_t}$ has finite variation over finite intervals. Hence by Theorems 3 and 4 with $\tilde{\nu}$ in (6), we identify $\{X'_t\}$ to be the semimartingale with characteristics $(\gamma_t, A_t, \tilde{\nu}(ds,dx))$. Since we have defined processes in this section to be stochastically continuous, then the last term in (6) is equal to 1 and the set $J=\emptyset$. The same conclusion also follows from (13).

If the additive process $\{X'_t\}$ has the stationary increments property (i.e. a Lévy process), then the condition in Theorem 3 is satisfied and it follows from Corollary 1 that its set of characteristics is $(t\gamma, tA, t\nu_1(dx))$. Conversely, given an infinitely divisible distribution μ on $\mathbb{R}^d$ with generating triplet (γ, A, ν), define the system of measures $\{\mu_{s,t}\}_{0\leq s\leq t<\infty}$ by the system of generating triplets $((s-t)\gamma,(s-t)A,(s-t)\nu)$. It follows easily from the representation

$$\widehat{\mu}_{s,t}(z)=\exp\biggl[(t-s)\biggl(-\frac{1}{2}\langle z,Az\rangle+i\langle\gamma,z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle}-1-i\langle z,x\rangle 1_D(x))\nu(dx)\biggr)\biggr] \tag{15}$$

for $0\leq s\leq t<\infty$ and $z\in\mathbb{R}^d$, that the conditions listed in Theorem 6 are satisfied. Then there is an additive process Y such that Y_t-Y_s has distribution $\mu_{s,t}$ and which, in this case, has the stationary increments property. In the sequel we will construct Lévy processes by stipulating that its increments are described by a given infinitely divisible distribution.

4.3 Subordination of Lévy Processes

Now we shall construct a driving process by the method of subordination. This can be seen as a generalisation of our model by substituting the physical time, indexed by t, with an increasing non-negative Lévy process. The resulting process X_T is said to be *subordinated* to the *noise process* X by the *subordinator* T. In what follows we will specify a subordinator to be the Lévy process whose increments follow a given non-negative infinitely divisible distribution. Subordination can be interpreted as a transformation of the physical time to the "intrinsic time" of the underlying market. In other words, T will rescale the time axis to model periods of high or low business activity. $T(t)$ is interpreted as a measure of the cumulative trading volume up to the physical time t (Hurst, Platen, Rachev [10]).

Our goal is to construct the set of characteristics of the subordinated process in terms of the characteristics of the two component processes. Madan and Seneta [13] developed the *Variance Gamma* (VG) process by subordinating a Brownian motion with a gamma process for stock prices. Hurst, Platen and Rachev [10] used an $\alpha/2$-stable subordinator with a Brownian motion. These will be presented as examples of our methodology.

Rachev, Mittnik [16] studied the USD-CHF exchange rate using a subordinated model $Z_t = S(T_t)$. They gathered a data sample of $N = 128400$ spanning the period of 499 business days from 20 May 1985 to 20 May 1987. The average time between observations is 2 minutes, 6 seconds. Denote by $p_{\text{bid}}(t)$ and $p_{\text{ask}}(t)$, respectively, the bid and ask quote at time t for the exchange rate. For each $i = 1, \ldots, N$ the ith observation $x(t_i)$ is the logarithmic price at time t_i, defined by

$$x(t_i) = \frac{\log p_{\text{bid}}(t_i) + \log p_{\text{ask}}(t_i)}{2}.$$

Note that the set $\{x(t) : t \in \{t_i : 1 \leq i \leq N\}\}$ can be regarded as a sample path of the price process in *physical* time $\{(Z_t)\}$. On the other hand, $\{x(t_i) : 1 \leq i \leq N\}$ can be regarded as a sample path of the price process in *intrinsic* time $\{S(t)\}$. Define the *return* $r(t_i; \Delta t)$ *at time* t_i over the period Δt by

$$r(t_i; \Delta t) = x(t_i) - x(t_i - \Delta t).$$

Note that the quantities $x(t_i) - x(t_{i-k})$, i.e. the returns at *k-quote* frequency, can be interpreted as price change in physical time or as price change in intrinsic (quote) time. The probabilistic structure of the process $S(t)$ was studied by estimating the pdf of the returns in intrinsic time. A stable model with an estimated $\alpha = 1.716$ provided an excellent fit for the returns at the 4-quote frequency. Given the average time elapsed between quotes, the relationship $t_i - t_k \approx 2(i - k)$ for $i > k$

was used to study the processes T and Z at the corresponding physical time scale. Define the *market time process* $\hat{T}$ by

$$\hat{T}(t) = \sum_{i=1}^{N} 1_{[t_i,\infty)}(t), \quad t \geq 0.$$

Then $\hat{T}(t)$ is the number of transactions up to time t, and $\hat{T}(t_i) = i$. The estimated pdf for the 8-minute time increments $\hat{T}(t) - \hat{T}(t-8)$ was studied to determine a model for the process T. The Weibull distribution provided the best fit. The Gamma distribution, which is infinitely divisible, also offered a good fit. In both cases the process $\{S(T_t)\}$ subordinated to the α-stable process S can be described in terms of stable distributions. On the other hand, the price process in physical time Z was similarly studied for 8-minute increments and obtained a stable fit with $\alpha = 1.3745$.

For any Lévy process X in this section it will be assumed that for *every* ω, $X(\omega)$ is cádlág and $X_0(\omega) = 0$. Let X and T be independent Lévy process defined on a stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$. We begin by specifying the characteristics of a subordinator (see Sato [18]).

Theorem 9. *Let $\{T_t\}_{t\geq 0}$ be a subordinator with Lévy measure ρ, drift β_0, and let $\lambda = P_{Z_1}$. Its second characteristic is zero and its Laplace transform is given by*

$$E[e^{-uZ_t}] = \int_{[0,\infty)} e^{-us}\lambda^t(ds) = e^{t\Psi(-u)}, \qquad u \geq 0,$$

where for any complex w with $\mathrm{Re}\ w \leq 0$,

$$\Psi(w) = \beta_0 w + \int_{(0,\infty)} (e^{ws} - 1)\rho(ds)$$

with

$$\beta_0 \geq 0 \qquad \textit{and} \qquad \int_{(0,\infty)} (1 \wedge s)\rho(ds) < \infty.$$

Note that the theorem implies that a subordinator can only display jumps in the positive direction. This is obviously necessary, since we cannot go backwards in time. Moreover, the diffusion component has to be zero since otherwise there will be a negative change over any interval with positive probability.

The following result gives the characteristics of the subordinated process.

Theorem 10. *Let $\{T_t\}_{t\geq 0}$ be a subordinator with Lévy measure ρ, drift β_0, and $P_{T_1} = \lambda$. Let $\{X_t\}$ be an $\mathbb{R}^d$-valued Lévy process with generating triplet (γ, A, ν) and let $\mu = P_{X_1}$. Suppose that $\{X_t\}$ and $\{T_t\}$ are independent. Define*

$$Y(\omega) = X_{T_t(\omega)}(\omega), \qquad t \geq 0.$$

Then $\{Y_t\}$ is a Lévy process and

$$P[Y_t \in B] = \int_{[0,\infty)} \mu^s(B)\lambda^t(ds), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

The generating triplet (γ', A', ν') of $\{Y_t\}$ is as follows:

$$\begin{aligned} \gamma' &= \beta_0\gamma + \int_{(0,\infty)} \rho(ds) \int_{|x|\leq 1} x\mu^s(dx), \\ A' &= \beta_0 A, \\ \nu'(B) &= \beta_0\nu(B) + \int_{(0,\infty)} \mu^s(B)\rho(ds), \qquad B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\}). \end{aligned} \tag{16}$$

4.3.1 Example: Variance-Gamma Process

We will now apply the previous result to obtain the characteristics for the Variance Gamma (VG) process (Madan, Seneta [13]). To this end, we first introduce the subordinator T which we define as the Lévy process such that

$$T_{t+s} - T_t \sim \Gamma\left(\frac{s}{\mu}, \frac{1}{\mu}\right) \tag{17}$$

where $\Gamma(c, \alpha)$ is the *gamma*-distribution with density

$$\frac{\alpha^c}{\Gamma(c)} x^{c-1} e^{-\alpha x}, \ x > 0 \quad \text{for } c > 0, \ \alpha > 0. \tag{18}$$

Lemma 1. *The generating triplet for the $\Gamma(c, \alpha)$ distribution is $(0, 0, \rho)$, where the Lévy measure ρ is given by*

$$\rho(dx) = cx^{-1}e^{-\alpha x}dx, \ x > 0. \tag{19}$$

It follows that the Γ-*subordinator* $\{T_t\}$ has characteristics $(0, 0, t\rho)$, with $c = 1/\mu$ and $\alpha = 1/\mu$.

Proof. Let μ be the probability measure with density (18). Denote its Laplace transform by $L_\mu(u)$. Then

$$L_\mu(u) = \left(1+\frac{u}{\alpha}\right)^{-c}, \quad u \geq 0. \tag{20}$$

We will now see that

$$L_\mu(u) = \exp\left[c\int_0^\infty (e^{ux}-1)\frac{e^{-\alpha x}}{x}dx\right]. \tag{21}$$

In fact,

$$\begin{aligned}\log(1+\alpha^{-1}u) &= \int_0^u \frac{dy}{\alpha+y} = \int_0^u dy \int_0^\infty e^{-\alpha x - yx}dx \\ &= \int_0^\infty e^{-\alpha x}\left(\frac{e^{-ux}-1}{-x}\right)dx,\end{aligned}$$

so that (21) now follows from (20).

For $w \in \mathbb{C}$, define $\Phi(w) = \int_0^\infty e^{wx}\mu(dx)$. Observe that Φ is analytic on $\{\mathrm{Re}\, w < 0\}$, continuous on $\{\mathrm{Re}\, w \leq 0\}$ and equal to $L_\mu(u)$ for $w = -u < 0$. Then Φ can be extended such that

$$\Phi(w) = \exp\left[c\int_0^\infty (e^{wx}-1)\frac{e^{-\alpha x}}{x}dx\right], \quad \mathrm{Re}\, w \leq 0.$$

For $z \in \mathbb{R}$, it follows that

$$\hat{\mu}(z) = \Phi(iz) = \exp\left[c\int_0^\infty (e^{izx}-1)\frac{e^{-\alpha x}}{x}dx\right],$$

and that the generating triplet of μ is $(0, 0, \rho)$ with $\rho(dx)$ given by (19). □

We state the following result for future reference. Let K_ν denote the modified Bessel function of the third kind with index ν (see, e.g., Watson [22]).

Lemma 2. *(Watson [22], p.80, 183)*

$$K_p(x) = \frac{1}{2}\left(\frac{x}{2}\right)^p \int_0^\infty e^{-t-x^2/(4t)}t^{-p-1}dt, \quad x > 0,\, p \in \mathbb{R}, \tag{22}$$

$$K_{n+\frac{1}{2}}(x) = \sqrt{\pi/2}\, x^{-1/2}e^{-x}\left(1+\sum_{i=1}^{n}\frac{(n+i)!}{(n-i)!i!}(2x)^{-i}\right), \quad x > 0,\, n \in \mathbb{N}. \tag{23}$$

Let X be the process defined by $X_t = \sigma W_t + \theta t$ where W is a standard Brownian motion and $\sigma > 0$, $\theta \in \mathbb{R}$ are volatility and drift parameters, respectively. The *Variance Gamma process (VG)* is defined as the process Y subordinated to X by the Γ-subordinator T. Equivalently,

$$Y_t := X_{T(t)} = \sigma W_{T(t)} + \theta T(t).$$

By Theorem 10 the VG process has characteristics $(t\beta, 0, t\nu)$ for some $\beta \in \mathbb{R}$ and ν given by (16), which we compute as follows:

$$\begin{aligned}
\nu(dx) &= \int_0^\infty P^s_{\sigma W_1+\theta}(dx) c s^{-1} e^{-\alpha s} ds \\
&= \frac{c}{\sqrt{2\pi}\sigma} dx \int_0^\infty e^{-\frac{(s-\theta s)^2}{2\sigma^2 s}} s^{-3/2} e^{-\alpha s} ds \\
&= \frac{c}{\sqrt{2\pi}\sigma} e^{x\theta/\sigma^2} dx \int_0^\infty s^{-3/2} \exp\left[-\left(\alpha + \frac{\theta^2}{2\sigma^2}\right) s - \left(\frac{x^2}{2\sigma^2}\right)\frac{1}{s}\right] ds.
\end{aligned}$$

Using (22) and the change of variable $s' = \beta s$ with $\beta = \left(\alpha + \frac{\theta^2}{2\sigma^2}\right)$, the last integral is equal to

$$K_{\frac{1}{2}}\left(\sqrt{2x^2\beta/\sigma^2}\right) \left[\frac{1}{2}\left(\frac{\sqrt{2x^2\beta/\sigma^2}}{2}\right)^{1/2}\right]^{-1}.$$

Now using (23) with $n = 0$, it follows that

$$\nu(dx) = \frac{c}{|x|} e^{x\theta/\sigma^2} e^{-\frac{|x|}{\sigma}\sqrt{2\beta}} dx,$$

and substituting $c = 1/\mu$, $\alpha = 1/\mu$, we conclude that

$$\nu(dx) = \frac{1}{|x|\mu} \exp\left(\frac{x\theta}{\sigma^2} - \frac{|x|}{\sigma}\sqrt{\frac{2}{\mu} + \frac{\theta^2}{\sigma^2}}\right) dx, \quad -\infty < x < \infty.$$

4.3.2 Example: Subordination of Brownian Motion by $\alpha/2$-Stable

Using the same procedure, we now compute the characterization of the process subordinated to Brownian motion by the stable subordinator (Hurst, Platen, Rachev [10]). Define the subordinator T to be the Lévy process such that

$$T_{t+s} - T_t \sim S_{\alpha/2}(cs^{\alpha/2}, 1, 0), \quad c > 0,\ s, t \geq 0.$$

where $S_{\alpha/2}(cs^{\alpha/2}, 1, 0)$ is the $\alpha/2$-*stable distribution* (Samorodnitsky, Taqqu [17]) with characteristic function

$$\exp\Big\{-sc^{\alpha/2}|z|^{\alpha/2}\Big(1 - i\tan\Big(\frac{\pi\alpha}{4}\Big)\operatorname{sgn} z\Big)\Big\}, \quad z \in \mathbb{R}. \tag{24}$$

In order to obtain the set of characteristics for T, we will use the following results (Sato [18]).

Lemma 3. *Let μ be an infinite divisible distribution on $\mathbb{R}^d$ with characteristics (β, A, ν). Then μ is α-stable if and only if $A = 0$ and there is a finite measure λ on $S = \{x \in \mathbb{R}^d : |x| = 1\}$ such that*

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{25}$$

Lemma 4. *Let μ be a non-trivial α-stable distribution on $\mathbb{R}^d$ with $0 < \alpha < 2$ and Lévy measure ν. Then $\int_{\{|x|\le 1\}} |x|\nu(dx)$ is finite if and only if $\alpha < 1$. Also, $\int_{\{|x|>1\}} |x|\nu(dx)$ is finite if and only if $\alpha > 1$. The mass of ν is always infinite.*

Lemma 5. *The generating triplet of the $\alpha/2$-stable distribution defined in (24) is $(0, 0, \rho)$, where*

$$\rho(dr) = \frac{\lambda\, dr}{r^{1+\alpha/2}}, \quad r > 0 \tag{26}$$

with

$$\lambda = \frac{-c^{\alpha/2}}{\Gamma\left(-\frac{\alpha}{2}\right)\cos(\frac{\alpha\pi}{4})}. \tag{27}$$

It follows that the $\alpha/2$-*stable subordinator* $\{T_t\}$ has characteristics $(0, 0, t\rho)$.

Proof. In what follows, the Γ-function is extended from $(0, \infty)$ to any $s \in \mathbb{R}$ with $s \neq 0, -1, -2, \cdots$ by $\Gamma(s + 1) = s\Gamma(s)$. The following auxiliary result will be used:

$$\int_0^\infty (e^{wr} - 1)\frac{dr}{r^{1+\alpha'}} = \Gamma(-\alpha')(-w)^{\alpha'} \quad \text{for } \alpha' \in (0, 1), \tag{28}$$

which is valid for $w \neq 0$ complex such that $\operatorname{Re} w \le 0$. Indeed, both sides of (28) are analytic on $\{w : \operatorname{Re} w < 0\}$ and continuous on $\{w : \operatorname{Re} w \leq 0,\, w \neq 0\}$. Since

$$\begin{aligned}\int_0^\infty (e^{-ur} - 1)\frac{dr}{r^{1+\alpha'}} &= -\int_0^\infty \int_0^r u\, e^{-uy} dy\, \frac{dr}{r^{1+\alpha'}} \\ &= -\frac{u}{\alpha'}\int_0^\infty e^{-uy} y^{-\alpha'} dy \\ &= \frac{\Gamma(1-\alpha')}{-\alpha'} u^{\alpha'} \\ &= \Gamma(-\alpha') u^{\alpha'} \quad \text{for } u > 0,\end{aligned}$$

then (28) holds for real $w = -u < 0$. Hence it also holds on $\{w : \operatorname{Re} w \leq 0,\, w \neq 0\}$. Since $d = 1$, observe that if $1_B(r\xi) > 0$ in (25) then $\xi \in \{-1, 1\}$. Then (25) reduces to

$$\begin{aligned}\rho(B) = \lambda_{-1} \int_0^\infty 1_B(-r)\frac{dr}{r^{1+\alpha/2}} \\ + \lambda_1 \int_0^\infty 1_B(r)\frac{dr}{r^{1+\alpha/2}} \quad \text{for } B \in \mathcal{B}(\mathbb{R}),\end{aligned} \tag{29}$$

where $\lambda_j := \lambda(\{j\}) \geq 0$ and $j \in \{-1, 1\}$ such that $\lambda_{-1} + \lambda_1 > 0$. It follows from Lemma 3, Lemma 4, and (11) that the characteristic function of μ is of the form

$$\log \widehat{\mu}(z) = \int_{\mathbb{R}} \left(e^{izx} - 1\right) \rho(dx) + i\gamma_0 z, \quad z \in \mathbb{R}. \tag{30}$$

We shall now compute the integral in (30) with ρ defined by (29). Let $\alpha' = \alpha/2$. Choose the branch $(-w)^{\alpha'} = |w|^{\alpha'} e^{i\alpha' \arg(-w)}$ with $\arg(-w) \in (-\pi, \pi]$ in (28), implying that

$$\begin{aligned}\int_0^\infty \left(e^{izr} - 1\right) \frac{dr}{r^{1+\alpha'}} &= \Gamma(-\alpha')|z|^{\alpha'} \exp\!\left(-i\frac{\pi\alpha'}{2}\operatorname{sgn}(z)\right) \\ &= \Gamma(-\alpha')|z|^{\alpha'} \cos\!\left(\frac{\pi\alpha'}{2}\right) \left[1 - i\tan\!\left(\frac{\pi\alpha'}{2}\right)\operatorname{sgn}(z)\right].\end{aligned}$$

Hence the integral in (30) with (29) is equal to

$$
\begin{aligned}
&\Gamma(-\alpha')|z|^{\alpha'}\cos\!\left(\frac{\pi\alpha'}{2}\right)\\
&\qquad\qquad \times\left\{\lambda_{-1}\left[1-i\tan\!\left(\frac{\pi\alpha'}{2}\right)\operatorname{sgn}(-z)\right]+\lambda_1\left[1-i\tan\!\left(\frac{\pi\alpha'}{2}\right)\operatorname{sgn}(z)\right]\right\}\\
&\quad=\Gamma(-\alpha')|z|^{\alpha'}\cos\!\left(\frac{\pi\alpha'}{2}\right)\\
&\qquad\qquad\qquad \times\,(\lambda_1+\lambda_{-1})\left\{1-i\left(\frac{\lambda_1-\lambda_{-1}}{\lambda_1+\lambda_{-1}}\right)\tan\!\left(\frac{\pi\alpha'}{2}\right)\operatorname{sgn}(z)\right\}.
\end{aligned}
$$

From the uniqueness in the Lévy-Khintchine representation it now follows from (24) and (30) that $\lambda_{-1}=0$, $\lambda_1=\lambda$ as defined in (27), and $\gamma_0=0$. Therefore (30) simplifies to

$$
\log\widehat{\mu}(z)=\lambda\int_0^\infty\left(e^{izr}-1\right)\frac{dr}{r^{1+\alpha/2}},\quad z\in\mathbb{R},
$$

from which (26) immediately follows. $\square$

From Theorem 10 the process $\{W_{T(t)}\}$ subordinated to Brownian motion has characteristics $(0,0,t\nu)$ with ν given by (16). Therefore we conclude that

$$
\begin{aligned}
\nu(dx)&=\int_0^\infty P^s_{W_1}(dx)\frac{\lambda\,ds}{s^{1+\alpha/2}}\\
&=\frac{\lambda}{\sqrt{2\pi}}\int_0^\infty s^{-\frac{3+\alpha}{2}}e^{-x^2/2s}ds\,dx\\
&=\frac{\lambda\,2^{\alpha/2}}{\sqrt{\pi}}\,\Gamma\!\left(\frac{\alpha+1}{2}\right)\frac{dx}{|x|^{1+\alpha}},\quad -\infty<x<\infty.
\end{aligned}
$$

4.3.3 Example: Subordination of α-Stable by Gamma

Motivated by the results in Hurst, Platen, Rachev [10] cited in Section 4.3, we provide an expression for the characteristics of the subordination of the α-stable Lévy process with $1<\alpha<2$ by the Γ subordinator. Although the stable distribution is absolutely continuous with respect to Lebesgue measure, there is no known closed-form expression for the pdf valid for a range of values of α. We will then leave the Lévy measure expressed in terms of the series representation of the pdf (see [6]). To this end, we begin with the following representation for the characteristic function of the α-stable distribution on $\mathbb{R}$.

Theorem 11. *Let $0 < \alpha \leq 2$. If μ is an α-stable distribution on $\mathbb{R}$, then*

$$\widehat{\mu}(z) = \exp(-c_1|z|^\alpha e^{-i(\pi/2)\theta\alpha \operatorname{sgn} z}), \tag{31}$$

where $c_1 > 0$ and $\theta \in \mathbb{R}$ with $|\theta| \leq (\frac{2-\alpha}{\alpha} \wedge 1)$. The parameters c_1 and θ are uniquely determined by μ. Conversely, for any c_1 and θ, there is an α-stable distribution μ satisfying (31).

Denote the parameters in (31) by $(\alpha, \theta, c_1)_Z$ and denote the density of μ by $p(x; (\alpha, \theta, c_1)_Z)$.

Theorem 12. *The density for the distribution μ on $\mathbb{R}$ defined in (31) with $1 < \alpha < 2$ is given by*

$$\begin{aligned} & p(x;(\alpha,\theta,c_1)_Z) = c_1^{-1/\alpha} p(c_1^{-1/\alpha} x; (\alpha,\theta,1)_Z) \quad \mathit{for}\ x>0 \\ \text{and} \quad & p(x;(\alpha,\theta,c_1)_Z) = p(-x;(\alpha,-\theta,c_1)_Z) \quad \mathit{for}\ x<0, \end{aligned}$$

where

$$p(x;(\alpha,\theta,1)_Z) = \frac{1}{\pi x}\sum_{k=1}^{\infty} \frac{\Gamma(1+k/\alpha)}{k!}(-x)^k \sin\!\left(\frac{k\pi}{2}(\theta-1)\right),\ x>0.$$

Let T be the $\Gamma(\gamma,\beta)$-subordinator (19). Let X be the Lévy process such that X_1 is α-stable with parameters c_1, θ in the representation (31). Then the Lévy measure ν of the subordinated process X_T is

$$\begin{aligned} \nu(dx) &= \int_0^\infty P^s_{X_1}(dx)\gamma s^{-1} e^{-\beta s} ds \\ &= \gamma c_1^{-1/\alpha} dx \int_0^\infty p'(s,x) e^{-\beta s} \frac{ds}{s^{1+1/\alpha}}, \end{aligned}$$

where

$$\begin{aligned} p'(s,x) = \Big[& p\big(-(sc_1)^{-1/\alpha}x;(\alpha,-\theta,1)_Z\big)1_{(-\infty,0)}(x) \\ & + p\big((sc_1)^{-1/\alpha}x;(\alpha,\theta,1)_Z\big)1_{(0,\infty)}(x)\Big]. \end{aligned}$$

5 Concluding Remarks

We have presented a summary of our earlier work regarding term structure models, where we expressed the results in terms of the characteristics of the driving process. Here we have described a methodology for constructing Lévy processes as potential drivers for our model. To illustrate, we derived the characteristics of some processes from the literature with infinite Lévy measure.

Acknowledgements

Prof. Rachev gratefully acknowledges research support by grants from Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, the Deutschen Forschungsgemeinschaft and the Deutscher Akademischer Austausch Dienst.

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Geometric and Approximation Properties of Some Complex Rotation-Invariant Integral Operators in the Unit Disk*

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Abstract. The purpose of this paper is to obtain Jackson-type estimates in approximation by some complex rotation-invariant integral operators in the unit disk. In addition, these operators preserve some sufficient conditions for starlikeness and univalence of analytic functions.

AMS 2000 Mathematics Subject Classification: 30E10, 30C45, 30C55, 41A25, 41A35.

Key words and phrases: Complex rotation-invariant integral operators, Jackson-type estimates, global smoothness preservation, shape preserving properties.

1 Introduction

Let us consider the open unit disk $D = \{z \in \mathbb{C}; |z| < 1\}$ and $A(\overline{D}) = \{f : \overline{D} \to \mathbb{C};$ f is analytic on D, continuous on $\overline{D}$, $f(0) = 0$, $f'(0) = 1\}$. Therefore, if

*This paper was written during the 2005 Spring Semester when the second author was a Visiting Professor at the Department of Mathematical Sciences, The University of Memphis, TN, U.S.A.

$f \in A(\overline{D})$ then we have $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, for all $z \in D$.

In a series of very recent papers [8], [2]–[4], geometric and approximation properties of some complex convolution polynomials and singular integrals attached to $f \in A(\overline{D})$ were proved.

Now, for $f \in A(\overline{D})$, $z \in \overline{D}$, let us consider the complex rotation-invariant integral operators given by

$$B_k(f)(z) = 2^k \int_{-\infty}^{+\infty} f(ze^{iv})\varphi(-2^k v)\, dv,$$

and generalized complex rotation-invariant integral operators given by

$$L_{k,j}(f)(z) = \frac{2^k}{j} \int_{-\infty}^{+\infty} \ell_k(f)(2^k z e^{iv})\varphi\left(-\frac{2^k}{j} v\right) dv, \quad k \in \mathbb{Z},\ j \in \mathbb{N}.$$

Here $i^2 = -1$, φ is a real-valued function of compact support $\subseteq [-a, a]$, $a > 0$, $\varphi(x) \geq 0$, $\int_{-\infty}^{+\infty} \varphi(x-u)\, du = 1$, $\forall x \in \mathbb{R}$, and $\{\ell_k\}_{k\in\mathbb{Z}}$ is a sequence of linear operators from $A_1(\overline{D}) = \{f\colon \overline{D} \to \mathbb{C};\ f$ is analytic on D and continuous on $\overline{D}\}$ into $A_1(\overline{D})$, defined by recurrence as $\ell_k(f)(z) = \ell_0(f_k)(z)$, $z \in \overline{D}$, where $f_k(z) = f\left(\frac{z}{2^k}\right)$, $z \in \overline{D}$ and $\ell_0\colon A_1(\overline{D}) \to A_1(\overline{D})$ is a linear operator.

Also, let us consider the Jackson-type generalization of $L_{k,j}(f)(z)$ given by

$$I_{k,q}(f)(z) = -\sum_{j=1}^{q} (-1)^j \binom{q}{j} L_{k,j}(f)(z), \quad \forall k \in \mathbb{Z},\ q \in \mathbb{N}$$

and its slightly modified variant

$$\mathcal{J}_{k,q}(f)(z) = \sum_{j=0}^{q} \alpha_j L_{k,j}(f)(z), \quad k \in \mathbb{Z},\ q \in \mathbb{N},$$

where $\alpha_j = (-1)^{r-j}\binom{r}{j}$, $j = \overline{1,q}$, $\alpha_0 = 1 - \sum_{j=1}^{q} \alpha_j$ (these last coefficients appear in the case of real smooth Picard operators in [1]).

Note that the real variants (for real-valued functions of a real variable) of these operators were studied in [7], [5]–[6].

The aim of this paper is to prove approximation and shape preserving properties (in geometric function theory) for the above complex rotation-invariant integral operators.

2 Approximation Properties

In this section we obtain Jackson-type rates in approximation by the complex operators $B_k(f)(z)$ and $L_{k,j}(f)(z)$ and global smoothness preservation properties of them.

We present

Theorem 2.1. (i) *For all* $f \in A_1(\overline{D})$, $z \in \overline{D}$ *and* $k \in \mathbb{Z}$, *we have*

$$|f(z) - B_k(f)(z)| \leq 3\omega_1 \left(f; \frac{a}{2^k}\right)_{\overline{D}};$$

(ii) *For* $f \in A_1(\overline{D})$, $z \in \overline{D}$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$, *we have*

$$|f(z) - L_{k,j}(f)(z)| \leq \omega_1 \left(f; \frac{mja+n}{2^{k+r}}\right)_{\overline{D}},$$

where for fixed $a > 0$ *it is assumed that*

$$\sup_{\substack{z,y \in \overline{D} \\ |z-y| \leq a}} |\ell_0(f)(z) - f(y)| \leq \omega_1 \left(f; \frac{ma+n}{2^r}\right)_{\overline{D}};$$

(iii) *For the hypothesis in (ii) and* $q \in \mathbb{N}$, *we have*

$$|f(z) - I_{k,q}(f)(z)| \leq (2^q - 1)\omega_1 \left(f; \frac{mqa+n}{2^{k+r}}\right)_{\overline{D}};$$

(iv)

$$\begin{gathered}
\omega_1(B_k(f);\delta)_{\overline{D}} \leq \omega_1(f;\delta)_{\overline{D}}, \quad \delta > 0,\ f \in A_1(\overline{D}),\ k \in \mathbb{Z}, \\
\omega_1(L_{k,j}(f);\delta)_{\overline{D}} \leq \omega_1(f;\delta)_{\overline{D}}, \quad \delta > 0,\ k \in \mathbb{Z},\ j \in \mathbb{N}, \\
\omega_1(I_{k,q}(f);\delta)_{\overline{D}} \leq (2^q - 1)\omega_1(f;\delta)_{\overline{D}}, \quad \delta > 0,\ k \in \mathbb{Z},\ q \in \mathbb{N},
\end{gathered}$$

in the hypothesis

$$|\ell_0(f)(x-u+h) - \ell_0(f)(x-u)| \leq \omega_1(f;h)_{\overline{D}}, \quad \forall h > 0,$$

$\forall x, u \in \overline{D}$ *with* $x - u$, $x - u + h \in \overline{D}$.

Proof. (i) Since

$$2^k \int_{-\infty}^{+\infty} \varphi(-2^k v)\, dv = \int_{-\infty}^{+\infty} \varphi(u)\, du = 1$$

we obtain

$$\begin{aligned}
|f(z) - B_k(f)(z)| &= \left| 2^k \int_{-\infty}^{+\infty} [f(z) - f(ze^{iv})]\varphi(-2^k v)\, dv \right| \\
&\leq 2^k \int_{-\infty}^{+\infty} |f(z) - f(ze^{iv})|\varphi(-2^k v)\, dv \\
&\leq 2^k \int_{-\infty}^{+\infty} \omega_1(f; |z| \cdot |1 - e^{iv}|)_{\overline{D}}\varphi(-2^k v)\, dv
\end{aligned}$$

$$\leq 2^k \int_{-\infty}^{+\infty} \omega_1\left(f; 2\left|\sin\frac{v}{2}\right|\right)_{\overline{D}} \varphi(-2^k v)\, dv$$
$$\leq 2^k \int_{-\infty}^{+\infty} \omega_1(f; |v|)_{\overline{D}} \varphi(-2^k v)\, dv$$
$$\leq \omega_1\left(f; \frac{a}{2^k}\right)_{\overline{D}} \cdot 2^k \int_{-\infty}^{+\infty} \left[\frac{2^k}{a}|v| + 1\right] \varphi(-2^k v)\, dv$$
$$= \omega_1\left(f; \frac{a}{2^k}\right)_{\overline{D}} \cdot \left[1 + \frac{2^k \cdot 2^k}{a} \int_{-\infty}^{+\infty} |v|\varphi(-2^k v)\, dv\right].$$

But

$$\frac{2^k \cdot 2^k}{a} \int_{-\infty}^{+\infty} |v|\varphi(-2^k v)\, dv = (\text{by } u = -2^k v)$$
$$= \frac{2^k \cdot 2^k}{a} \cdot \int_{-\infty}^{+\infty} \frac{|u|}{2^k} \cdot \varphi(u) \cdot \frac{du}{2^k} = \frac{1}{a} \int_{-\infty}^{+\infty} |u|\varphi(u)\, du$$
$$= \frac{1}{a} \int_{-a}^{a} |u|\varphi(u)\, du \leq \frac{1}{a} \int_{-a}^{a} |u|\, du = \frac{2}{a} \cdot a = 2,$$

which immediately proves (i).

(ii) By

$$\frac{2^k}{j} \int_{-\infty}^{+\infty} \varphi\left(-\frac{2^k}{j} v\right) dv = \int_{-\infty}^{+\infty} \varphi(u)\, du = 1,$$

we get

$$L_{k,j}(f)(z) - f(z) = \frac{2^k}{j} \int_{-\infty}^{+\infty} \left[\ell_k(f)(2^k z e^{iv}) - f(z)\right] \varphi\left(-\frac{2k}{j} v\right) dv$$
$$= \frac{2^k}{j} \int_{-\infty}^{+\infty} \left[\ell_0(f_k)(2^k z e^{iv}) - f_k(2^k z)\right] \varphi\left(-\frac{2k}{j} v\right) dv$$
$$\left(\text{by } -\frac{2k}{j} v = u\right)$$
$$\leq \int_{-\infty}^{+\infty} \left|\ell_0(f_k)\left(2^k z e^{i\left(-\frac{j}{2^k} u\right)}\right) - f_k(2^k z)\right| \varphi(u)\, du$$
$$= \int_{-a}^{a} \left|\ell_0(f_k)\left(2^k z e^{i\left(-\frac{j}{2^k} u\right)}\right) - f_k(2^k z)\right| \varphi(u)\, du.$$

But

$$\left|2^k z e^{i\left(-\frac{j}{2^k} u\right)} - 2^k z\right| \leq 2^k \cdot 2 \sin\left|\frac{j}{2 \cdot 2^k} u\right| \leq 2^k \cdot \frac{j|u|}{2^k} = j|u| \leq ja,$$

for all $|z| \leq 1$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$, which implies (reasoning as in [5, p. 9])

$$\int_{-a}^{a} \left|\ell_0(f_k)\left(2^k z e^{i\left(-\frac{j}{2^k} u\right)}\right) - f_k(2^k z)\right| \varphi(u)\, du$$
$$\leq \int_{-a}^{a} \sup\{|\ell_0(f_k)(w) - f_k(y)|; |w - y| \leq ja\} \varphi(u)\, du \leq \omega_1\left(f; \frac{mja + n}{2^{k+r}}\right)_{\overline{D}},$$

which proves (ii).

(iii) By the relation $-\sum\limits_{j=1}^{q}(-1)^j\binom{q}{j}=1$, we get

$$\begin{aligned}
|I_{k,q}(f)(z)-f(z)| &= \left|-\sum_{j=1}^{q}(-1)^j\binom{q}{j}L_{k,j}(f)(z)-\left(-\sum_{j=1}^{q}(-1)^j\binom{q}{j}\right)f(z)\right|\\
&= \left|\sum_{j=1}^{q}(-1)^j\binom{q}{j}[L_{k,j}(f)(z)-f(z)]\right|\\
&\le \sum_{j=1}^{q}\binom{q}{j}\cdot|L_{k,j}(f)(z)-f(z)|\\
&\le \sum_{j=1}^{q}\binom{q}{j}\omega_1\left(f;\frac{mja+n}{2^{k+r}}\right)_{\overline{D}}\\
&\le (2^q-1)\omega_1\left(f;\frac{mqa+n}{2^{k+r}}\right)_{\overline{D}},
\end{aligned}$$

which proves (iii) too.

(iv) Let $|z_1-z_2|\le\delta$, $z_1,z_2\in\overline{D}$. We get

$$\begin{aligned}
|B_k(f)(z_1)-B_k(f)(z_2)| &\le 2^k\int_{-\infty}^{+\infty}|f(z_1e^{iv})-f(z_2e^{iv})|\varphi(-2^kv)\,dv\\
&\le \omega_1(f;|z_1-z_2|)_{\overline{D}}\le\omega_1(f;\delta)_{\overline{D}},
\end{aligned}$$

where from passing to supremum with $|z_1-z_2|\le\delta$, we obtain

$$\omega_1(B_k(f):\delta)_{\overline{D}}\le\omega(f;\delta)_{\overline{D}},\quad\forall\delta>0,\ k\in\mathbb{Z}.$$

Then,

$$\begin{aligned}
&|L_{k,j}(f)(z_1)-L_{k,j}(f)(z_2)|\\
&\quad\le\ \frac{2^k}{j}\int_{-\infty}^{+\infty}|\ell_k(f)(2^kz_1e^{iv})-\ell_k(f)(2^kz_2e^{iv})|\varphi\left(-\frac{2^k}{j}v\right)dv\\
&\qquad\qquad\left(\text{by }-\frac{2^k}{j}v=u\right)\\
&\quad\le\ \int_{-\infty}^{+\infty}\left|\ell_0(f_k)\big(2^kz_1e^{i\left(-\frac{j}{2^k}u\right)}\big)-\ell_0(f_k)\big(2^kz_2e^{i\left(-\frac{j}{2^k}u\right)}\big)\right|\varphi(u)\,du\\
&\quad\le\ \omega_1(f;|z_1-z_2|)_{\overline{D}}\le\omega_1(f;\delta)_{\overline{D}},
\end{aligned}$$

where from passing to supremum with $|z_1-z_2|\le\delta$, we obtain

$$\omega_1(L_{k,j}(f);\delta)_{\overline{D}}\le\omega_1(f;\delta)_{\overline{D}}.$$

The inequality

$$\omega_1(I_{k,q}(f);\delta)_{\overline{D}} \le (2^q-1)\omega_1(f;\delta)_{\overline{D}}$$

follows immediately from the above inequality for $L_{k,j}$ and from the relation

$$\sum_{j=1}^{q}\binom{q}{j} = 2^q - 1,$$

which proves the theorem. ∎

Remark. Reasoning exactly as for $I_{k,q}(f)$, we get

$$|\mathcal{J}_{k,q}(f)(z) - f(z)| \le (2^q-1)\omega_1\left(f;\frac{mqa+n}{2^{k+r}}\right)_{\overline{D}}, \quad \forall z\in\overline{D}$$

and

$$\omega_1(\mathcal{J}_{k,q}(f);\delta)_{\overline{D}} \le (2^q-1)\omega_1(f;\delta)_{\overline{D}}, \quad \forall\delta>0.$$

3 Geometric Properties

In this section we will prove some geometric properties of $B_k(f)$, $L_{k,j}(f)$ and $I_{k,q}(f)$ with respect to geometric function theory.

First, let us consider the following classes of functions:

$S_3 = \{f\in A(\overline{D});\ |f''(z)|\le 1,\ \forall z\in D\}$,
$\mathcal{P} = \{f\colon\overline{D}\to\mathbb{C};\ f$ is analytic on D, $f(0)=1$ and $\mathrm{Re}[f(z)]>0,\ \forall z\in D\}$,
$S_M = \{f\in A(\overline{D});\ |f'(z)|<M,\ \forall z\in D\}$, $M>1$.

According to [10], if $f\in S_3$ then f is starlike (and univalent) in D and by e.g. [9, p. 111, Exercise 5.4.1] if $f\in S_M$ then f is univalent in $\left\{z\in\mathbb{C};\ |z|<\frac{1}{M}\right\}\subset D$.

We present

Theorem 3.1. (i) *If $f(z)=\sum_{p=0}^{\infty}a_pz^p$ is analytic in D and continuous in $\overline{D}$, then $B_k(f)(z)$, $L_{k,j}(f)(z)$ and $I_{k,q}(f)(z)$ are analytic in D and continuous in $\overline{D}$. The analyticity of $L_{k,j}(f)(z)$ and $I_{k,q}(f)(z)$ is proved only for $\ell_0(f)\equiv f$.*

Also we can write

$$B_k(f)(z) = \sum_{p=0}^{\infty}a_pb_{p,k}z^p, \quad z\in D,$$

where

$$b_{p,k} = \int_{-\infty}^{+\infty}\cos\left(\frac{pu}{2^k}\right)\varphi(u)\,du, \quad p=0,1,\ldots, \quad k\in\mathbb{Z}.$$

If $\ell_0(f) \equiv f$ *then*

$$L_{k,j}(f)(z) = \sum_{p=0}^{\infty} a_p b_{p,k,j} z^p, \quad z \in D, \quad k \in \mathbb{Z}, \ j \in \mathbb{N}$$

with

$$b_{p,k,j} = \int_{-\infty}^{\infty} \cos\left(\frac{pju}{2^k}\right) \varphi(u)\, du,$$

and

$$I_{k,q}(f)(z) = \sum_{p=0}^{\infty} a_p c_{p,k,q} z^p, \quad z \in D, \quad k \in \mathbb{Z}, \ q \in \mathbb{N}$$

with

$$c_{p,k,q} = \sum_{j=1}^{q} (-1)^{j+1} \binom{q}{j} b_{p,k,j}.$$

If $\varphi(x) = 1 - x$, $x \in [0,1]$, $\varphi(x) = 1 + x$, $x \in [-1,0]$, $\varphi(x) = 0$, $x \in \mathbb{R} \setminus (0,1)$, *then*

$$b_{1,k} = 2^{2k+1}\left(1 - \cos\frac{1}{2^k}\right) > 0, \quad \forall k \in \mathbb{Z},$$
$$b_{1,k,j} = \frac{2^{2k+1}}{j^2}\left(1 - \cos\frac{j}{2^k}\right) > 0, \quad \forall k \in \mathbb{Z}, \ j \in \mathbb{N},$$
$$c_{1,k,q} = \sum_{j=1}^{q} (-1)^{j+1} \binom{q}{j} \frac{2^{2k+1}}{j^2}\left(1 - \cos\frac{j}{2^k}\right), \quad \forall k \in \mathbb{Z}, \ q \in \mathbb{N}.$$

(ii) *It also holds that* $B_k(\mathcal{P}) \subset \mathcal{P}$, $\forall k \in \mathbb{N}$,

$$\frac{1}{b_{1,k}} B_k\big(S_{3,b_{1,k}}\big) \subset S_3, \frac{1}{b_{1,k}} B_k(S_M) \subset S_{M/|b_{1,k}|} \ \ \forall k \in \mathbb{Z}.$$

If $\ell_0(f) \equiv f$ *then*

$$L_{k,j}(\mathcal{P}) \subset \mathcal{P}, \quad \frac{1}{b_{1,k,j}} L_{k,j}\big(S_{3,b_{1,k,j}}\big) \subset S_3,$$

$$\frac{1}{b_{1,k,j}} L_{k,j}(S_M) \subset S_{M/|b_{1,k,j}|}, \quad \forall k \in \mathbb{Z}, \ j \in \mathbb{N}.$$

Here in all the cases we take $\varphi(x) = 1 - x$, $x \in [0,1]$, $\varphi(x) = 1 + x$, $x \in [-1,0]$, $\varphi(x) = 0$, $x \in \mathbb{R} \setminus (0,1)$ *and we denote by* $S_{3,a} = \{f \in S_3;\ |f''(z)| \le |a|\}$ *and* $S_B = \{f \in A(\overline{D});\ |f'(z)| < B,\ z \in D\}$.

Proof. (i) Let $z_0, z_n \in \overline{D}$ be with $\lim_{n\to\infty} z_n = z_0$. We get (as in the proof of Theorem 2.1, (iv))

$$|B_k(f)(z_n) - B_k(f)(z_0)| \le \omega_1(f; |z_n - z_0|)_{\overline{D}},$$
$$|L_{k,j}(f)(z_n) - L_{k,j}(f)(z_0)| \le \omega_1(f; |z_n - z_0|)_{\overline{D}},$$
$$|I_{k,q}(f)(z_n) - I_{k,q}(f)(z_0)| \le \omega_1(f; |z_n - z_0|)_{\overline{D}},$$

which proves the continuity of these operators in $\overline{D}$.

It remains to prove that $B_k(f)(z)$, $L_{k,j}(f)(z)$ and $I_{k,q}(f)(z)$ are analytic in D.

By hypothesis we have $f(z) = \sum_{p=0}^{\infty} a_p z^p$, $z \in D$. Let $z \in D$ be fixed. We get

$$f(ze^{iv}) = \sum_{p=0}^{\infty} a_p e^{ipv} z^p$$

and since $|a_p e^{ipv}| = |a_p|$ for all $v \in \mathbb{R}$ and the series $\sum_{p=0}^{\infty} a_p z^k$ is convergent, it follows that the series $\sum_{p=0}^{\infty} a_p e^{ipv} z^p$ is uniformly convergent with respect to $v \in \mathbb{R}$. This immediately implies that the series can be integrated term by term, i.e.

$$\begin{aligned}
B_k(f)(z) &= \int_{-\infty}^{+\infty} f\left(ze^{i\left(-\frac{u}{2^k}\right)}\right) \varphi(u)\, du \\
&= \sum_{p=0}^{\infty} a_p \left[\int_{-\infty}^{+\infty} e^{i\left(-\frac{pu}{2^k}\right)} \varphi(u)\, du\right] z^p \\
&= \sum_{p=0}^{\infty} a_p \left[\int_{-\infty}^{+\infty} \cos\left(-\frac{pu}{2^k}\right) \varphi(u)\, du\right] z^p = \sum_{p=0}^{\infty} a_p b_{p,k} z^k,
\end{aligned}$$

since cos is even function.

If $\ell_0(f) \equiv f$ then $\ell_k(f)(2^k z e^{iv}) = f(ze^{iv})$ and we obtain

$$L_{k,j}(f)(z) = \frac{2^k}{j} \int_{-\infty}^{+\infty} f(ze^{iv}) \varphi\left(-\frac{2^k v}{j}\right) dv$$

and reasoning as for $B_k(f)(z)$ we immediately obtain

$$L_{k,j}(f)(z) = \sum_{p=0}^{\infty} a_p b_{p,k,j} z^p, \quad z \in D,$$

with

$$b_{k,p,j} = \int_{-\infty}^{+\infty} \cos\left(\frac{pju}{2^k}\right) \varphi(u)\, du.$$

The development for $I_{k,q}(f)(z)$ follows easily from above, which proves (i).

For the particular choice of $\varphi(x)$, we have:

$$\begin{aligned}
b_{1,k,j} &= \int_{-1}^{0} \cos\left(\frac{ju}{2^k}\right) \cdot (1+u)\, du + \int_{0}^{1} \cos\left(\frac{ju}{2^k}\right) \cdot (1-u)\, du \\
&= 2\int_0^1 (1-u) \cos\frac{ju}{2^k}\, du = 2\left[\sin\left(\frac{ju}{2^k}\right) \cdot \frac{2^k}{j}\right]\Bigg|_0^1 - 2\int_0^1 u \cos\frac{ju}{2^k}\, du
\end{aligned}$$

$$= \frac{2^{k+1}}{j}\sin\frac{j}{2^k} - 2\left[\frac{2^{2k}}{j^2}\cos\frac{ju}{2^k} + \frac{2^k}{j}u\sin\frac{ju}{2^k}\right]\Bigg|_0^1$$
$$= \frac{2^{2k+1}}{j^2}\left(1-\cos\frac{j}{2^k}\right) > 0, \quad \forall k\in\mathbb{Z},\ j\in\mathbb{N}.$$

For $j=1$ we get $b_{1,k,1} := b_{1,k} = 2^{2k+1}\big(1-\cos\frac{1}{2^k}\big) > 0$. Therefore,

$$c_{1,k,q} = \sum_{j=1}^{q}(-1)^{j+1}\binom{q}{j}b_{1,k,j} = 2^{2k+1}\sum_{j=1}^{q}(-1)^{j+1}\binom{q}{j}\frac{1}{j^2}\left(1-\cos\frac{j}{2^k}\right).$$

(ii) Since from (i) we have

$$b_{0,k} = b_{0,k,j} = \int_{-\infty}^{+\infty}\varphi(u)\,du = 1, \quad \forall k\in\mathbb{Z}, \quad j\in\mathbb{N},$$

it follows $B_k(f)(0) = L_{k,j}(f)(0) = a_0$, i.e. if $f\in\mathcal{P}$, $f=U+iV$ then $a_0=1$ and $U>0$ on D, which implies $B_k(f)(0) = L_{k,j}(f)(0) = 1$,

$$\mathrm{Re}[B_k(f)(z)] = 2^k\int_{-\infty}^{+\infty}U(r\cos(x+v), r\sin(x+v))\varphi(-2^k v)\,dv > 0$$

, $\forall z = re^{ix}\in D$, and for $\ell_0(f)\equiv f$,

$$\mathrm{Re}[L_{k,j}(f)(z)] = \frac{2^k}{j}\int_{-\infty}^{+\infty}U(r\cos(x+v), r\sin(x+v))\varphi\left(-\frac{2^k v}{j}\right)dv > 0,$$

for all $z=re^{ix}\in D$, i.e. $B_k(\mathcal{P})$, $L_{k,j}(\mathcal{P})\subset\mathcal{P}$.

Let $f(0) = f'(0)-1 = 0$. From (i) we get

$$\frac{1}{b_{1,k}}\cdot B_k(f)(0) = \frac{1}{b_{1,k}}B_k'(f)(0) - 1 = 0$$

and if $\ell_0(f)\equiv f$ then

$$\frac{1}{b_{1,k,j}}L_{k,j}(f)(0) = \frac{1}{b_{1,k,j}}\cdot L_{k,j}'(f)(0) - 1 = 0.$$

Also, for $f\in S_{3,b_{1,k}}$ we get

$$\left|\frac{1}{b_{1,k}}B_k''(f)(z)\right| \le \frac{1}{|b_{1,k}|}2^k\int_{-\infty}^{+\infty}|f''(ze^{iv})e^{2iv}|\varphi(-2^k v)\,dv$$
$$\le 2^k\int_{-\infty}^{+\infty}\varphi(-2^k v)\,dv = 1,$$

i.e. $\frac{1}{b_{1,k}}B_k(f)\in S_3$, then for $f\in S_M$ it follows

$$\left|\frac{1}{b_{1,k}}B_k'(f)(z)\right| \le \frac{1}{|b_{1,k}|}2^k\int_{-\infty}^{+\infty}|f'(ze^{iv})e^{iv}|\varphi(-2^k v)\,dv < \frac{M}{|b_{1,k}|}, \ \ z\in D,$$

i.e. $\frac{1}{b_{1,k}}B_k(f) \in S_{M/|b_{1,k}|}$.

The proof in the case of $L_{k,j}$ is similar. The above proves the theorem. ■

Remarks. 1) From the proof of Theorem 3.1 we obtain the following geometric properties: if $f \in S_{3,b_{1,k}}$ then $B_k(f)$ is starlike (and univalent) on D, if $f \in S_M$ then $B_k(f)$ is univalent in

$$\left\{z \in \mathbb{C};\ |z| < \frac{|b_{1,k}|}{M}\right\} \subset \left\{z \in \mathbb{C};\ |z| < \frac{1}{M}\right\},$$

and by (i)

$$|b_{1,k}| \le \int_{-\infty}^{+\infty} \left|\cos\left(\frac{u}{2^k}\right)\right| \varphi(u)\, du \le \int_{-\infty}^{+\infty} \varphi(u)\, du = 1;$$

if $\ell_0(f) \equiv f$ then $f \in S_{3,b_{1,k,j}}$ implies that $L_{k,j}(f)$ is starlike (and univalent) on D and $f \in S_M$ implies that $L_{k,j}(f)$ is univalent in

$$\left\{z \in \mathbb{C};\ |z| < \frac{|b_{1,k,j}|}{M}\right\} \subset \left\{z \in \mathbb{C};\ |z| < \frac{1}{M}\right\},$$

since by (i)

$$|b_{1,k,j}| \le \int_{-\infty}^{+\infty} \left|\cos\left(\frac{pju}{2^k}\right)\right| \varphi(u)\, du \le \int_{-\infty}^{+\infty} \varphi(u)\, du = 1.$$

2) Let $\ell_0(f) \equiv f$. If $c_{1,k,q} \neq 0$ then similarly we get

$$\frac{1}{c_{1,k,q}} I_{k,q}(S_M) \subset S_{M(2^q-1)/|c_{1,k,q}|},$$

that if $f \in S_M$ implies $I_{k,q}(f)$ is univalent in

$$\left\{z \in \mathbb{C};\ |z| < \frac{|c_{1,k,q}|}{M(2^q-1)}\right\} \subset \left\{z \in \mathbb{C};\ |z| < \frac{1}{M}\right\},$$

since by (i)

$$\begin{aligned} |c_{1,k,q}| &= \left|\sum_{j=1}^{q} (-1)^{j+1} \binom{q}{j} b_{1,k,j}\right| \\ &\le \sum_{j=1}^{q} \binom{q}{j} |b_{1,k,j}| \le \sum_{j=1}^{q} \binom{q}{j} = 2^q - 1. \end{aligned}$$

3) For $\varphi(x) = 1 - x$, $\forall x \in [0,1]$, $\varphi(x) = 1 + x$, $\forall x \in [-1,0]$, $\varphi(x) = 0$, $x \in \mathbb{R} \setminus (0,1)$, let us consider

$$b_1 = \inf\{|b_{1,k}|; k \in \mathbb{N}\} = \inf\left\{2^{2k+1}\left(1 - \cos\frac{1}{2^k}\right); k \in \mathbb{N}\right\},$$

$$b_1^* = \inf\{|b_{1,k,j}|; k, j \in \mathbb{N}, j \le 2^{k+1}\} \text{ and } c_{1,q} = \inf\{|c_{1,k,q}|; k \in \mathbb{N}\}.$$

We have:

$$|b_{1,k}| = 2^{2k+1}\left(1-\cos\frac{1}{2^k}\right) = 2^{2k+2}\sin^2\frac{1}{2^{k+1}} = \left(2^{k+1}\sin\frac{1}{2^{k+1}}\right)^2,$$

$$|b_{1,k,j}| = \frac{2^{2k+1}}{j^2}\left(1-\cos\frac{j}{2^k}\right) = \frac{2^{2k+2}}{j^2}\sin^2\frac{j}{2^{k+1}} = \left(\frac{2^{k+1}}{j}\cdot\sin\frac{j}{2^{k+1}}\right)^2,$$

which by the fact that $f(t) = t\sin\frac{1}{t}$ is increasing for $t \geq 1$, $f(1) = \sin 1$, implies

$$0 < b_1 = \left(4\sin\frac{1}{4}\right)^2 = 16\sin^2\frac{1}{4}.$$

Also, since $1 \leq \frac{2^{k+1}}{j}$, $j = \overline{1, 2^{k+1}}$, we get $b_1^* = \sin^2 1$. Therefore, it is immediate the following.

Corollary 3.2. (i) *If $f \in A(\overline{D})$, $|f''(z)| \leq 16\sin^2\frac{1}{4}$, $\forall z \in D$ then $B_k(f) \in S_3$, for all $k \in \mathbb{N}$ and if $f \in S_M$, $M > 1$, then $B_k(f)$ is univalent in $\{z \in \mathbb{C};$ $|z| < \frac{16\sin^2\frac{1}{4}}{M}\}$, for all $k \in \mathbb{N}$;*

(ii) *If $f \in A(\overline{D})$, $|f''(z)| \leq \sin^2 1$, $\forall z \in D$, then $L_{k,j}(f) \in S_3$ and if $f \in S_M$, $M > 1$, then $L_{k,j}(f)$ is univalent in $\{z \in \mathbb{C};$ $|z| < \frac{\sin^2 1}{M}\}$, for all $k, j \in \mathbb{N}$, $j \leq 2^{k+1}$.*

Remarks. 1) Let $\ell_0(f) \equiv f$. Reasoning as in the case of $I_{k,q}(f)(z)$ (see Remark 2 after the proof of Theorem 3.1), we get that $f \in S_M$ implies $\mathcal{J}_{k,q}(f)(z)$ is univalent in

$$\left\{z \in \mathbb{Z};\ |z| < \frac{|c^*_{1,k,q}|}{M(2^q-1)}\right\},$$

where $c^*_{1,k,q}$ is the coefficient of z in the development in series of $\mathcal{J}_{k,q}(f)(z)$.

2) It would be of interest to find other geometric properties of the operators B_k, $L_{k,j}$, $I_{k,q}$ and $\mathcal{J}_{k,q}$.

3) Let $f \in A(\overline{D})$ we define $f_\alpha(z) := f(\alpha z)$ for all $\alpha, z \in \overline{D}$. The operator Φ is called *rotation invariant* iff $\Phi(f_\alpha) = (\Phi(f))_\alpha$. We assume that

$$\ell_0(f(2^{-k}\bullet))(az) = \ell_0(f(2^{-k}\alpha\bullet))(z), \quad k \in \mathbb{Z},$$

a condition fulfilled trivially by B_k operators, case of $\ell_0(f) = f$. Then easily one proves that $\ell_k(f_\alpha) = (\ell_k(f))_\alpha$ and $B_k(f_\alpha) = (B_k f)_\alpha$,

$$(L_{k,j}(f_\alpha)) = (L_{k,j}(f))_\alpha, \quad I_{k,q}(f_\alpha) = (I_{k,q}(f))_\alpha, \quad \mathcal{J}_{k,q}(f_\alpha) = (\mathcal{J}_{k,q}f)_\alpha.$$

So all operators we are dealing with here are *rotation invariant.*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.4,369-377,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

BERNSTEIN POLYNOMIALS AND OPERATIONAL METHODS

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ABSTRACT

We combine the properties of Bernestein polynomials with methods of operational nature to obtain new identities for classical polynomials (Hermite, Laguerre, Jacobi….)

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3 INTRODUCTION

In this note we combine a well known identity from the theory of approximation with Bernstein polynomials [1] with other, of operational nature, involving classical orthogonal polynomials [2].

We remind that the function [1]

$$F_n(x,\alpha) = \left[(1-x) + xe^{\alpha/n}\right]^n \tag{1}$$

in the limit of large n, for $x \in (0,1)$ and $|\alpha| \leq 1$ converges to the exponential function, namely

$$\lim_{n\to\infty} F_n(x,\alpha) = e^{\alpha x} \quad . \tag{2}$$

After this remark, we remind that a set of operational identities (O.I.), which will be given in the following, have been proved to play a useful role within the context of the theory of classical and generalized orthogonal polynomials [3].

a) *O.I. and Hermite Polynomials*

For variables x and $y \in C$ and $n \in N$, the following identity holds [2],

$$\begin{gathered} e^{y\frac{\partial^2}{\partial x^2}} x^n = H_n(x,y), \\ H_n(x,y) = n! \sum_{s=0}^{[n/2]} \frac{x^{n-2s} y^s}{(n-2s)!s!} \end{gathered} \tag{3}$$

with $H_n(x,y)$ satisfying the properties

$$
\begin{aligned}
&H_n(x,y) = \left(-i\sqrt{2y}\right)^n He_n\left(\frac{ix}{\sqrt{2y}}\right) \\
&H_n(2x,-1) = H_n(x) \\
&H_n(x,-\frac{1}{2}) = He_n(x) \\
&\alpha^n H_n(x,y) = H_n(\alpha x, \alpha^2 y)
\end{aligned}
\tag{4}
$$

b) O.I. and Laguerre Polynomials

Analogous formulae hold for Laguerre like polynomials too, in this case we have

$$
\begin{aligned}
&e^{-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}\left[\frac{(-1)^n x^n}{n!}\right] = \mathcal{L}_n(x,y) \\
&\qquad \mathcal{L}_n(x,y) = n!\sum_{s=0}^{n}\frac{(-x)^s y^{n-s}}{(n-s)!(s!)^2}
\end{aligned}
\tag{5}
$$

where the $\mathcal{L}_n(x,y)$ are linked to the ordinary Laguerre by

$$
\mathcal{L}_n(x,y) = y^n \mathcal{L}_n\left(\frac{x}{y}\right) \quad . \tag{6}
$$

It is also worth noting that

$$
e^{-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}} e^x = \frac{e^{\frac{x}{1+y}}}{1+y} \quad . \tag{7}
$$

Still within the context of Laguerre polynomials we must underline that the use of operators involving the negative derivative [2]

$$
\hat{\mathcal{D}}_x^{-n}(1) = \frac{x^n}{n!} \tag{8}
$$

yields the further O.I. [2]

$$\left(y-\hat{\mathcal{D}}_x^{-1}\right)^n(1)=\mathcal{L}_n(x,y)$$
$$e^{(y-\hat{\mathcal{D}}_x^{-1})}(1)=e^y J_0(2\sqrt{x}) \tag{9}$$

where $J_0(x)$ denotes the 0^{th} order cylindrical Bessel Function.

In the following section we will exploit the previous identities to derive further relations relevant to the families of polynomials we have dealt with. The possibility of extending the results to Legendre and Jacobi-like polynomials will be discussed in the concluding section.

4 OPERATIONAL IDENTITIES AND BERNSTEIN POLYNOMIALS

It is convenient, for the purposes of the present paper to recast Eq. (1) in the polynomial form

$$F_n(x,\alpha)=\sum_{s=0}^{n}\frac{n!}{(n-s)!s!}x^s\left[\Phi_n(\alpha)\right]^s$$
$$\Phi_n(\alpha)=e^{\alpha/n}-1 \qquad . \tag{10}$$

According to the identities relevant to Hermite-like polynomials we find (see Eq. (3))

$$e^{y\frac{\partial^2}{\partial x^2}}F_n(x,\alpha)=\sum_{s=0}^{n}\frac{n!}{(n-s)!s!}\left[\Phi_n(\alpha)\right]^s H_s(x,y) \tag{11}$$

which, on account of the last of Eqs. (4) and of the Hermite polynomials addition theorem, yields

$$e^{y\frac{\partial^2}{\partial x^2}}F_n(x,\alpha)=\sum_{s=0}^{n}\frac{n!H_s\left(x\Phi_n(\alpha),y\left[\Phi_n(\alpha)\right]^2\right)}{(n-s)!s!}=H_s\left(x\Phi_n(\alpha)+1,y\left[\Phi_n(\alpha)\right]^2\right)\ . \tag{12}$$

Now since

$$e^{y\frac{\partial^2}{\partial x^2}} e^{\alpha x} = e^{\alpha x + \alpha^2 y} \tag{13}$$

we end up with the following asymptotic property of Hermite-like polynomials

$$\lim_{n\to\infty} H_n\left(x\Phi_n(\alpha) + 1, y\left[\Phi_n(\alpha)\right]^2 \right) = e^{\alpha x + \alpha^2 y} \\ |\alpha x + \alpha^2 y| < 1 \tag{14}$$

Further relations involving generalized forms of Hermite polynomials will be discussed in the following.

Let us now apply the developed technique to the case of Laguerre type polynomials. The identities (5-7) yield

$$\begin{aligned} e^{-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}} F_n(x,\alpha) &= n!\sum_{s=0}^{n} \frac{(-1)^s\left[\Phi_n(\alpha)\right]^s \mathcal{L}_s(x,y)}{(n-s)!} \\ &= n!\sum_{s=0}^{n} \frac{(-1)^s \mathcal{L}_s\left(x\Phi_n(\alpha), y\Phi_n(\alpha)\right)}{(n-s)!} \end{aligned} \tag{15}$$

and since

$$e^{-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}} e^{\alpha x} = \frac{e^{\frac{\alpha x}{1+\alpha y}}}{1+\alpha y} \tag{16}$$

we end up with the asymptotic relation

$$\lim_{n\to\infty}\left[n!\sum_{s=0}^{n} \frac{(-1)^s \mathcal{L}_s\left(x\Phi_n(\alpha), y\Phi_n(\alpha)\right)}{(n-s)!} \right] = \frac{e^{\frac{\alpha x}{1+\alpha y}}}{1+\alpha y}, \quad |x|<1, \quad |y|<\frac{1}{\alpha} \quad . \tag{17}$$

The use of the first of the identities in Eq. (9), allows the derivation of the further relations

$$F_n(y-\hat{D}_x^{-1},\alpha)(1)=n!\sum_{s=0}^{n}\frac{\mathcal{L}_s(x,y)\left[\Phi_n(\alpha)\right]^s}{(n-s)!s!}=\mathcal{L}_n\left(x\Phi_n(\alpha),y\Phi_n(\alpha)+1\right) \quad . \tag{18}$$

Therefore according to the second of Eqs. (9), we obtain the aymptotic relation

$$\lim_{n\to\infty}\mathcal{L}_n\left(x\Phi_n(\alpha),y\Phi_n(\alpha)+1\right)=e^{\alpha y}J_0(2\sqrt{\alpha x}) \tag{19}$$

in the case of y=0 Eq. (19) reduces to

$$\lim_{n\to\infty}L_n\left(x(e^{\frac{\alpha}{n}}-1)\right)=J_0\left(2\sqrt{\alpha x}\right) \quad . \tag{20}$$

By recalling that [4]

$$\begin{aligned}&\frac{d^m}{dx^m}L_n(x)=(-1)^m L_{n-m}^{(m)}(x)\\&\frac{d^m}{dx^m}J_0(2\sqrt{x})=(-1)^m x^{-\frac{m}{2}}J_m(2\sqrt{x})\end{aligned} \tag{21}$$

we find from Eq. (20))

$$\lim_{n\to\infty}\left[\Phi_n(\alpha)\right]^m L_{n-m}^{(m)}(x\Phi_n(\alpha))=(\alpha x)^{-\frac{m}{2}}\alpha^m J_m(2\sqrt{\alpha x}) \quad . \tag{22}$$

In the forthcoming section we will see how the so far obtained results can be extended to other families of polynomials.

5 CONCLUDING REMARKS

In the previous section we have used the O.I. of Hermite and Laguerre-like polynomials to obtain asymptotic relations in the polynomial index. In this section we will see that the method can be extended to other families of polynomials.

Legendre type polynomials

$$ {}_2\mathcal{L}_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^{n-2r} x^r}{(n-2r)!(r!)^2} \tag{23} $$

have been shown in ref. (5) to be derivable from the O.I.

$$ \left(y + 2\hat{\mathcal{D}}_x^{-1} \frac{\partial}{\partial x} \right)^n (1) = \left({}_2\mathcal{L}_n(x,y) \right) \tag{24} $$

and to satisfy the generating function

$$ \sum_{n=0}^{\infty} \frac{t^n}{n!} \left({}_2\mathcal{L}_n(x,y) \right) = e^{yt} I_0\left(2t\sqrt{x}\right) \tag{25} $$

where $I_0(x)$ is the 0^{th} order modified Bessel function. The use of the method described before yields

$$ \lim_{n\to\infty} \left({}_2\mathcal{L}_n\left(x\left[\Phi_n,\alpha)\right]^2, y\Phi_n(\alpha)+1 \right) \right) = e^{y\alpha} I_0\left(2\alpha\sqrt{x}\right) \quad . \tag{26} $$

By noting that the polynomials (23) reduce to the ordinary Legendre for [5]

$$ {}_2\mathcal{L}_n\left(-\frac{1}{4}(1-y^2), y \right) = P_n(y) \tag{27} $$

we also find

$$\lim_{n\to\infty}\left(\sum_{s=0}^{n}\frac{n!\left[\Phi_n(\alpha)\right]^s}{(n-s)!s!}P_s\left(\cos(\phi)\right)\right)=e^{\alpha\cos(\phi)}J_0\left(\alpha\sin(\phi)\right) \quad . \tag{28}$$

The results of the present investigation will be extended, in a forthcoming note, to many index special polynomials.

ACKNOWLEDGEMENTS

The authors express their sincere appreciation to Profs. P.E. Ricci and H.M. Srivastava for helpful suggestions and stimulating discussions.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.8,NO.4,379-388,2006,COPYRIGHT 2006 EUDOXUS PRESS,LLC

On Ostrowski Like Integral Inequality for the Čebyšev Difference and Applications.MISPRINTS FREE.

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Abstract. Some integral inequalities similar to the Ostrowski's result for Čebyšev's difference and applications for perturbed generalized Taylor's formula are given.

Key Words: Ostrowski's inequality, Čebyšev's difference, Taylor's formula.

AMS Subj. Class.: Primary 26D15; Secondary 26D10

1. Introduction

In [5], A. Ostrowski proved the following inequality of Grüss type for the difference between the integral mean of the product and the product of the integral means, or *Čebyšev's difference*, for short:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{8}(b-a)(M-m)\|f'\|_{[a,b],\infty} \tag{1.1}$$

provided g is measurable and satisfies the condition

$$-\infty<m\leq g(x)\leq M<\infty \quad \text{for a.e. } x\in[a,b]; \tag{1.2}$$

and f is absolutely continuous on $[a,b]$ with $f'\in L_\infty[a,b]$.

The constant $\frac{1}{8}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller constant.

In this paper we establish some similar results. Applications for perturbed generalized Taylor's formulae are also provided.

2. Integral Inequalities

The following result holds.

Theorem 1. *Let $f:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ be an absolutely continuous function with $f'\in L_\infty[a,b]$ and $g\in L_1[a,b]$. Then one has the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \|f'\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx. \tag{2.1}$$

The inequality (2.1) is sharp in the sense that the constant $c=1$ in the left hand side cannot be replaced by a smaller one.

Proof. We observe, by simple computation, that one has the identity

$$\begin{aligned} T(f,g) &:= \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \\ &= \frac{1}{b-a}\int_a^b\left[f(x)-f\left(\frac{a+b}{2}\right)\right]\left[g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right]dx. \end{aligned} \tag{2.2}$$

Since f is absolutely continuous, we have

$$\int_{\frac{a+b}{2}}^x f'(t)\,dt=f(x)-f\left(\frac{a+b}{2}\right)$$

and thus, the following identity that is in itself of interest,

$$T(f,g)=\frac{1}{b-a}\int_a^b\left(\int_{\frac{a+b}{2}}^x f'(t)\,dt\right)\left[g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right]dx \tag{2.3}$$

holds.

Since

$$\left|\int_{\frac{a+b}{2}}^x f'(t)\,dt\right|\le\left|x-\frac{a+b}{2}\right| ess \sup_{\substack{t\in\left[x,\frac{a+b}{2}\right]\\ \left(t\in\left[\frac{a+b}{2},x\right]\right)}}|f'(t)|=\left|x-\frac{a+b}{2}\right|\|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}$$

for any $x\in[a,b]$, then taking the modulus in (2.3), we deduce

$$\begin{aligned} |T(f,g)| &\le \frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\ &\le \sup_{x\in[a,b]}\left\{\|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}\right\}\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\ &= \max\left\{\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty},\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right\} \\ &\qquad\times\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\ &= \|f'\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \end{aligned}$$

and the inequality (2.1) is proved.

To prove the sharpness of the constant $c = 1$, assume that (2.1) holds with a positive constant $D > 0$, i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq D\,\|f'\|_{[a,b],\infty} \cdot \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| \left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx. \tag{2.4}$$

If we choose $\mathbb{K} = \mathbb{R}$, $f(x) = x - \frac{a+b}{2}$, $x \in [a,b]$ and $g : [a,b] \to \mathbb{R}$,

$$g(x) = \begin{cases} -1 & \text{if} \quad x \in \left[a, \frac{a+b}{2}\right] \\ 1 & \text{if} \quad x \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

then

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \\ = \; & \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| dx = \frac{b-a}{4}, \end{aligned}$$

$$\frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| \left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx = \frac{b-a}{4},$$

$$\|f'\|_{[a,b],\infty} = 1$$

and by (2.4) we deduce

$$\frac{b-a}{4} \leq D \cdot \frac{b-a}{4},$$

giving $D \geq 1$, and the sharpness of the constant is proved. □

The following corollary may be useful in practice.

Corollary 1. *Let $f : [a,b] \to \mathbb{K}$ be an absolutely continuous function on $[a,b]$ with $f' \in L_\infty[a,b]$. If $g \in L_\infty[a,b]$, then one has the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(b-a)\,\|f'\|_{[a,b],\infty} \left\|g - \frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}. \tag{2.5}$$

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Obviously,

$$\begin{aligned}
(2.6)\qquad & \frac{1}{b-a}\int_a^b \left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\
& \le \left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|dx \\
& = \frac{b-a}{4}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}.
\end{aligned}$$

Using (2.1) and (2.6) we deduce (2.5).

Assume that (2.5) holds with a constant $E>0$ instead of $\frac{1}{4}$, i.e.,

$$\begin{aligned}
(2.7)\qquad & \left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \\
& \qquad\le E\,(b-a)\,\|f'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}.
\end{aligned}$$

If we choose the same functions as in Theorem 1, then we get from (2.7)

$$\frac{b-a}{4}\le E\,(b-a)\,,$$

giving $E\ge\frac{1}{4}$. □

Corollary 2. *Let f be as in Theorem 1. If $g\in L_p[a,b]$ where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then one has the inequality:*

$$\begin{aligned}
(2.8)\qquad & \left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \\
& \le \frac{(b-a)^{\frac{1}{q}}}{2\,(q+1)^{\frac{1}{q}}}\,\|f'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],p}.
\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. By Hölder's inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, one has

$$\begin{aligned}
(2.9)\qquad & \frac{1}{b-a}\int_a^b \left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\
& \le \frac{1}{b-a}\left(\int_a^b\left|x-\frac{a+b}{2}\right|^q dx\right)^{\frac{1}{q}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \\
& = \frac{1}{b-a}\left[\frac{(b-a)^{q+1}}{2^q\,(q+1)}\right]^{\frac{1}{q}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \\
& = \frac{(b-a)^{\frac{1}{q}}}{2\,(q+1)^{\frac{1}{q}}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Using (2.1) and (2.9), we deduce (2.8).

Now, if we assume that the inequality (2.8) holds with a constant $F > 0$ instead of $\frac{1}{2}$ and choose the same functions f and g as in Theorem 1, we deduce

$$\frac{b-a}{4} \leq \frac{F}{(q+1)^{\frac{1}{q}}}(b-a), \ q > 1$$

giving $F \geq \frac{(q+1)^{\frac{1}{q}}}{4}$ for any $q > 1$. Letting $q \to 1+$, we deduce $F \geq \frac{1}{2}$, and the corollary is proved. □

Finally, we also have

Corollary 3. *Let f be as in Theorem 1. If $g \in L_1[a,b]$, then one has the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{2}\|f'\|_{[a,b],\infty}\left\|g - \frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1}. \tag{2.10}$$

Proof. Since

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right|\left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx \\ \leq\ & \sup_{x\in[a,b]}\left|x - \frac{a+b}{2}\right|\left\|g - \frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1} \\ =\ & \frac{b-a}{2}\left\|g - \frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1} \end{aligned}$$

the inequality (2.10) follows by (2.1). □

Remark 1. *Similar inequalities may be stated for weighted integrals. These inequalities and their applications in connection to Schwartz's inequality will be considered in* [3].

3. Applications to Taylor's Formula

In the recent paper [4], M. Matić, J. E. Pečarić and N. Ujević proved the following generalized Taylor formula.

Theorem 2. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is, $P_n'(t) = P_{n-1}(t)$ for $n \geq 1$, $n \in \mathbb{N}$, $P_0(t) = 1$, $t \in \mathbb{R}$. Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is a function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then*

$$f(x) = \tilde{T}_n(f;a,x) + \tilde{R}_n(f;a,x), \quad x \in I, \tag{3.1}$$

where

$$\tilde{T}_n(f;a,x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)\right] \tag{3.2}$$

and

$$\tilde{R}_n(f;a,x) = (-1)^n\int_a^x P_n(t)f^{(n+1)}(t)\,dt. \tag{3.3}$$

For some particular instances of harmonic sequences, they obtained the following Taylor-like expansions:

$$f(x) = T_n^{(M)}(f;a,x) + R_n^{(M)}(f;a,x), \quad x \in I, \tag{3.4}$$

where

$$T_n^{(M)}(f;a,x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x) \right], \tag{3.5}$$

$$R_n^{(M)}(f;a,x) = \frac{(-1)^n}{n!} \int_a^x \left(t - \frac{a+x}{2} \right)^n f^{(n+1)}(t)\, dt; \tag{3.6}$$

and

$$f(x) = T_n^{(B)}(f;a,x) + R_n^{(B)}(f;a,x), \quad x \in I, \tag{3.7}$$

where

$$\begin{aligned} T_n^{(B)}(f;a,x) &= f(a) + \frac{x-a}{2}\left[f'(x) + f'(a) \right] \\ &\quad - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(x-a)^{2k}}{(2k)!} B_{2k} \left[f^{(2k)}(x) - f^{(2k)}(a) \right], \end{aligned} \tag{3.8}$$

and $[r]$ is the integer part of r. Here, B_{2k} are the Bernoulli numbers, and

$$R_n^{(B)}(f;a,x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x B_n\left(\frac{t-a}{x-a} \right) f^{(n+1)}(t)\, dt, \tag{3.9}$$

where $B_n(\cdot)$ are the Bernoulli polynomials, respectively.

In addition, they proved that

$$f(x) = T_n^{(E)}(f;a,x) + R_n^{(E)}(f;a,x), \quad x \in I, \tag{3.10}$$

where

$$\begin{aligned} & T_n^{(E)}(f;a,x) \\ &= f(a) + 2 \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(x-a)^{2k-1}\left(4^k - 1\right)}{(2k)!} B_{2k} \left[f^{(2k-1)}(x) + f^{(2k-1)}(a) \right] \end{aligned} \tag{3.11}$$

and

$$R_n^{(E)}(f;a,x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n\left(\frac{t-a}{x-a} \right) f^{(n+1)}(t)\, dt, \tag{3.12}$$

where $E_n(\cdot)$ are the Euler polynomials.

In [1], S.S. Dragomir was the first author to introduce the perturbed Taylor formula

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!} \left[f^{(n)}; a, x \right] + G_n(f;a,x), \tag{3.13}$$

where

$$T_n(f;a,x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{3.14}$$

and

$$\left[f^{(n)}; a, x \right] := \frac{f^{(k)}(x) - f^{(k)}(a)}{x-a};$$

and had the idea to estimate the remainder $G_n(f;a,x)$ by using Grüss and *Čebyšev* type inequalities.

In [4], the authors generalized and improved the results from [1]. We mention here the following result obtained via a pre-Grüss inequality (see [4, Theorem 3]).

Theorem 3. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose $f : I \to \mathbb{R}$ is as in Theorem 2. Then for all $x \in I$ we have the perturbed generalized Taylor formula:*

$$\begin{aligned} f(x) &= \tilde{T}_n(f;a,x) + (-1)^n [P_{n+1}(x) - P_{n+1}(a)] \left[f^{(n)};a,x\right] \\ &\quad + \tilde{G}_n(f;a,x). \end{aligned} \tag{3.15}$$

For $x \geq a$, the remainder $\tilde{G}(f;a,x)$ satisfies the estimate

$$\left|\tilde{G}_n(f;a,x)\right| \leq \frac{x-a}{2}\sqrt{T(P_n,P_n)}\,[\Gamma(x) - \gamma(x)], \tag{3.16}$$

provided that $f^{(n+1)}$ is bounded and

$$\Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t) < \infty, \quad \gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t) > -\infty, \tag{3.17}$$

where $T(\cdot,\cdot)$ is the Čebyšev functional on the interval $[a,x]$, that is, we recall

$$T(g,h) := \frac{1}{x-a}\int_a^x g(t)\,h(t)\,dt - \frac{1}{x-a}\int_a^x g(t)\,dt \cdot \frac{1}{x-a}\int_a^x h(t)\,dt. \tag{3.18}$$

In [2], the author has proved the following result improving the estimate (3.16).

Theorem 4. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L_2(I)$. If $x \geq a$, then we have the inequality*

$$\begin{aligned} &\left|\tilde{G}_n(f;a,x)\right| \\ &\leq (x-a)\,[T(P_n,P_n)]^{\frac{1}{2}} \left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n)};a,x\right]\right)^2\right]^{\frac{1}{2}} \\ &\quad \left(\leq \frac{x-a}{2}\,[T(P_n,P_n)]^{\frac{1}{2}}\,[\Gamma(x) - \gamma(x)], \quad \text{if } f^{(n+1)} \in L_\infty[a,x]\right), \end{aligned} \tag{3.19}$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $[a,x]$, i.e.,

$$\left\|f^{(n+1)}\right\|_2 = \left(\int_a^x \left|f^{(n+1)}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Remark 2. *If $f^{(n+1)}$ is unbounded on (a,x) but $f^{(n+1)} \in L_2(a,x)$, then the first inequality in (3.19) can still be applied, but not the Matić-Pečarić-Ujević result (3.16) which requires the boundedness of the derivative $f^{(n+1)}$.*

The following corollary [2] improves Corollary 3 of [4], which deals with the estimation of the remainder for the particular perturbed Taylor-like formulae (3.4), (3.7) and (3.10).

Corollary 4. *With the assumptions in Theorem 4, we have the following inequalities*

$$\left|\tilde{G}_n^{(M)}(f;a,x)\right| \leq \frac{(x-a)^{n+1}}{n!2^n\sqrt{2n+1}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.20}$$

$$\left|\tilde{G}_n^{(B)}(f;a,x)\right| \leq (x-a)^{n+1}\left[\frac{|B_{2n}|}{(2n)!}\right]^{\frac{1}{2}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.21}$$

$$\begin{aligned}\left|\tilde{G}_n^{(E)}(f;a,x)\right| & \\ \leq\ & 2(x-a)^{n+1}\left[\frac{\left(4^{n+1}-1\right)|B_{2n+2}|}{(2n+2)!} - \left[\frac{2\left(2^{n+2}-1\right)B_{n+2}}{(n+1)!}\right]^2\right]^{\frac{1}{2}} \\ & \times\sigma\left(f^{(n+1)};a,x\right),\end{aligned} \tag{3.22}$$

and

$$|G_n(f;a,x)| \leq \frac{n(x-a)^{n+1}}{(n+1)!\sqrt{2n+1}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.23}$$

where, as in [4],

$$\tilde{G}_n^{(M)}(f;a,x) = f(x) - T_n^M(f;a,x) - \frac{(x-a)^{n+1}\left[1+(-1)^n\right]}{(n+1)!2^{n+1}}\left[f^{(n)};a,x\right];$$

$$\tilde{G}_n^{(B)}(f;a,x) = f(x) - T_n^B(f;a,x);$$

$$\tilde{G}_n^{(E)}(f;a,x) = f(x) - \frac{4(-1)^n(x-a)^{n+1}\left(2^{n+2}-1\right)B_{n+2}}{(n+2)!}\left[f^{(n)};a,x\right],$$

$G_n(f;a,x)$ *is as defined by (3.13),*

$$\sigma\left(f^{(n+1)};a,x\right) := \left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n+1)};a,x\right]\right)^2\right]^{\frac{1}{2}}, \tag{3.24}$$

and $x \geq a$, $f^{(n+1)} \in L_2[a,x]$.

Note that for all the examples considered in [1] and [4] for f, the quantity $\sigma\left(f^{(n+1)};a,x\right)$ can be completely computed and then those particular inequalities may be improved accordingly. We omit the details.

Now, observe that (for $x > a$)

$$\tilde{G}_n(f;a,x) = (-1)^n(x-a)T_n\left(P_n, f^{(n+1)};a,x\right),$$

where $T_n(\cdot,\cdot;a,x)$ is the Čebyšev's functional on $[a,x]$, i.e.,

$$\begin{aligned} T_n\left(P_n, f^{(n+1)};a,x\right) &= \frac{1}{x-a}\int_a^x P_n(t)f^{(n+1)}(t)\,dt \\ &\quad -\frac{1}{x-a}\int_a^x P_n(t)\,dt\cdot\frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt \\ &= \frac{1}{x-a}\int_a^x P_n(t)f^{(n+1)}(t)\,dt - [P_{n+1};a,x]\left[f^{(n)};a,x\right]. \end{aligned}$$

In what follows we will use the following lemma that summarizes some integral inequalities obtained in the previous section.

Lemma 1. *Let $h : [x, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ with $h' \in L_\infty [a, b]$. Then*

$$(3.25)\quad |T_n (h, g; a, b)| \le \begin{cases} \frac{1}{4} (b-a) \|h'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y)\, dy \right\|_{[a,b],\infty} & \text{if } g \in L_\infty [a, b]; \\ \frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \|h'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y)\, dy \right\|_{[a,b],p} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and } g \in L_p [a, b]; \\ \frac{1}{2} \|h'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y)\, dy \right\|_{[a,b],1} & \text{if } g \in L_1 [a, b]; \end{cases}$$

where

$$T_n (h, g; a, b) := \frac{1}{b-a} \int_a^b h(x) g(x)\, dx - \frac{1}{b-a} \int_a^b h(x)\, dx \cdot \frac{1}{b-a} \int_a^b g(x)\, dx.$$

Using the above lemma, we may obtain the following new bounds for the remainder $\tilde{G}_n (f; a, x)$ in the Taylor's perturbed formula (3.15).

Theorem 5. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on any compact subinterval of I. Then, for $x, a \in I, x > a$, we have that*

$$(3.26)\quad \left|\tilde{G}_n (f; a, x)\right| \le \begin{cases} \frac{1}{4} (x-a)^2 \|P_{n-1}\|_{[a,x],\infty} \left\| f^{(n+1)} - \left[f^{(n)}; a, x\right] \right\|_{[a,x],\infty} & \text{if } f^{(n+1)} \in L_\infty [a, x]; \\ \frac{(x-a)^{\frac{1}{q}+1}}{2(q+1)^{\frac{1}{q}}} \|P_{n-1}\|_{[a,x],\infty} \left\| f^{(n+1)} - \left[f^{(n)}; a, x\right] \right\|_{[a,x],p} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and } f^{(n+1)} \in L_p [a, x]; \\ \frac{1}{2} (x-a) \|P_{n-1}\|_{[a,x],\infty} \left\| f^{(n+1)} - \left[f^{(n)}; a, x\right] \right\|_{[a,x],1}. \end{cases}$$

The proof follows by Lemma 1 on choosing $h = P_n$, $g = f^{(n+1)}$, $b = x$.

The dual result is incorporated in the following theorem.

Theorem 6. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n+1)}$ is absolutely continuous on any compact subinterval of I. Then, for $x, a \in I, x > a$, we have that*

$$(3.27)\quad \left|\tilde{G}_n (f; a, x)\right|$$

$$(3.28)\quad \le \begin{cases} \frac{1}{4} (x-a)^2 \left\| f^{(n+2)} \right\|_{[a,x],\infty} \|P_n - [P_{n+1}; a, x]\|_{[a,x],\infty} \\ \frac{(x-a)^{\frac{1}{q}+1}}{2(q+1)^{\frac{1}{q}}} \left\| f^{(n+2)} \right\|_{[a,x],\infty} \|P_n - [P_{n+1}; a, x]\|_{[a,x],p} \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2} (x-a) \left\| f^{(n+2)} \right\|_{[a,x],\infty} \|P_n - [P_{n+1}; a, x]\|_{[a,x],1}. \end{cases}$$

The proof follows by Lemma 1.

The interested reader may obtain different particular instances of integral inequalities on choosing the harmonic polynomials mentioned at the beginning of this section. We omit the details.

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Reference:Journal of Computational Analysis and Applications,Vol.7,No.1,49-69,2005.

Title:Iterative Algorithms for Multi-Valued Variational Inclusions in Banach Spaces.

Author appearing there:K.R.Kazmi.

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